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Résumé

Dans cette thèse, nous étudions les propriétés de la surface d'un champ alétoire. Plus précisément, nous nous intéressons à la loi du maximum d'un champ gaussien centré stationnaire et au volume de l'ensemble d'excursion (le temps de séjour). Nous améliorons la "méthode des records" en dimension 2 et la prolongeons à dimension 3 pour donner des bornes supérieures pour la queue de la distribution du maximum. Nous donnons aussi la formule asymptotique de cette queue en dimension 2. Il y a une correspondance entre la formule asymptotique et les coefficients de la formule de Steiner du domaine considéré. Il s'agit d'une prolongation du résultat de Adler. Nous étudions la vitesse de convergence dans le théorème de la limite centrale pour le temps de séjour dans deux cas: à niveau fixe et à niveau variable.

Abstract

In this thesis, we study the properties of the paths of random fields. More precisely, we are interested in the distribution of the maximum of stationary centered Gaussian field and the volume of the excursion set (sojourn time). We extend slightly the "record method" in dimension 2 and developpe it in dimension 3 to give an upper bound for the tail of the distribution of the maximum. We also give an asymptotic formula for this tail in dimension 2. There is a correspondence between the asymptotic formula and the coefficients of the Steiner formula of the domain considered. This can be viewed as an extension of some results of Adler. We study the rate of convergence of the central limit theorems of the sojourn time in both cases: fixed and moving level.

Introduction générale

La théorie des champs aléatoires est une branche importante des probabilités. Elle est très utile pour modéliser des phénomènes aléatoires spatiaux ou temporels. On trouve de nombreux domaines d'application : sciences de l'environnement et de la terre, épidémiologie, agronomie, météorologie, traitement d'image, neurosciences, etc. Dans cette thèse, nous nous intéressons aux deux sujets classiques suivants de la théorie des champs aléatoires :

1. La loi de la variable aléatoire $M_S = \sup\{X(t) : \ t \in S\}$ où S est un sous-ensemble de $\mathbb{R}^d$, $d > 1$ et X est un champ indexé par S.

2. Les ensembles d'excursion $\{t \in S : \ X(t) \geq u\}$ pour chaque niveau u.

La loi du maximum M_S est connue dans environ une dizaine cas, voir Azaïs et Wschebor [6, p.4]. Dans les autres cas, trouver une formule exacte reste un problème ouvert. Il faut donc construire des approximations de la distribution de M_S.

Les bornes en dimension 2 et 3 dans le cas Gaussien

Certaines inégalités générales, par example, l'inégalité de Borell-Sudakov-Tsirelson donnent sous des conditions peu restrictives, pour tout $u > 0$,

$$\mathrm{P}(|M_S - \mu(M_S)| > u) \leq \exp\left(-\frac{u^2}{2\sigma^2}\right), \qquad (1)$$

où $\mu(M_S)$ est la médiane et $\sigma^2 = \sup\{\mathrm{Var}(X(t)) : \ t \in S\}$. On peut également remplacer la médiane par l'espérance. Cependant, (1) est très général donc faible sur le plan numérique et insuffisante pour les applications statistiques. L'objet du deuxième chapitre de cette thèse est d'utiliser la régularité du champ aléatoire pour améliorer (1) et d'obtenir des inégalités signifiantes au point de vue numérique.

L'outil principal est la formule de Rice qui est développée par Adler et Taylor [1] ou Azaïs et Wschebor [6] dont voici un énoncé : soit $Z : U \to \mathbb{R}^d$ un champ aléatoire, U un sous-ensemble ouvert de $\mathbb{R}^d$ et $u \in \mathbb{R}^d$ un oint fixé ; alors sous certaines conditions, pour tout ensemble borélien

B inclus dans U,

$$\mathrm{E}(\mathrm{card}\{t \in B : Z(t) = u\}) = \int_B \mathrm{E}\left(|\det(Z'(t))| \mid Z(t) = u\right) p_{Z(t)}(u)dt,$$

où $p_{Z(t)}$ est la densité de variable aléatoire $Z(t)$. Elle permet de calculer l'espérance du nombre de maximal locaux et permet une bonne estimation de la fonction de répartition du maximum. Lorsqu'on cherche à appliquer cette formule, la difficulté est d'évaluer la quantité

$$\mathrm{E}(|\det(X''(t))|.\mathbb{I}_{X''(t)\preceq 0} \mid X(t) = x,\, X'(t) = 0).$$

Quand x tends vers l'infini, cette quantité est approximée par

$$\mathrm{E}(\det(X''(t)) \mid X(t) = x,\, X'(t) = 0) = H_{d-1}(x),$$

où $H_{d-1}(x)$ est le $d-1$ ème polynôme d'Hermite, ce qui donne la formule asymptotique dans les travaux de Adler et Taylor.

Combinant la formule de Rice et quelque techniques de la théorie des matrices aléatoires, Azaïs et Wschebor [5] trouvent la "méthode directe" pour calculer l'espérance

$$\mathrm{E}(|\det(X''(t))| \mid X(t) = x,\, X'(t) = 0)$$

dans le cas où le champ est isotrope, c-à-d. lorsque la fonction de covariance peut être exprimée comme fonction de la distance entre deux points :

$$\mathrm{Cov}(X(s), X(t)) = \rho(\|t-s\|^2)$$

où ρ est dans classe $\mathcal{C}^4$ et satisfait

$$\rho''(0) - \rho'(0)^2 \geq 0,$$

et donnent une borne supérieure pour n'importe quelle dimension d. Néanmoins, ces bornes sont exprimées sous un forme très complexe. Quand S est un polytope convexe,

$$\mathrm{P}(M_S \geq u) \leq \int_u^\infty \overline{p}(x)dx,$$

où

$$\overline{p}(x) = \varphi(x)\left\{\sum_{t\in S_0}\widehat{\sigma}_0(t) + \sum_{j=1}^{d}\left[(2\pi)^{-j/2}\, H_j(x) + R_j(x)\right] g_j\right\},$$

où $\widehat{\sigma}_0(t)$ et les R_j, g_j ont une expression relativement compliquée, voir Sousection 1.1.6.

Afin d'éviter le calcul de l'espérance de la valeur absolue du déterminant, Mercadier [35] donne la "méthode des records" pour avoir de meilleures bornes, plus simples. Elle cherche le point d'ordonnée minimale dans la courbe de niveau u et applique la formule de Rice au champ $Z(t) = (X(t), X'_1(t))$. En appliquant la formule de Rice, sa méthode permet de factoriser

$$\det(Z'(t)) = X''_{11}(t)X'_2(t),$$

sous la condition $Z(t) = (u, 0)$, d'où un calcul facile d'espérance. Malheureusement, elle ne s'applique qu'en dimension 2.

Nous nous intéressons à l'amélioration de la "méthode des records" en dimension 2 et la prolongeons en dimensions 3. En dimension 2, sans avoir besoin de paramétrer la frontière du domaine S, nous obtenons le

Théorème 0.1. *Soit* $\{X(t),\ t \in NS \subset \mathbb{R}^2\}$ *un champ gaussien centré stationnaire à trajectoires* $\mathcal{C}^1$*, défini dans un voisinage* NS *de* S*. Supposons que*

i. Il existe une direction supposée arbitrairement la première coordonnée telle que la dérivé $X''_{11}(t)$ *existe.*

ii. $\mathrm{E}(X(t)) = 0,\ \mathrm{Var}(X(t)) = 1,\ \mathrm{Var}X'(t) = I_2$ *et* $\mathrm{Var}(X''_{11}(t)) > 1$.

iii. S *est la limite Hausdorff d'une suite de polygones connexes* S_n.

Alors,

$$\mathrm{P}\{M_S \geq u\} \leq \overline{\Phi}(u) + \frac{\liminf \sigma_1(\partial S_n)\varphi(u)}{2\sqrt{2\pi}} + \frac{\sigma_2(S)}{2\pi}\left[c\varphi(u/c) + u\Phi(u/c)\right]\varphi(u), \tag{2}$$

où c, $\varphi(u)$, $\Phi(u)$, $\overline{\Phi}(u)$ *et* σ_i *sont respectivement* $\sqrt{\mathrm{Var}(X''_{11}) - 1}$*, la densité et la fonction de répartition d'une variable normale réduite,* $1 - \Phi(u)$ *et la mesure surface de dimension* i.

En dimension 2, sans utiliser la théorie des matrices aléatoires et certaines conditions restrictives comme Azaïs and Wschebor, nous pouvons calculer l'espérance de la valeur absolue du déterminant

Proposition 0.2. *Soit* X *un champ gaussien centré isotrope de variance unité en dimension 2. On a*

$$\mathrm{E}\left(|\det(X''(t))| \mid (X, X'_1, X'_2)(t) = (u, 0, 0)\right) = u^2 - 1 + 2\frac{(8\rho''(0))^{\frac{3}{2}}\exp(-u^2.(24\rho''(0) - 2)^{-1})}{\sqrt{24\rho''(0) - 2}}. \tag{3}$$

Et on donne la borne supérieure en dimension 3 :

Théorème 0.3. *Soit* S *un sous ensemble compact convexe de* $\mathbb{R}^3$ *et* $\{X(t),\ t \in NS \subset \mathbb{R}^2\}$ *un champ gaussien centré stationnaire, défini dans un voisinage* NS *de* S*. De plus, supposons que* *i. X est isotrope par rapport aux deux premières coordonées, c-à-d.* $\mathrm{Cov}(X(t_1,t_2,t_3);X(s_1,s_2,t_3)) = \rho((t_1-s_1)^2+(t_2-s_2)^2)$.

ii. Les trajectoires sont de classe $\mathcal{C}^2$.

iii. Pour tout $x \in \mathbb{R}^3$*, presque surement il n'existe pas de point* $t \in S$ *tel que* $Z(t) = (X(t), X_1'(t), X_2'(t)) = x$ *et* $\det(Z'(t)) = 0$.

Alors, pour tout réel u,

$$\begin{aligned}\mathrm{P}\{M_S \geq u\} \leq \quad & 1-\Phi(u) + \frac{2\lambda(S)}{\sqrt{2\pi}}\varphi(u) + \frac{\sigma_2(S)\varphi(u)}{4\pi}\left[\sqrt{12\rho''-1}\varphi(\frac{u}{\sqrt{12\rho''-1}}) + u\Phi(\frac{u}{\sqrt{12\rho''-1}})\right] \\ & + \frac{\sigma_3(S)\varphi(u)}{(2\pi)^{\frac{3}{2}}}\left[u^2-1+\frac{(8\rho'')^{\frac{3}{2}}\exp(-u^2.(24\rho''-2)^{-1})}{\sqrt{24\rho''-2}}\right],\end{aligned}$$

où λ *est le diamètre de calibre (caliper diameter).*

Étude asymptotique en dimension 2

Nous nous intéressons à l'étude asymptotique de la queue du maximum en dimension 2. Nous étudions la pertinence de la borne supérieure précédente. En comparant avec des résultats des méthodes connues comme les

- "Méthode de la caractéristique d'Euler", d'Adler et Taylor [1],
- "Méthode directe", d'Azaïs et Delmas [4], d'Azaïs et Wschebor [5],

qui donnent une bonne approximation dans le cas où S est convexe

$$\overline{\Phi}(u) + \frac{\sigma_1(\partial S)\varphi(u)}{2\sqrt{2\pi}} + \frac{\sigma_2(S)}{2\pi}u\varphi(u), \tag{4}$$

dont l'erreur est $o(\varphi((1+\delta)u))$ pour un certain δ positif. (2) est aussi une bonne approximation. Quand S satisfait seulement l'hypothèse minimale d'avoir un "outer Minkowski content" (OMC) défini par

$$\mathrm{OMC(S)} = \lim_{\epsilon\to 0}\frac{\sigma_2(\mathrm{S}^{+\epsilon}\setminus \mathrm{S})}{\epsilon}$$

si la limite existe, où $S^{+\epsilon}$ est le ϵ-voisinage de S défini par

$$S^{+\epsilon} = \{s \in \mathbb{R}^2 : \mathrm{dist}(s,S) \leq \epsilon\},$$

alors Azaïs et Wschebor [7] donnent un développement à deux terms

$$\mathrm{P}(M_S \geq u) = \frac{\mathrm{OMC}(S)\varphi(u)}{2\sqrt{2\pi}} + \frac{\sigma_2(S)}{2\pi}u\varphi(u) + o(\varphi(u)). \tag{5}$$

Sous certaines conditions de régularité, notre but est de raffiner (5) en calculant le coefficient de $\overline{\Phi}(u)$. On peut le voir comme une extension de (4).

Définition 0.1. *S est dit vérifier l'heuristique de la formule de Steiner (SFH) si*

- *Pour ϵ suffisamment petit, le terme principal du volume de $S^{+\epsilon}$ est exprimé comme un polynôme en ϵ, c-à-d.*

$$\sigma_2(S^{+\epsilon}) = \sigma_2(S) + \epsilon L_1(S) + \pi\epsilon^2 L_0(S) + o(\epsilon^2). \tag{6}$$

- $$\mathrm{P}(M_S \geq u) = L_0(S)\overline{\Phi}(u) + L_1(S)\frac{\varphi(u)}{2\sqrt{2\pi}} + \sigma_2(S)\frac{u\varphi(u)}{2\pi} + o(u^{-1}\varphi(u)). \tag{7}$$

Notre résultat principal est le suivant : quand S est un domaine connexe inclus dans $\mathbb{R}^2$ avec une frontière de classe $\mathcal{C}^2$ par morceau et un nombre fini de points irrégulièrs et $X(t)$ est un champ gaussien centré stationnaire de variance unité tel que $\mathrm{Var}(X') = I_2$ vérifiant certaines hypothèses additionnelles, alors S a la propriété SFH.

La constante L_0 est déterminée par la contribution des points concaves qui sont définis précisements au Chapitre 2 et dont un exemple est donné dans le figure 3.3.

Un exemple d'ensemble localement convexe mais irrégulier est proposé ; pour cet ensemble, le développement asymptotique contient des puissances entières et fractionaires, ce qui est nouveau. Notre méthode permet aussi de donner un développement à trois termes pour un champ indexé par un polytope dans $\mathbb{R}^3$.

Théorème central limite pour le volume de l'ensemble d'excursion

Considérons dans $\mathbb{R}^d$ l'hypercube $[0,T]^d$ et un niveau u_T qui peut éventuellement tendre vers l'infini. On définit le volume de l'ensemble d'excursion (ou temps de séjour) au dessus de u_T par

$$S_T = \int_{[0,T]^d} \mathbb{I}(X(t) \geq u_T)dt.$$

Nous cherchons un théorème central limite pour S_T. Ce sujet est à l'intersection de l'étude des propriétés géométriques des surfaces aléatoires et des formes fonctionnelles non linéaires des champs gaussiens. Bien qu'il existe de nombreux résultats qui montrent le théorème central limite, rien n'est dit sur la vitesse de convergence. Nous nous y intéressons dans deux cas: celui du niveau fixe et celui du niveau variable.

Nous considérons la distance de Wasserstein

$$d(X,Y) = \sup_{h \in \mathrm{Lip}(1)} |\mathrm{E}(h(X)) - \mathrm{E}(h(Y))|,$$

où Lip(1) est ensemble des fonctions Lipschitziennes de coefficient plus petit que 1; supposons que

(B). $\{X(t) : t \in \mathbb{R}^d\}$ un champ gaussien centré de variance unité et de fonction de covariance $\rho(t)$ telle que

$$\int_{\mathbb{R}^d} |\rho(t)|dt < \infty.$$

On a alors les résultats principaux

Théorème 0.4 **(Niveau fixe).** *Soit* $u_T = u = constante$. *Soit* $\{X(t) : t \in \mathbb{R}^d\}$ *un champ aléatoire satisfaisant la condition (B). Supposons que la fonction de covariance* ρ *satisfasse*

$$\int_{\mathbb{R}^d \setminus [-a,a]^d} |\rho(t)|dt \leq (const)(\log a)^{-1}, \text{ pour } a \to \infty. \tag{8}$$

Alors,

$$d\left(\frac{S_T - \mathrm{E}(S_T)}{\sqrt{T^d}}, \mathcal{N}(0, \sigma^2)\right) \leq C(\log T)^{-1/4},$$

où

$$0 < \sigma^2 = \sum_{n=1}^{\infty} \frac{\varphi^2(u)H_{n-1}^2(u)}{n!} \int_{\mathbb{R}^d} \rho^n(t)dt < \infty,$$

les H_n *étant les polynômes d'Hermite et* C *une constante qui ne dépend que du champ et du niveau.*

Théorème 0.5 **(Niveau variable).** *Soit* u_T *tel que* $u_T \to \infty$ *quand* $T \to \infty$. *Soit* $\{X(t) : t \in \mathbb{R}^d\}$ *un champ aléatoire satisfaisant la condition (B). Supposons qu'il existe une constante positive* $\alpha \in]0; 2]$ *telle que dans un voisinage de* 0, *la fonction de covariance* ρ *satisfasse*

$$1 - \rho(t) \cong (const)\|t\|^{\alpha} \text{ pour } t \to 0.$$

Alors, pour tout $\beta \in (0; d/2)$, *il existe une constante* C_β *qui ne dépend que du champ telle que*

$$d\left(\frac{S_T - \mathrm{E}(S_T)}{\sqrt{\mathrm{Var}(S_T)}}, \mathcal{N}(0,1)\right) \leq C_\beta \left[\sqrt{\frac{u_T^{\frac{2+\alpha}{\alpha}}}{(\log T)^{1/6}}} + \frac{1}{T^\beta \varphi(u_T)u_T}\right].$$

Author's articles

- *On the rate of convergence for central limit theorems of sojourn times of Gaussian fields*, Stochastic Processes and their Applications 123 (2013), pp. 2158-2174.

- *The record method for two and three dimensional parameters random fields*, with Jean Marc Azaïs, submitted.

- *Asymptotic formula for the tail of the maximum of smooth Gaussian fields on non locally convex sets*, with Jean Marc Azaïs, submitted.

Contents

Notation

M_S	$\sup\{X(t) :\ t \in S\}$
∂S	the boundary of S
$\overset{\circ}{S}$	the interior of S
σ_i	the surface measure of dimension i, can be defined as a Hausdorff measure
$X',\ X''$	the first and second derivatives of the process $X(t)$
X'_α	the derivative along the direction α
X''_{ij}	$\dfrac{\partial^2 X}{\partial x_i \partial x_j}$
$M \preceq 0$	the square matrix M is semi-definite negative
$S^{+\epsilon}$	$\{s \in \mathbb{R}^2 :\ \mathrm{dist}(s, S) \leq \epsilon\}$
d_H	the Hausdorff distance between sets, $d_H(S,T) = \inf\{\epsilon :\ S \subset T^{+\epsilon}, T \subset S^{+\epsilon}\}$
$\varphi(u)$	$\dfrac{exp(-u^2/2)}{\sqrt{2\pi}}$, the density of a standard normal variable
$\Phi(u)$	$\displaystyle\int_{-\infty}^{u} \varphi(x)dx$, distribution function of a standard normal variable
$\overline{\Phi}(u)$	$1 - \Phi(u)$
$p_Z(x)$	the value of the density function of random vector Z at point x
I_d	identity $d \times d$ matrix
$H_n(x)$	$(-1)^n e^{\frac{x^2}{2}} \dfrac{d^n}{dx^n} e^{\frac{-x^2}{2}}$, the n-th Hermite polynomial
$\Gamma(z)$	$\displaystyle\int_0^\infty t^{z-1} e^{-t} dt$, the Gamma function
x^+	$\sup(x, 0)$
x^-	$\sup(-x, 0)$

Chapter 1

Introduction

1.1 The excursion probability

The most important contribution of this thesis is to study the random variable

$$M_S = \sup\{X(t),\ t \in S \subset \mathbb{R}^d\},$$

where X is a stationary centered Gaussian process indexed by S which is a compact subset of $\mathbb{R}^d$. This problem is classic in probability theory and has many applications in statistics and related areas.

One famous example is when X is the standard Brownian motion and S is the interval $[0, T]$, then

$$\mathrm{P}\{M_S \geq u\} = 2\overline{\Phi}(u/\sqrt{T}) \text{ for } u > 0.$$

Such an explicit formula can be derived for some special processes listed in the page 4 in the book of Azaïs and Wschebor [6]. However, a general exact formula does not exist. Therefore, one needs to approximate the excursion probability

$$\mathrm{P}\{M_S \geq u\}.$$

In this section, we introduce some existing results for this problem.

1.1.1 The order of the tail and the isoperimetric inequalities

In the 70s, Landau, Marcus and Shepp [28], [33] proved that if the paths of the field are almost surely bounded then

$$\lim_{u\to\infty} u^{-2} \log \mathrm{P}\{M_S \geq u\} = -(2\sigma^2)^{-1},$$

where $\sigma^2 = \sup_{t\in S} \mathrm{Var}(X(t))$. Later on, Borell [12] and Ibragimov, Sudakov and Tsirelson [23] provided more precise results. They stated that

Theorem 1.1. *If* $\mathrm{P}\{M_S < \infty\} = 1$*, then*

1. $\sigma^2 = \sup_{t \in S} \mathrm{Var}(X(t)) < +\infty$.
2. $\mathrm{E}(M_S) < +\infty$.
3. *For every* $u > 0$,
$$\mathrm{P}\{M_S - \mathrm{E}(M_S) \geq u\} \leq \exp(-u^2/(2\sigma^2)).$$
4. *For every* $u > 0$,
$$\mathrm{P}\{M_S - m(M_S) \geq u\} \leq \exp(-u^2/(2\sigma^2)),$$
where $m(M_S)$ *is the median of random variable* M_S.

These inequalities are quite general and not sharp. So they can not be applied directly in the statistics tests. However, they are very useful to prove that a quantity is exponentially smaller.

1.1.2 Processes with unique point of maximal variance

Now, we consider the processes with unique point of maximal variance. From the equality in the above subsection, we see that for u large enough, the point of maximum value must concentrate at this unique point. Indeed, if there exists only $t_0 \in S$ such that $\mathrm{Var}(X(t_0)) = \sigma^2$, then
$$\mathrm{P}\{M_S \geq u\} \geq \mathrm{P}\{X(t_0) \geq u\} \geq \frac{\exp(-u^2/(2\sigma^2))}{\sqrt{2\pi}} \left(\frac{\sigma}{u} - \frac{\sigma^3}{u^3} \right),$$
this implies that
$$\lim_{u \to \infty} u^{-2} \log \mathrm{P}\{M_S \geq u\} \geq \lim_{u \to \infty} u^{-2} \log \mathrm{P}\{X(t_0) \geq u\} = -(2\sigma^2)^{-1}.$$
In this case, with some additional conditions, Talagrand [56] gave a stronger result

Theorem 1.2. *Let* $\{X(t),\, t \in S\}$ *be a centered, bounded Gaussian process. Assume that there is a unique point* $t_0 \in S$ *such that*
$$\mathrm{Var}(X(t_0)) = \sigma^2 = \sup_S \mathrm{Var}(X(t)).$$
For $\delta > 0$*, set*
$$T_\delta = \left\{ t \in S \,:\, \mathrm{E}(X(t)X(t_0)) \geq \sigma^2 - \delta^2 \right\}.$$
Suppose that
$$\lim_{\delta \to 0} \frac{\mathrm{E}(\sup_{t \in T_\delta} X(t))}{\delta} = 0.$$
Then
$$\lim_{u \to \infty} \frac{\mathrm{P}\{M_S \geq u\}}{\mathrm{P}\{X(t_0) \geq u\}} = 1.$$

We remark that under some other conditions, Berman [8] proved the same result.

1.1.3 The double sum method

Pickands [43] proposed an elegant way named "The double sum method" to find the asymptotic behavior of the excursion probability. The main idea of this method is: the indexed set is divided into the subsets $\{T_i,\, i = 1, \ldots, N\}$ that depend on the level u; and thanks to Bonferroni inequality,

$$\sum_i \mathrm{P}\{\sup_{t\in T_i} X(t) \geq u\} \geq \sum_i \mathrm{P}\{M_S \geq u\} \geq \sum_i \mathrm{P}\{\sup_{t\in T_i} X(t) \geq u\} - \sum_{i<j} \mathrm{P}\{\sup_{t\in T_i} X(t) \geq u,\ \sup_{t\in T_j} X(t) \geq u\}.$$

The name "double sum" comes from the sum of the probability $\mathrm{P}\{\sup_{t\in T_i} X(t) \geq u,\ \sup_{t\in T_j} X(t) \geq u\}$. This sum is proved to be exponentially smaller than the first term. So, asymptotically,

$$\mathrm{P}\{M_S \geq u\} = \sum_i \mathrm{P}\{\sup_{t\in T_i} X(t) \geq u\}, \text{ for } u \to \infty.$$

We give some examples.

We consider a centered stationary Gaussian process $X(t)$ with a covariance function satisfying the conditions

$$r(t) = 1 - |t|^\alpha + o(|t|^\alpha),\ \ t \to 0,$$

and $r(t) \neq \pm 1$ for all $t > 0$. Now we divide the interval $[0, T]$ into the sub-intervals of length $u^{-2/\alpha}T$. Applying the double-sum method and by the stationarity, we need to study

$$\begin{aligned} \mathrm{P}\left\{\max_{[0,u^{-2/\alpha}T]} X(t) > u\right\} &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-v^2/2} \mathrm{P}\left\{\max_{[0,u^{-2/\alpha}T]} X(t) > u \mid X(0) = v\right\} dv \\ &= \frac{e^{-u^2/2}}{\sqrt{2\pi}u} \int_{\mathbb{R}} e^{w - \frac{w^2}{2u^2}} \mathrm{P}\left\{\max_{[0,u^{-2/\alpha}T]} X(t) > u \mid X(0) = u - \frac{w}{u}\right\} dw, \end{aligned}$$

Here we use the change of variable $v = u - w/u$. We denote

$$\chi_u(t) = u(X(u^{-2/\alpha}t) - u) + w.$$

Then, the event

$$\max_{[0,u^{-2/\alpha}T]} X(t) > u$$

can be expressed as

$$\max_{[0,T]} \chi_u(t) > w.$$

Note that, when u tends to infinity, the conditional distribution of χ_u converges to the distribution of the fractional Brownian motion with a shift $\chi(t)$ defined as follows: $\{\chi(t),\, t \in \mathbb{R}\}$ is a Gaussian process with continuous paths, having the expectation function

$$\mathrm{E}\chi(t) = -|t|^\alpha$$

and the covariance function

$$\mathrm{cov}(\chi(t), \chi(s)) = |t|^\alpha + |s|^\alpha - |t-s|^\alpha.$$

Thus, as $u \to \infty$,

$$\mathrm{P}\left\{\max_{[0,u^{-2/\alpha}T]} X(t) > u\right\} \sim \frac{\varphi(u)}{u}\int_{\mathbb{R}} e^w \mathrm{P}\left(\max_{[0,T]}\chi(t) > w\right) dw = \frac{\varphi(u)}{u}\mathrm{E}\exp(\max_{t\in[0,T]}\chi(t)).$$

Let us define the Pickands constant

$$H_\alpha = \lim_{T\to\infty}\frac{\mathrm{E}\exp(\max_{t\in[0,T]}\chi(t))}{T},\ 0 < H_\alpha < \infty.$$

Then from the above observations, we have the following theorem

Theorem 1.3. *Let $\{X(t),\, t \in [0,T]\}$ be a centered stationary Gaussian process. Assume that the covariance function satisfies the conditions*

$$r(t) = 1 - |t|^\alpha + o(|t|^\alpha),\ \ t \to 0,$$

and $r(t) \neq \pm 1$ for all $t > 0$. Then,

$$\mathrm{P}\{M_T \geq u\} = H_\alpha T u^{2/\alpha}\overline{\Phi}(u)(1+o(1)).$$

Now we return to the process with unique point of maximal variance. We have

Theorem 1.4. *Let $\{X(t),\, t \in [0,T]\}$ be a centered Gaussian process having continuous paths. Assume that the covariance function attains its maximum (equals to 1) at the unque point t_0 in the interior of $[0,T]$. Futher assume that X satisfies the conditions*

- *For some positive a, β,*

$$\mathrm{Var}(X(t)) = 1 - a|t-t_0|^\beta(1+o(1)),\ \ t \to t_0.$$

- *For some positive α,*

$$\mathrm{Cov}(X(t), X(s)) = 1 - |t-s|^\alpha(1+o(1)),\ \ t, s \to t_0.$$

- *For some positive γ, G,*

$$\mathrm{E}(X(t) - X(s))^2 \geq G|t-s|^\gamma.$$

One has

i. If $\beta > \alpha$, then

$$\mathrm{P}\{M_T \geq u\} = \frac{2H_\alpha\Gamma(1/\beta)}{\beta a^{1/\beta}}u^{\frac{2}{\alpha}-\frac{2}{\beta}}\overline{\Phi}(u)(1+o(1)).$$

ii. If $\beta = \alpha$, then

$$\mathrm{P}\{M_T \geq u\} = 2H_\alpha^a \overline{\Phi}(u)(1 + o(1)),$$

where

$$0 < H_\alpha^a = \lim_{K\to\infty} H_\alpha^a(K) < \infty,$$

$$H_\alpha^a(K) = \mathrm{E}\exp(\max_{t\in[-K,K]}(\chi(t) - a|t|^\alpha)).$$

iii. If $\beta < \alpha$, then

$$\mathrm{P}\{M_T \geq u\} = \overline{\Phi}(u)(1 + o(1)).$$

There exist also the multidimensional versions of the above theorems. For more details, we refer to the book of Piterbarg [44]. Now the difficulty is the Pickands constant H_α since it is not easy to calculate. In some special cases we have the exact value of H_α, for example, let $\{X(t),\, t \in S \subset \mathbb{R}^d\}$ be a standard isotropic Gaussian field, i.e a stationary isotropic centered Gaussian field with unit variance and paths of class $\mathcal{C}^2$, whose the covariance matrix of $X'(t)$ equals to the identity matrix, then under some mild conditions,

$$\mathrm{P}\{M_S \geq u\} = \lambda_d(S)u^{d-1}\overline{\Phi}(u)(1 + o(1)).$$

1.1.4 Rice method for one-parameter stationary Gaussian processes

Now we consider a very special class of one-parameter Gaussian processes, that is a stationary process with $\mathcal{C}^1-$paths (in this case $\alpha = 2$). Bulinskaya[15] showed that almost surely there is no critical point with value u. Then we have an observation that

$$\begin{aligned}
\mathrm{P}\{M_T \geq u\} = \;& \mathrm{P}\{X(0) \geq u\} + \mathrm{P}\{X(0) < u,\; M_T \geq u\} \\
\leq \;& \mathrm{P}\{X(0) \geq u\} + \mathrm{P}\{\exists t \in [0,T]:\; X(t) = u,\; X'(t) > 0\} \\
\leq \;& \mathrm{P}\{X(0) \geq u\} + \mathrm{E}(\mathrm{card}\{t \in [0,T]:\; X(t) = u,\; X'(t) > 0\}).
\end{aligned}$$

In the 40s, Rice[46] gave a formula to calculate the mean number of level crossings and also the mean number of upcrossings

Theorem 1.5 (Rice formula). *Let $\{X(t),\, t \in [0,T]\}$ be a Gaussian process having $\mathcal{C}^1$-paths then*

$$\mathrm{E}(card\{t \in [0,T]:\; X(t) = u\}) = \int_0^T \mathrm{E}\left(|X'(t)| \mid X(t) = u\right) p_{X(t)}(u)dt,$$

and

$$\mathrm{E}(card\{t \in [0,T]:\; X(t) = u,\; X'(t) > 0\}) = \int_0^T \mathrm{E}\left(X'^{+}(t) \mid X(t) = u\right) p_{X(t)}(u)dt.$$

We obtain the upper bound

$$P\{M_T \geq u\} \leq \overline{\Phi}(u) + \frac{e^{-u^2/2}}{2\pi} T \sqrt{\mathrm{Var}(X'(t))}.$$

This upper bound is proved to be a good approximation in the sense that the difference between it and the excursion probability is exponentially smaller under the additional conditions that

1. The covariance function is of class $\mathcal{C}^4$.
2. $r(s) \neq \pm 1 \; \forall s > 0$.

1.1.5 The Euler characteristic method

Many authors investigated in the general case when S is a subset of $\mathbb{R}^d$. When $d > 1$, the number of level upcrossings is infinite, so the above observation does not work. However, instead of considering the level upcrossings, we can search for some special points. Then, we need a formula to calculate the expectation. This is the origin of the generalized Rice formula of Azaïs and Wschebor [6] or the Metatheorem of Adler and Taylor [1].

Theorem 1.6 (Generalized Rice formula). *Let $X : U \to \mathbb{R}^d$ be a Gaussian random field, U an open subset of $\mathbb{R}^d$, and $u \in \mathbb{R}^d$ a fixed point. Assume that:*

(i) Almost surely the path of X is of class $\mathcal{C}^1$.

(ii) For each $t \in U$, $X(t)$ has a nondegenerate distribution.

(iii) $P\{\exists t \in U : \; X(t) = u, \; \det(X'(t)) = 0\} = 0$.

Then for every Borel subset B of U,

$$\mathrm{E}(card\{t \in B : \; X(t) = u\}) = \int_B \mathrm{E}\left(|\det(X'(t))| \mid X(t) = u\right) p_{X(t)}(u)\, dt. \tag{1.1}$$

If B is compact, both sides of (1.1) are finite.

Adler and Taylor [1] suggested a nice and surprising way to estimate the asymptotic behavior. That is "The Euler characteristic method".

- **Some conditions about the indexed set**

  We will clarify the notation of a locally convex, regular stratified manifold set that is considered by Adler and Taylor.

  Definition 1.1. Let $\tilde{M}$ be a $\mathcal{C}^k$ manifold. A subspace $M \subset \tilde{M}$ is called a $\mathcal{C}^l$, $l \leq k$, stratified space if it has a partition $\mathcal{Z}$ such that

1. Each piece $S \in \mathcal{Z}$ is an embedded $\mathcal{C}^l$ submanifold without boundary of $\tilde{M}$.
2. For $R, S \in \mathcal{Z}$, if $R \cap \overline{S} \neq \emptyset$, then $R \subset \overline{S}$.

From this definiton, a stratified M can be written as

$$M = \bigcup_{l=0}^{\dim M} \partial_l M,$$

where $\partial_l M$ is the l-dimensional boundary of M.

Definition 1.2. A stratified space $(M, \mathcal{Z})$ is said to satisfy the Whitney condition if for every point $t \in S \in \mathcal{Z}$, for all piece $\tilde{S} \supset S$ and chart $(U, \varphi) \ni t$, one has:
Consider two convergent sequences $t_n \to t$ and $s_n \to t$ such that $t_n \in S$ and $s_n \in \tilde{S}$ for all n. Suppose that the sequence of line segments $\overline{\varphi(t_n)\varphi(s_n)}$ converges in projective space to a line l and the sequence of tangent spaces $T_{s_n}\tilde{S}$ converges to a subspace $\tau \subset T_t\tilde{M}$. Then $\varphi_*^{-1}(l) \subset \tau$.

Definition 1.3. A Whitney stratified space $(M, \mathcal{Z})$ embedded in a manifold $\tilde{M}$ is called locally convex if for every point $t \in M$, the support cone at t obtained from the collection of limiting directions

$$\left\{c'(0) : \ c \in \mathcal{C}^1([-1, 1], M), \, c(0) = t\right\}$$

is convex.

For the definitions of cone space and $C-$tame manifold, see [1, p. 198]. We are able to define a locally convex, regular stratified manifold.

Definition 1.4. A $\mathcal{C}^2$ Whitney stratified manifold M is called regular stratified if it is a $\mathcal{C}^{2,1}$ cone space of arbitrary depth and also C-tame for some finite C.
If, in addition, M is locally convex, then it is called a locally convex, regular stratified manifold.

- **What is Euler characteristic?**

  1. Let us recall some notations from the theory of integral geometry

     Definition 1.5. A compact subset A of $\mathbb{R}^d$ is *basic* if the intersection $E \cap A$ is simply connected for all affine subspaces E of the form

     $$E = \{t \in \mathbb{R}^d : \ t_j = a_j \ \forall j \in J \subset \{1, 2, \ldots, d\}\}.$$

     Definition 1.6. A subset A of $\mathbb{R}^d$ is *basic complex* if it is the union of a finite number of basic sets $\{A_1, A_2, \ldots, A_k\}$ such that any intersection of all the elements in a subset of $\{A_1, A_2, \ldots, A_k\}$ is also basic.

Definition 1.7. The *Euler characteristic* of a basic complex A is defined by

$$\varphi(A) = \begin{cases} 0 \ \text{ if } \ A = \emptyset, \\ 1 \ \text{ if } \ A \neq \emptyset \text{ is basic}, \\ \sum\limits_{i=1}^{k} (-1)^{i-1} \sum\limits_{\{j_1,\dots,j_i\} \subset \{1,2,\dots,d\}} \varphi(A_{j_1} \cap \dots \cap A_{j_i}) \ \text{ otherwise}. \end{cases}$$

We can prove that the Euler characteristic does not depend on the partition. And an important remark is that a Whitney stratified space has a well-defined Euler characteristic.

The formulas of the Euler characteristic of compact subsets in dimension 1 and 2 are respectively number of disjoint intervals and number of connected components minus number of holes.

2. **Morse function**

Definition 1.8. Let $\tilde{M}$ be a $\mathcal{C}^k$ manifold. For $\tilde{f} \in C^2(\tilde{M})$, a critical point of $\tilde{f}$ is a point $t \in \tilde{M}$ such that $\nabla \tilde{f}(t) = 0$.

Now we can define a Morse function as

Definition 1.9. Let M be a $\mathcal{C}^2$ Whitney stratified manifold embedded in a $\mathcal{C}^3$ manifold $\tilde{M}$. A function $f \in \mathcal{C}^2(\tilde{M})$ is called a Morse function on M if for all $k = 0, \dots, \dim(M)$,

1. On the k-dimensional boundary $\partial_k M$, $f_{|\partial_k M}$ is nondegenerate in the sense that at all critical point $t \in \partial_k M$, the Hessian $\nabla^2 f_{|T_t \partial_k M}$ is nondegenerate.
2. The restriction of f on $\overline{\partial_k M}$ has no critical point on $\cup_{j=0}^{k-1} \partial_j M$.

The most important properties of a Morse function are:

* The Euler characteristic of M is calculable through the function $f = \tilde{f}_{|M}$.
* The Euler characteristic of the exursion set $f^{-1}[u, \infty)$ is well-defined.

3. For example, we consider a function on the cube I^d. Let I_k be the set of faces of dimension k in I^d. For each face J in I_k, there are a set $\sigma(J) \subset \{1, 2, \dots, d\}$ with k elements and a set $\epsilon(J) = \{\epsilon_1, \dots, \epsilon_{d-k}\} \subset \{0,1\}^{d-k}$ such that

$$J = \{t \in I^d : \ 0 < t_j < 1 \quad \text{if } j \in \sigma(J), \quad t_j = \epsilon_j \quad \text{if } j \notin \sigma(J)\}.$$

Then,

$$\varphi(\{t \in I^d : \ f(t) \geq u\}) = \sum_{k=0}^{d} \sum_{J \in I_k} \sum_{i=0}^{k} (-1)^i \mu_i(J),$$

with

$$\mu_i(J) = \{t \in J : \ \ f(t) \geq u, \ f_j(t) = 0 \ \forall j \in \sigma(J), \ (2\epsilon_j - 1) f_j(t) > 0 \ \forall j \notin \sigma(J), \\ \text{index}(\nabla^2 f_{|J}) = i\},$$

where the index of a symmetric matrix is the number of its negative eigenvalues.

- **The result**

  The main idea of the "Euler characteristic method" is for u large enough, we can approximate the excursion probability by the expectation of the Euler characteristic of the excursion set and calculate this expectation by using the Metatheorem.

  Before stating the main theorem, we recall the definition of an isotropic Gaussian field

  Definition 1.10. A centered Gaussian field X is called to be isotropic if the covariance function can be expressed under the form

  $$\mathrm{E}(X(s)X(t)) = \rho(\|t-s\|^2),$$

  where $\rho : \mathbb{R}^+ \to \mathbb{R}$ is of class $\mathcal{C}^4$.

  A standard isotropic Gaussian field is an isotropic field satisfying

  1. $\rho(0) = 1$.
  2. $\rho'(0) = -1/2$, i.e $\mathrm{Var}(X'(t)) = I_d$.

  Theorem 1.7. *Let $S \subset \mathbb{R}^d$ be a locally convex regular stratified manifold and X be a standard isotropic Gaussian field defined on a neighborhood of S. Then,*

  – $\mathrm{E}(\varphi(\{t \in S : X(t) \geq u\})) = C_0\overline{\Phi}(u) + u^d e^{-u^2/2}\sum_{i=1}^{d} C_i u^{-i}$,
  where the C_j are the Lipschitz-Killing curvatures of S and in particular, C_0 is the Euler characteristic of S.

  – *There exists a positive constant $\alpha > 1$ such that*

  $$|\mathrm{P}(M_S \geq u) - \mathrm{E}(\varphi(\{t \in S : X(t) \geq u\}))| < O(e^{-\alpha u^2/2}).$$

  In addition, when S is a compact convex set with non-empty interior, α can be chosen equal to

  $$1 + \frac{1}{12\rho''(0) - 1}.$$

1.1.6 The direct method

Azaïs and Wschebor[5] considered the problem of finding the upper bound for the excursion probability which is very important in statistics tests. Their method is based on the idea of counting the number of the local maxima on each face and using the Rice formula.

Theorem 1.8. *Let $\{X(t) : t \in S \subset \mathbb{R}^d\}$ be a standard isotropic Gaussian field. Assume that S has polyhedral shape with the decomposition $S = S_0 \cup \ldots \cup S_d$ where S_i is a manifold of dimension i without boundary and X satisfies*

1. *X can be defined on an open neighborhood of S and has $\mathcal{C}^2$ paths.*

2. *For every $s, t \in S$, $s \neq t$, the distribution of $(X(s), X(t))$ does not degenerate.*

3. *Almost surely the maximum of $X(t)$ on S is attained at a single point.*

4. *Almost surely for every $j = 1, \ldots, d$ there is no point $t \in S_j$ such that $X'_{|S_j}(t) = 0$ and $X''_{|S_j}(t) = 0$.*

Then the random variable M_S has the density function $p(x)$ and $p(x) \leq \overline{p}(x)$ where $\overline{p}(x)$ equals to

$$\overline{p}(x) = \varphi(x) \left\{ \sum_{t \in S_0} \widehat{\sigma}_0(t) + \sum_{j=1}^{d} \left[(2\pi)^{-j/2} H_j(x) + R_j(x) \right] g_j \right\},$$

where

- *$H_j(x)$ is the Hermite polynomial.*
- *g_j is the geometric parameter of the face S_j defined by*

$$g_j = \int_{S_j} \widehat{\sigma}_j(t) \sigma_j(dt),$$

with

  - *$\sigma_j(dt)$ is the geometric measure of S_j.*
  - *$\widehat{\sigma}_j(t)$ is the normalized solid angle of the cone $\widehat{C}_{t,j}$, i.e*

$$\widehat{\sigma}_j(t) = \frac{\sigma_{d-j-1}(\widehat{C}_{t,j} \cap \mathcal{S}^{d-j-1})}{\sigma_{d-j-1}(\mathcal{S}^{d-j-1})} \text{ for } j = 0, \ldots, d-1$$
$$\widehat{\sigma}_d(t) = 1.$$

  Here we define $\widehat{C}_{t,j}$ by dual cone of $C_{t,j}$, i.e

$$\widehat{C}_{t,j} = \{ z \in \mathbb{R}^d : \langle z, \lambda \rangle \geq 0 \ \forall \lambda \in C_{t,j} \},$$

  where

$$C_{t,j} = \left\{ \lambda \in \mathbb{R}^d : \exists s_n \in S \text{ such} \quad \text{that } s_n \to t \text{ and } \lim_{n \to \infty} \frac{t - s_n}{\|t - s_n\|} = \lambda \right\}$$

  whenever this set is not empty and $C_{t,j} = \{0\}$ the otherwise; and $\mathcal{S}^{d-j-1}$ is the $(d-j-1)$-dimensional unit sphere.

- *For $j = 1, \ldots, d$,*

$$R_j(x) = \left(\frac{4\rho''(0)}{\pi} \right)^{j/2} \frac{\Gamma((j+1)/2)}{\pi} \int_{\mathbb{R}} T_j(v) \, e^{-y^2/2} \, dy,$$

where

– Γ *is the Gamma function.*

– $v = \dfrac{-1}{\sqrt{2}}\left(\sqrt{1-\dfrac{1}{4\rho''(0)}}y - \dfrac{x}{2\sqrt{\rho''(0)}}\right).$

– $T_j(v) = \left(\sum_{k=0}^{j-1}\dfrac{\overline{H}_k^2(v)}{2^k k!}\right)e^{-v^2/2} - \dfrac{\overline{H}_j(v)}{2^j(j-1)!}I_{j-1}(v),$

with

$$\overline{H}_j(v) = e^{v^2}\left(-\frac{\partial}{\partial v}\right)^j e^{-v^2},$$

$$I_j(v) = 2e^{-v^2/2}\sum_{k=0}^{[(j-1)/2]} 2^k\frac{(j-1)!!}{(j-1-2k)!!}\overline{H}_{j-1-2k}(v) + \mathbb{I}_{\{j|2\}}2^{j/2}(j-1)!!\overline{\Phi}(v).$$

Note that when we integrate the main term

$$\varphi(x)\left\{\sum_{t\in S_0}\widehat{\sigma}_0(t) + \sum_{j=1}^{d}\left[(2\pi)^{-j/2}H_j(x)\right]g_j\right\},$$

from u to infinity, we see again the approximation of the Euler characteristic method.

1.1.7 The record method

The main difficulty to apply the Rice formula is the expectation of the absolute value of the determinant since it is not easy to calculate explicitly. Adler and Taylor ignored the absolute sign since as u tends to infinity, the probability that the determinant is negative, is very small; and they obtained the asymptotic behaviour. To deal with, Azaïs and Wschebor assumed the isotropic condition and used the random matrix theory to obtain a quite complicated upper bound that works for all dimension.

There is another method to find the upper bound which is true for every level that is the "record method". However, it works only for low dimension, since for higher dimension, it faces the same difficulty of calcaluting the expectation of absolute value of the determinant. It was first raised by I. Rychlik[48] in 1990 for Gaussian process and lately refined and extended to two-parameter case by Mercadier[35] in 2006. The main idea of this method is finding the record point, that is the smallest point in the level curve in some order, and also using the Rice formula. The advantage of this method is to give a bound in a simpler form than the direct method and very efficient for the numerical computation.
For one-parameter process, we have the following theorem

Theorem 1.9 (Rychlik). *Let $\{X(t),\, t\in[0,T]\}$ be a Gaussian process having $\mathcal{C}^1$-paths and satisfying*

1. *For all $s,t\in[0,T],\ s<t$, the distribution of $(X(s),X(t))$ does not degenerate.*

2. *For all* $t \in [0,T]$, *the distribution of* $(X(t), X'(t))$ *does not degenerate.*

Then

$$\mathrm{P}\{M_T \geq u\} = \mathrm{P}\{X(0) \geq u\} + \int_0^T \mathrm{E}\left(X'^{+}(t)\mathbb{I}_{\{X(s)<X(t),\ \forall s<t\}}\right) p_{X(t)}(u)\, dt.$$

If we remove the condition $\{X(s) < X(t),\ \forall s < t\}$ under the conditional expectation, then we will obtain the upper bound that we have seen before. Another meaning of the above formula is that we can give a discrete approximation of the condition $\{X(s) < X(t) = u,\ \forall s < t\}$ by $\{X(kt/n) < u,\ k = 0,\dots,n-1\}$ and therefore we can use some computer programs for the calculation.

For two-dimensional field, Mercadier also gave two versions: one is an exact formula that is useful for numerical purpose and the other one is an upper bound. Here we recall the second one

Theorem 1.10. *Let* $\{X(t),\ t \in S \subset \mathbb{R}^2\}$ *be a Gaussian field with* $\mathcal{C}^2-$*paths. Assume that*

1. S *is compact;* S *and its complement are connected.*
2. *There exists a continous parametrization* $\rho:\ [0,L] \to \partial S$ *of the boundary of* S *by its length that is of class* $\mathcal{C}^1$ *except at some finite points.*

Then, under some mild conditions, for every u,

$$\begin{aligned}\mathrm{P}\{M_S \geq u\} \leq \mathrm{P}\{Y(0) \geq u\} \quad &+ \int_0^L \mathrm{E}\left(|Y'(t)| \mid Y(t) = u\right) p_{Y(t)}(u)\, dt \\ &+ \int_S \mathrm{E}\left(|X''_{11}(t)^{+} X'_2(t)^{-}| \mid X(t) = u,\ X'_1(t) = 0\right) p_{X(t),X'_1(t)}(u,0)\, dt,\end{aligned}$$

where $Y(t) = X(\rho(t))$.

1.1.8 Another approach

In all the results above, the authors assumed that the indexed set S is a submanifold of $\mathbb{R}^d$. Azaïs and Wschebor[7] studied the problem when S is a fractal set. We recall the definition of the Minkowski dimension and content of a fractal set

Definition 1.11. Let S be a subset of $\mathbb{R}^d$. For each $\epsilon > 0$, define the $\epsilon-$neighborhood $S^{+\epsilon}$ of S is the set $\{t \in \mathbb{R}^d :\ \mathrm{dist}(t,S) \leq \epsilon\}$.

1. The Minkowski dimension of S is defined as

$$n = d - \lim_{\epsilon \to 0} \frac{\log \lambda_d(S^{+\epsilon})}{\log \epsilon},$$

whenever the limit exists.

2. The Minkowski content of a fractal set with Minkowski dimension n is a positive constant C such that

$$\lambda_d(S^{+\epsilon}) \cong C\epsilon^{d-n} \text{ for } \epsilon \to 0.$$

Azaïs and Wschebor have proved the following theorem

Theorem 1.11. *Let $\{X(t):\ t \in S \subset \mathbb{R}^d\}$ be a centered stationary Gaussian field satisfying*

1. $\mathrm{Var}(X(t)) = 1$ *and* $\mathrm{Var}(X'(t)) = I_d$.
2. *The paths are of class $\mathcal{C}^3$.*
3. *For all $s \neq t$, the distribution of $(X(s), X(t), X'(s), X'(t))$ does not degenerate.*
4. *For all $\lambda \in \mathcal{S}^{d-1}$, the distribution of $(X(t), X'(t), X''(t)\lambda)$ does not degenerate.*

Then,

i. *If the Minkowski dimension of S is well-defined, then as $u \to +\infty$,*

$$\log \mathrm{P}\{M_S \geq u\} = \frac{-u^2}{2} + (n - 1 + o(1)) \log u.$$

ii. *If the Minkowski dimension and content of S are well-defined, then as $u \to +\infty$,*

$$\mathrm{P}\{M_S \geq u\} \cong \frac{C}{2^{n/2}\pi^{d/2}} \Gamma(1 + (d-n)/2) u^{n-1} \varphi(u),$$

where Γ is the Gamma function.

In the usual case, when S is a body in $\mathbb{R}^n$ with positive volume, then the Minkowski dimension of S is n and the content is the volume of S, so we have a well-known result.

1.1.9 The tube method

This is an interesting method proposed by J. Sun[51] and later developped by Takemura and Kuriki [53] to find the approximation of the excursion probabilities of Gaussian processes with unit variance. It has two versions.

- When the process has the finite Karhunen-Loève expansion, it can be written as

$$X(t) = \sum_{i=1}^{d} a_{t,i} Z_i,$$

where $Z_i \overset{i.i.d}{\sim} \mathcal{N}(0,1)$ and $a_{t,1}^2 + \ldots + a_{t,d}^2 = 1\, \forall t \in S$. It means that the original process can be seen as a process indexed by a subset $\Omega(S)$ of the sphere $\mathcal{S}^{d-1}$, i.e, with $Z = (Z_1, \ldots, Z_d)$,

$$\{X(t),\ t \in S\} = \{\langle a_t, Z\rangle,\ a_t \in \Omega(S) \subset \mathcal{S}^{d-1}\}.$$

Let

$$U = \left(\frac{Z_1}{\|Z\|}, \ldots, \frac{Z_d}{\|Z\|} \right)$$

be the random vector with uniform distribution on $\mathcal{S}^{d-1}$. Then,

$$\begin{aligned} \mathrm{P}(M_S \geq u) &= \int_u^\infty \mathrm{P}(\max_{a_t \in \Omega(S)} \langle a_t, U \rangle \geq u/z) \, \mathrm{P}(\|Z\| \in dz) \\ &= \int_u^{(1+r)u} + \int_{(1+r)u}^\infty, \end{aligned} \tag{1.2}$$

where r is the critical radius which can be defined as the supremum of all ϵ positive such that in $\Omega(S)^{+\epsilon}$ all points have the unique projection on $\Omega(S)$.

Now, from the observation that for a fixed point $a \in \mathcal{S}^{d-1}$, the set

$$\{b \in S^{d-1} : \langle a, b \rangle \geq u/z\}$$

is just a tube around a in $\mathcal{S}^{d-1}$ with the geodesic distance, we have

$$\max_{a_t \in \Omega(S)} \langle a_t, b \rangle \geq u/z \Leftrightarrow b \in \Omega(S)^{+\epsilon},$$

with ϵ depends on z. Therefore, it needs to calculate the surface measure of $\Omega(S)^{+\epsilon}$. From the Weyl formula, if $\Omega(S)$ is a manifold embedded in $\mathcal{S}^{d-1}$ and $\epsilon < r$, then the volume (surface measure) of its $\epsilon-$neighborhood can be written as

$$\frac{2\pi^{m/2}}{\Gamma(m/2)} \sum_{0 \leq i \leq n, \, i \text{ even}} \kappa_i J_i(\arccos(1 - \epsilon^2/2)),$$

where $m = d - 1 - n$, κ_i's are the invariant constants of $\Omega(S)$ and $J_i(x)$'s are defined as

$$J_0(x) = \int_0^x \sin^{m-1}(x) \cos^n(x) \, dx,$$

$$m(m+2)\ldots(m+i-2) J_i(x) = \int_0^x \sin^{m+i-1}(x) \cos^{n-i}(x) \, dx \quad \text{for } i = 2, 4, \ldots$$

Substituting into 1.2, we have

$$\mathrm{P}(M_S \geq u) = \sum_{0 \leq i \leq n, \, i \text{ even}} \kappa_i \psi_i(u) + o(\psi_{2[n/2]}(u)),$$

where

$$\psi_i(u) = \frac{1}{2^{1+i/2} \pi^{(n+1)/2}} \int_{u^2/2}^\infty x^{(n+1-i)/2-1} e^{-x} \, dx \quad \text{for } i = 0, 2, \ldots$$

- When the Karhunen-Loève expansion is not finite, one can use a truncate version with finite expansion and using the method as above to get an approximation with two terms.

1.1.10 Main results in the thesis

A main direction in this thesis is the study about the excursion probability. We extend the record method to the two and three dimensional fields. For two-parameter field, we can remove the condition about the parametrization of the boundary and we can obtain a more accurate upper bound. We apply the record method for three-parameter field to obtain an upper bound in the case where the indexed set S is a convex body. There is a corresponding of the coefficients between our bound and the asymptotic approximation of Adler and Taylor. This can be viewed in Chapter 2.

Another important contribution in this direction is that we can derive an asymptotic formula for the excursion probability for some cases when the indexed set S is not locally convex. From our results, we observe that in some cases the expectation of the Euler characteristic is not a good approximation; and we conjecture that there is a relation between the coefficients of the asymptotic formula and the expansion of the general Steiner formula for domain S, then the formula of Adler and Taylor is just a special case. We give a full expansion for the expansion in dimension 2 and 3-term expansion in dimension 3, see Chapter 3.

1.2 Central limit theorem for the sojourn time

In this thesis, we also study a related object with the excursion probability; that is the volume of the excursion set (sojourn time). More precisely, let $X = \{X(t), t \in \mathbb{R}^d\}$ be a stationary centered Gaussian field and T be a measurable subset of $\mathbb{R}^d$; then the sojourn time of X above the level u_T in T is defined as

$$\int_T \mathbb{I}(X(t) \geq u_T)dt.$$

It is clear that the excursion set is nonempty if and only if the sojourn time is positive.

The sojourn time is used to approximate the distribution of the maximum as in Berman[8]. It is also used to study the random surface. Many authors investigated in the central limit theorem

Theorem 1.12. *Let $\{X(t) : t \in \mathbb{R}^d\}$ be a stationary centered Gaussian field with unit variance and covariance function $\rho(t)$ such that*

$$\int_{\mathbb{R}^d} |\rho(t)|dt < \infty.$$

For a fixed real-valued u, define the sojourn time as

$$S_T = \int_{[0,T]^d} \mathbb{I}(X(t) \geq u)dt. \quad (1.3)$$

Then, as T tends to infinity,

$$\frac{S_T - T^d\overline{\Phi}(u)}{\sqrt{T^d}} \xrightarrow{d} \mathcal{N}(0,\sigma^2),$$

where

$$0 < \sigma^2 = \sum_{n=1}^{\infty} \frac{\varphi^2(u)H_{n-1}^2(u)}{n!} \int_{\mathbb{R}^d} \rho^n(t)dt < \infty. \tag{1.4}$$

One of the using method is due to Berman: he proved the central limit theorem for the m-dependent processes and then approximated the original process by the m-dependent ones.

However, we did not know much information about the rate of convergence. Recently, a new method is introduced by Nourdin, Nualart and Peccati [36], [37], [38], [41]. They represent the sojourn time in the Wiener chaos expansion; use the "fourth moment condition" to normalize each chaos; and sum up. Their method also allows us to study about the rate of convergence.

In this thesis, we give the results about the rate of convergence for the central limit theorem of sojourn time in two cases: the fixed and moving level; see Chapter 4.

Chapter 2

The record method

2.1 Introduction

The problem of computing the tail of the maximum has a lot of applications in spatial statistics, image processing, oceanography, genetics etc ..., see for example Cressie and Wikle [17]. It is exactly solved only for about ten processes with parameter of dimension 1, see Azaïs and Wschebor [6, p. 4] for a complete list. In the other cases, one has to use some approximations. Several methods have been used, in particular

- The tube method, Sun [51].
- Double sum method, Piterbarg [44].
- Euler characteristic method see, for example, Adler and Taylor [1].
- Rice or direct method, Azaïs and Delmas [4], Azaïs and Wschebor [5].

With respect to these methods, the record method which is the main subject of this chapter and which is detailed in Section 2.2 has the advantage of simplicity and also the advantage of giving a bound which is non asymptotic: it is true for every level and not for large u only.

It has been introduced for one-parameter random processes by Rychlik [48] and extended to two-parameter random fields by Mercadier [35] to study the tail of the maximum of smooth Gaussian random fields on rather regular sets.

It has two version, one is an exact implicit formula : Theorem 2 in [35] that is interesting for numerical purpose and that will not be considered here; the other form is a bound for the tail, see inequality (2.1) hereunder.

This bound has the advantage of its simplicity. In particular it avoids the computation of the expectation of the absolute value of the Hessian determinant as in the direct method of [5] but it works only dimension 2.

For practical applications, the dimensions 2 and 3 (for the parameter set) are the most relevant, so there is a need of an extension to dimension 3 and this is done in Section 2.3 using results on quadratic forms by Li and Wei [31].

The bound also has the drawback of demanding a parameterization of the boundary. For example, if we consider the version of Azaïs and Wschebor ([6], Theorem 9.5) of the result of Mercadier, under some mild conditions on the set $S \subset \mathbb{R}^2$ and on the Gaussian process X, we have

$$\begin{aligned} \mathrm{P}\{M_S \geq u\} \leq & \mathrm{P}\{Y(O) \geq u\} + \int_0^L \mathrm{E}(|Y'(l)| \mid Y(l) = u) p_{Y(l)}(u)\, dl \\ & + \int_S \mathrm{E}(|X''_{11}(t)^- X'_2(t)^+| \mid X(t) = u,\, X'_1(t) = 0) p_{X(t), X'_1(t)}(u, 0)\, dt, \end{aligned} \tag{2.1}$$

where

- $Y(l) = X(\rho(l))$ with $\rho : [0, L] \to \partial S$ is a parameterization of the boundary ∂S by its length.
- $X''_{ij} = \dfrac{\partial^2 X}{\partial x_i \partial x_j}$.
- $p_Z(x)$: the value of the density function of random vector Z at point x.
- $x^+ = \sup(x, 0), \quad x^- = \sup(-x, 0)$.

The proof is based on considering the point with minimal ordinate (second coordinate) on the level curve. As we will see, this point can be considered as a "record point".
So the second direction of generalizations is to propose nicer and stronger forms of the inequality (2.1). This is done in Section 2.2. The result on quadratic form is presented in Section 2.4 and some numerical experiment is presented in Section 2.5.

2.2 The record method in dimension 2 revisited

We will work essentially under the following assumption:
Assumption 1: $\{X(t),\ t \in NS \subset \mathbb{R}^2\}$ is a Gaussian stationary field, defined in a neighborhood NS of S with $\mathcal{C}^1$ paths and such that there exists some direction, that will be assumed (without loss of generality) to be the direction of the first coordinate, in which the second derivative $X''_{11}(t)$ exists.
We assume moreover the following normalizing conditions that can always be obtained by a scaling

$$\mathrm{E}(X(t)) = 0,\ \mathrm{Var}(X(t)) = 1,\ \mathrm{Var} X'(t) = I_2.$$

Finally we assume that $\mathrm{Var}(X''_{11}(t)) > 1$ which is true as soon as the spectral measure of the process restricted to the first axis is not concentrated on two opposite atoms.

In some cases we will assume in addition

Assumption 2: $X(t)$ is isotropic, i.e $\mathrm{Cov}(X(s), X(t)) = \rho(\|t-s\|^2)$, with $\mathcal{C}^2$ paths and S is a convex polygon.

Under Assumption 1 and 2 plus some light additional hypotheses, the Euler Characteritic (EC) method [1] gives

$$\mathrm{P}\{M_S \geq u\} = \mathrm{P}_E(u) + \mathrm{Rest},$$

with

$$\mathrm{P}_E(u) = \overline{\Phi}(u) + \frac{\sigma_1(\partial S)}{2\sqrt{2\pi}}\varphi(u) + \frac{\sigma_2(S)}{2\pi}u\varphi(u),$$

where the rest is super exponentially smaller.

The direct method gives [5]

$$\begin{aligned}\mathrm{P}\{M_S \geq u\} \leq \mathrm{P}_M(u) = \overline{\Phi}(u) + \frac{\sigma_1(\partial S)}{2\sqrt{2\pi}}\int_u^\infty \left[c\varphi(x/c) + x\Phi(x/\mathrm{c})\right]\varphi(x)dx \\ + \frac{\sigma_2(S)}{2\pi}\int_u^\infty \left[x^2 - 1 + \frac{(8\rho''(0))^{3/2}\exp(-x^2.(24\rho''(0)-2)^{-1})}{\sqrt{24\rho''(0)-2}}\right]\varphi(x)dx, \qquad (2.2)\end{aligned}$$

where $c = \sqrt{\mathrm{Var}(X''_{11}) - 1} = \sqrt{12\rho''(0) - 1}$.

The record method gives [35]

$$\mathrm{P}\{M_S \geq u\} \leq \overline{\Phi}(u) + \frac{\sigma_1(\partial S)}{\sqrt{2\pi}}\varphi(u) + \frac{\sigma_2(S)}{2\pi}\left[c\varphi(u/c) + u\Phi(u/\mathrm{c})\right]\varphi(u).$$

A careful examination of these equations shows that the main terms are almost the same except that in the record method the coefficient of $\sigma_1(\partial S)$ is twice too large. When S is a rectangle $[0, T_1] \times [0, T_2]$, it is easy to prove that this coefficient 2 can be removed, see for example Exercise 9.2 in [6].

The goal of this section is to extend the result above to more general sets and to fields satisfying Assumption 1 only. The main result of this section is the following

Theorem 2.1. *Let X satisfy the Assumption 1 and suppose that S is the Hausdorff limit of connected polygons S_n. Then,*

$$\mathrm{P}\{M_S \geq u\} \leq \overline{\Phi}(u) + \frac{\liminf_n \sigma_1(\partial S_n)\varphi(u)}{2\sqrt{2\pi}} + \frac{\sigma_2(S)}{2\pi}\left[c\varphi(u/c) + u\Phi(u/c)\right]\varphi(u), \qquad (2.3)$$

where $c = \sqrt{\mathrm{Var}(X''_{11}) - 1}$.

Remark: the choice of the direction of ordinates is arbitrary and is a consequence of the arbitrary choice of the the second derivative X''_{11}. When the process $X(t)$ admits derivative in all direction, the choice that gives the sharpest bound consists in chosing as first axis, the direction α such that $\mathrm{Var}(X''_{\alpha\alpha})$ is minimum.

Unfortunately the proof it is based on an exotic topological property of the set S that will be called "emptyable".

Definition 2.1. The compact set S is emptyable if there exists a point $O \in S$ which has minimal ordinate, and such that for every $s \in S$ there exists a continuous path inside S from O to s with non decreasing ordinate.

In other word, suppose that S is filled with water and that gravity is in the usual direction; S is emptyable if after making a small hole at O, all the water will empty out, see Figure 2.1.

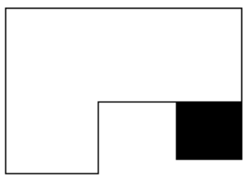


Figure 2.1: Example of non-emptyable set. The non-emptyable part is displayed in black.

Proof. Step 1 : Suppose for the moment that X has $\mathcal{C}^\infty$ paths and that S is an emptyable polygon. Considering the event $\{M_S \geq u\}$, we have

$$\mathrm{P}\{M_S \geq u\} = \mathrm{P}\{X(O) \geq u\} + \mathrm{P}\{X(O) < u, M_S \geq u\}. \tag{2.4}$$

It is clear that if $X(O) < u$ and $M_S \geq u$, because S is connected, the level curve

$$\mathcal{C}_u = \{t \in S : \ X(t) = u\}$$

is not empty, and there is at least one point T on $\mathcal{C}_u$ with minimal ordinate. There are two possibilities:

- T is in the interior of S. In that case, suppose that there exists a point $s \in S$ with smaller ordinate than T ($s_2 < T_2$), such that $X(s) > u$. Then, due to the emptyable property, on the continuous path from O to s there would exist one point s' with smaller ordinate than T, and with $X(s') = u$. This is in contradiction with the definition of T. So we have proved that for every $s \in S$, $s_2 < T_2$ we have $X(s) \leq u$. It is in the sense that T can be considered as a record point. It implies that

$$\{X'_1(T) = 0, \ X'_2(T) \geq 0, \ X''_{11}(T) \leq 0\}.$$

The probability that there exists such a point is clearly bounded, by the Markov inequality, by

$$\mathrm{E}\left(\mathrm{card}\{t \in S:\ X(t)=u,\ X'_1(t)=0,\ X'_2(t)\geq 0,\ X''_{11}(t)\leq 0\}\right).$$

Applying the Rice formula to the field $Z=(X,X'_1)$ from $\mathbb{R}^2$ to $\mathbb{R}^2$, we get that

$$\begin{aligned}
& \mathrm{P}\{\exists\, t \in \overset{\circ}{S}:\ X(t)=u,\ t \text{ has minimal ordinate on } \mathcal{C}_u\} \\
\leq\ & \int_S \mathrm{E}\left(|\det(Z'(t))| \mathbb{I}_{X'_2(t)\geq 0}\mathbb{I}_{X''_{11}(t)\leq 0} \mid Z(t)=(u,0)\right) \times p_{Z(t)}(u,0)\, dt \\
=\ & \sigma_2(S)\frac{\varphi(u)}{\sqrt{2\pi}}\, \mathrm{E}\left(X''^{-}_{11}(t) X'^{+}_2(t) \mid X(t)=u, X'_1(t)=0\right) \\
=\ & \sigma_2(S)\frac{\varphi(u)}{\sqrt{2\pi}}\, \mathrm{E}\left(X'^{+}_2(t)\right)\ \mathrm{E}\left(X''^{-}_{11}(t) \mid X(t)=u, X'_1(t)=0\right) \\
=\ & \sigma_2(S)\frac{\varphi(u)}{2\pi}\left[c\varphi(u/c)+u\Phi(u/c)\right]. \qquad (2.5)
\end{aligned}$$

Note that the validity of the Rice formula holds true because the paths are of class $\mathcal{C}^\infty$ and that $X(t)$ and $X'_1(t)$ are independent. The computations above use some extra independences that are a consequence of the normalization of the process. The main point is that, under the conditioning

$$\det(Z'(t)) = X''_{11}(t)X'_2(t).$$

- T is on the boundary of S that is the union of the edges $(F_1,\ldots,F_n)$. It is with probability 1 not located on a vertex. Suppose that, without loss of generality, it belongs to F_1. Using the reasoning we have done in the preceding case, because of the emptyable property, it is easy to see that

$$\{X(T)=u,\ X'_\alpha(T)\geq 0,\ X'_\beta(T)\leq 0\},$$

where α is the upward direction on F_1 and β is the inward horizontal direction. Then, apply the Markov inequality and Rice formula in the edge F_1,

$$\begin{aligned}
& \mathrm{P}\{\exists\, t \in F_1:\ X(t)=u,\ t \text{ has minimal ordinate on } \mathcal{C}_u\} \\
\leq\ & \mathrm{P}\{\exists\, t \in F_1:\ X(t)=u,\ X'_\alpha(t)\geq 0,\ X'_\beta(t)\leq 0\} \\
\leq\ & \mathrm{E}\left(\mathrm{card}\{t \in F_1:\ X(t)=u,\ X'_\alpha(t)\geq 0,\ X'_\beta(t)\leq 0\}\right) \\
=\ & \int_{F_1} \mathrm{E}\left(|X'_\alpha(t))| \mathbb{I}_{X'_\alpha(t)\geq 0}\mathbb{I}_{X'_\beta(t)\leq 0} \mid X(t)=(u)\right) \times p_{X(t)}(u)\, dt \\
=\ & \sigma_1(F_1)\varphi(u)\, \mathrm{E}\left(X'^{+}_\alpha(t)\, \mathbb{I}_{X'_\beta(t)\leq 0}\right).
\end{aligned}$$

Denote by θ_1 the angle (α,β). X'_β can be expressed as

$$\cos\theta_1\, X'_\alpha + \sin\theta_1\, Y,$$

with Y is a standard normal variable that is independent with X'_α. Then

$$\begin{aligned} & \mathrm{E}(X'^+_\alpha(t)\, \mathbb{I}_{X'_\beta(t)\le 0}) \\ = \; & \mathrm{E}(X'^+_\alpha\, \mathbb{I}_{\cos\theta_1 X'_\alpha + \sin\theta_1 Y \le 0}) \\ = \; & \frac{1-\cos\theta_1}{2\sqrt{2\pi}}. \end{aligned}$$

Summing up, the term corresponding to the boundary of S is at most equal to

$$\varphi(u)\sum_{i=1}^{n}\frac{(1-\cos\theta_i)\sigma_1(F_i)}{2\sqrt{2\pi}} = \frac{\varphi(u)\sigma_1(\partial S)}{2\sqrt{2\pi}}, \tag{2.6}$$

since $\sum_{i=1}^{n}\sigma_1(F_i)\,\cos\theta_i$ is just the length of the oriented projection of the boundary of S on the x -axis, so it is zero.

Hence, summing up (2.5), (2.6) and substituting into (2.4), we obtain the desired upper-bound in our particular case.

Step 2: Suppose now that S is a general connected polygon such that the vertex O with minimal ordinate is unique. We define S_1 as the maximal emptyable subset of S that contains O. It is easy to prove that S_1 is still a polygon with some horizontal edges and that $S\backslash S_1$ consists of several polygons with horizontal edges, say $S_2^1, \dots, S_2^{n_2}$, see Figure 2.2.

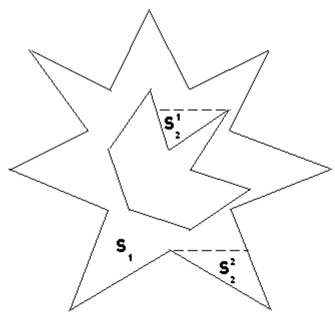


Figure 2.2: Example on construction of S_1.

So we write

$$\mathrm{P}\{M_S \ge u\} \le \mathrm{P}\{X(O)\ge u\} + \mathrm{P}\{M_{S_1}\ge u,\ X(O)<u\} + \sum_{i=1}^{n_2}\mathrm{P}\{M_{S_1}<u,\ M_{S_2^i}\ge u\}. \tag{2.7}$$

Suppose for the moment that all the $S_2^i,\ i = 1, \dots, n$ are emptyable. Then, to give bounds to the event

$$\{M_{S_1} < u,\ M_{S_2^i} \geq u\},$$

we can apply the reasoning of the preceding proof but inverting the direction: in S_2^i, we search points on the level curve with **maximum ordinate**. Let E be the common edge of S_1 and S_2^i. Clearly, when $\{M_{S_1} < u,\ M_{S_2^i} \geq u\}$, the level curve is non empty and by the same arguments as in Theorem 4.1, there exists $t \in S_2^i$ satisfying whether (except events with zero probability)

- t is in the interior of S_2^i and

$$\{X(t) = u,\ X_1'(t) = 0,\ X_2'(t) \leq 0,\ X_{11}''(t) \leq 0\}.$$

From Markov inequality and Rice formula, this probability is at most equal to

$$\frac{\varphi(u)\sigma_2(S_2^i)}{2\pi}\left[c\varphi(u/c) + u\Phi(u/c)\right]. \tag{2.8}$$

- t lies on some edges of S_2^i. Note that t can not belong to E. Then, as in Step 1, we consider the event t is on each edge and sum up the bounds to obtain

$$\begin{aligned} & \mathrm{P}\left(\{\exists\, t \in \partial S_2^i :\ X(t) = u,\ t\,\text{has maximal second ordinate on the level curve}\,\} \cap \{M_{S_1} < u\}\right) \\ \leq \ & \frac{\varphi(u)[\sigma_1(\partial S_2^i) - 2\sigma_1(E)]}{2\sqrt{2\pi}}. \end{aligned} \tag{2.9}$$

From (2.8) and (2.9) we have

$$\mathrm{P}\{M_{S_1} < u,\ M_{S_2^i} \geq u\} \leq \frac{\varphi(u)\sigma_2(S_2^i)}{2\pi}\left[c\varphi(u/c) + u\Phi(u/c)\right] + \frac{\varphi(u)[\sigma_1(\partial S_2^i) - 2\sigma_1(E)]}{2\sqrt{2\pi}}. \tag{2.10}$$

Summing up all the bounds as in (2.10), considering the upper bound for $\mathrm{P}\{X(O) < u,\ M_{S_1} \geq u\}$ as in Step 1 and substituting into (2.7), we get the result.
In the general case, when some S_2^i is not emptyable, we can decompose S_2^i as we did for S and by induction. Since the number of vertices is decreasing, we get the result.

Step 3: Passing to the limit. The extension to process with non $\mathcal{C}^\infty$ paths is direct by an approximation argument. Let $\overline{X}_\epsilon(t)$ be the Gaussian field obtained by convolution of $X(t)$ with a size ϵ convolution kernel (for example a Gaussian density with variance $\epsilon^2 I_2$). We can apply the preceding bound to the process

$$X_\epsilon(t) := \frac{1}{\sqrt{\mathrm{Var}(\overline{X}_\epsilon(t))}}\overline{X}_\epsilon\left(\Sigma_\epsilon^{-1/2}t\right),$$

where $\Sigma_\epsilon = \mathrm{Var}(\overline{X}_\epsilon'(t))$. Since $\mathrm{Var}(\overline{X}_\epsilon(t)) \to 1$ and $\Sigma_\epsilon \to I_2$, $\max_{t\in S} X_\epsilon(t) \to M_S$ and we are done.

The passage to the limit for S_n tending to S is direct. □

Some examples

- If S is compact convex with non-empty interior then it is easy to construct a sequence of polygons S_n converging to S and such that $\liminf_n \sigma_1(\partial S_n) = \sigma_1(\partial S)$, giving

$$\mathrm{P}\{M_S \geq u\} \leq \mathrm{P}_R(u) = \overline{\Phi}(u) + \frac{\sigma_1(\partial S)}{2\sqrt{2\pi}}\varphi(u) + \frac{\sigma_2(S)}{2\pi}\left[c\varphi(u/c) + u\Phi(u/\mathrm{c})\right]\varphi(u). \quad (2.11)$$

- More generaly, if S is compact and has a boundary that is piecewise-$\mathcal{C}^2$ except for a finite number of points and the closure of the interior of S equals to S, we get (2.11) by the same tools.

- Let us now get rid of the condition $\overline{\overset{\circ}{S}} = S$ but still assuming the piecewise-$\mathcal{C}^2$ condition. Define the "outer Minkowski content" of a closed subset $S \subset \mathbb{R}^2$ as (see [20])

$$\mathrm{OMC}(S) = \lim_{\epsilon \to 0} \frac{\sigma_2(S^{+\epsilon} \setminus S)}{\epsilon},$$

whenever the limit exists (for more treatment in this subject, see [2]). This definition of the perimeter differs from the quantity $\sigma_1(\partial S)$. A simple counter-example is a set corresponding to the preceding example with some "whiskers" added. Using approximation by polygons, we get

$$\mathrm{P}\{M_S \geq u\} \leq \mathrm{P}_R(u) = \overline{\Phi}(u) + \frac{\mathrm{OMC}(S)}{2\sqrt{2\pi}}\varphi(u) + \frac{\sigma_2(S)}{2\pi}\left[c\varphi(u/c) + u\Phi(u/\mathrm{c})\right]\varphi(u). \quad (2.12)$$

- The next generalization concerns compact r-convex sets with a positive r in the sense of [18]. These sets satisfy

$$S = \bigcap_{\overset{\circ}{B}(x,r) \cap S = \emptyset} \mathbb{R}^2 \setminus \overset{\circ}{B}(x,r).$$

This condition is slightly more general than the condition of having positive reach in the sense of Federer [20]. Suppose in addition that S satisfies the interior local connectivity property: there exists $\alpha_0 > 0$ such that for all $0 < \alpha < \alpha_0$ and for all $x \in S$, $\mathrm{int}\,(B(x,\alpha) \cap S)$ is a non-empty connected set. Then we can construct a sequence of approximating polygons in the following way.

Let $X_1, X_2, \ldots, X_n$ be a random sample drawn from a uniform distribution on S and S_n be the r-convex hull of this sample, i.e

$$S_n = \bigcap_{\overset{\circ}{B}(x,r) \cap \{X_1, X_2, \ldots, X_n\} = \emptyset} \mathbb{R}^2 \setminus \overset{\circ}{B}(x,r),$$

which can be approximated by polygons with an arbitrary error. By Theorem 6 of Cuevas et al [18], S_n is a fully consistent estimator of S, it means that $d_H(S_n, S)$ and $d_H(\partial S_n, \partial S)$ tend to 0 as n tends to infinity. This implies $\sigma_2(S_n) \to \sigma_2(S)$ and $\mathrm{OMC}(S_n) \to \mathrm{OMC}(S)$. Hence, we obtain (2.12).

- A complicated case: a "Swiss cheese". Here, we consider an unit square and inside it, we remove a sequence of disjoint disks of radius r_i such that $\pi \sum_{i=1}^{\infty} r_i^2 < 1$ to obtain the set S. When $\sum_{i=1}^{\infty} r_i < \infty$ the bound (2.3) makes sense directly. But examples can be constructed from the Sierpinski carpet (see Figure 2.3) such that $\sum_{i=1}^{\infty} r_i = \infty$: divide the square into 9 subsquares of the same size and instead of removing the central square, remove the disk inscribed in this square and do the same procedure for the remaining 8 subsquares, ad infinitum.

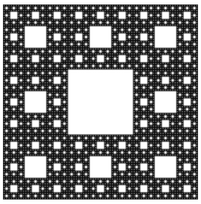


Figure 2.3: Sierpinski carpet (source: Wikipedia).

In our case,

$$\sum_{i=1}^{\infty} r_i^2 = \frac{1}{4} \sum_{i=1}^{\infty} \frac{8^{i-1}}{3^{2i}} = \frac{1}{4}$$

This proves that the obtained set S has positive Lebesgue measure and is not fractal. We have on the other hand

$$\sum_{i=1}^{\infty} r_i = \frac{1}{2} \sum_{i=1}^{\infty} \frac{8^{i-1}}{3^{i}} = \infty.$$

Let S_n be the set obtained after removing the n-th disk. Since $S \subset S_n$,

$$\mathrm{P}\{M_S \geq u\} \leq \mathrm{P}\{M_{S_n} \geq u\} \leq \overline{\Phi}(u) + \frac{\varphi(u)}{2\sqrt{2\pi}}(4+2\pi\sum_{i=1}^{n} r_i) + (1-\pi\sum_{i=1}^{n} r_i^2)\,[c\varphi(u/c) + u\Phi(u/c)]\,\varphi(u)/(2\pi).$$

Hence,

$$\mathrm{P}\{M_S \geq u\} \leq \overline{\Phi}(u) + \min_n \left[\frac{\varphi(u)}{2\sqrt{2\pi}}(4+2\pi\sum_{i=1}^{n} r_i) + (1-\pi\sum_{i=1}^{n} r_i^2)\,[c\varphi(u/c) + u\Phi(u/c)]\,\varphi(u)/(2\pi) \right].$$

Remarks:

1. In comparison with other results, all the examples considered here are new. Firstly the conditions on the process are minimal and weaker than the ones of the other methods. Secondly the considered sets are not covered by any other methods. Even for the first

example, because we do not assume that the number of irregular points is finite, which is needed, for example, for the convex set to be a stratified manifold as in [1].

2. Theorem 4.1 can be extended directly to non connected sets using sub-additivity

$$\mathrm{P}\{M_{S_1 \cup S_2} \geq u\} \leq \mathrm{P}\{M_{S_1} \geq u\} + \mathrm{P}\{M_{S_2} \geq u\}.$$

This implies that the coefficient of $\overline{\Phi}(u)$ in (2.3) must be the number of components.

Is the bound sharp?

- Under Assumption 2, Adler and Taylor [1] show that

$$\liminf_{u \to +\infty} -2u^{-2} \log |\mathrm{P}\{M_S \geq u\} - \mathrm{P}_E(u)| \geq 1 + 1/c^2.$$

From

$$0 \leq \mathrm{P}_R(u) - \mathrm{P}_E(u) = \frac{\sigma_2(S)}{2\pi} \varphi(u) \left[c\varphi(u/c) - u\overline{\Phi}(u/c)\right]$$

and the elementary inequality for $x > 0$,

$$\varphi(x) \left(\frac{1}{x} - \frac{1}{x^3} \right) < \overline{\Phi}(x) < \varphi(x) \left(\frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5} \right),$$

it is easy to see that

$$\liminf_{u \to +\infty} -2u^{-2} \log(\mathrm{P}_R(u) - \mathrm{P}_E(u)) \geq 1 + 1/c^2.$$

So the upper bound $\mathrm{P}_R(u)$ is as sharp as $\mathrm{P}_E(u)$.

- Let S be a compact and simply connected domain in $\mathbb{R}^2$ having a $\mathcal{C}^3$-piecewise boundary. Assume that all the discontinuity point are convex, in the sense that if we parametrize the boundary in the direction of positive rotation, then at each discontinuity point, the angle of the tangent has a positive discontinuity. Then, it is easy to see that the quantity

$$\kappa(S) = \sup_{t \in S} \sup_{s \in S,\, s \neq t} \frac{\mathrm{dist}(s - t, \mathcal{C}_t)}{\|s - t\|^2}$$

is finite, where dist is the Euclidean distance and C_t is the cone generated by the set of directions

$$\left\{ \lambda \in \mathbb{R}^2 : \ \|\lambda\| = 1, \exists s_n \in S \text{ such that } s_n \to t \text{ and } \frac{s_n - t}{\|s_n - t\|} \to \lambda \right\}.$$

In order to apply the Theorem 8.12 in [6], besides the Assumption 1, we make some additional assumptions on the field X such that it satisfies the conditions (A1)-(A5) page 185 in [6]. Assume that

- X has $\mathcal{C}^3$ paths.
- The covariance function $r(t)$ satisfies $|r(t)| \neq 1$ for all $t \neq 0$.
- For all $s \neq t$, the distribution of $(X(s), X(t), X'(s), X'(t))$ does not degenerate.

With these hypotheses, we can see that

- The conditions (A1)-(A3) are easily verified.
- The condition (A4) which states that the maximum is attained at a single point, can be deduced from Proposition 6.11 in [6] since for $s \neq t$, $(X(s), X(t), X'(s), X'(t))$ has a nondegenerate distribution.
- The condition (A5) which states that almost surely there is no point $t \in S$ such that $X'(t) = 0$ and $\det(X''(t)) = 0$, can be deduced from Proposition 6.5 in [6] applied to the process $X'(t)$.

Since all the required conditions are met, by Theorem 8.12 in [6], we have

$$\liminf_{x \to +\infty} -2x^{-2} \log \left[\mathrm{P}_M(x) - \mathrm{P}\{M_S \geq x\} \right] \geq 1 + \inf_{t \in S} \frac{1}{\sigma_t^2 + \kappa_t^2} > 1, \tag{2.13}$$

where

$$\sigma_t^2 = \sup_{s \in S \setminus \{t\}} \frac{\mathrm{Var}\left(X(s) \mid X(t), X'(t)\right)}{(1 - r(s,t))^2}$$

and

$$\kappa_t = \sup_{s \in S \setminus \{t\}} \frac{\mathrm{dist}\left(\frac{\partial}{\partial t} r(s,t), C_t\right)}{1 - r(s,t)}.$$

Note that the condition $\kappa(S)$ is finite implies that $\kappa(t)$ is also finite for every $t \in S$. (2.13) is true also for P_R, since as $x \to +\infty$, $\mathrm{P}_R(x)$ is smaller than $\mathrm{P}_M(x)$ (see Section 2.5 for the easy proof). As a consequence P_R is super exponentially sharp.

- Suppose that S is a circle in $\mathbb{R}^2$. Then $\{X(t) : t \in S\}$ can be viewed as a periodic process on the line. In that case, it is easy to show, see for example Exercise 4.2 in [6], that as $u \to \infty$

$$\mathrm{P}(M_S \geq u) = \frac{\sigma_1(S)}{\sqrt{2\pi}} \varphi(u) + O(\varphi(u(1+\delta)) = \frac{\mathrm{OMC}(S)}{2\sqrt{2\pi}} \varphi(u) + O(\varphi(u(1+\delta))$$

for some $\delta > 0$; while Theorem 4.1 gives with a standard approximation of the circle by polygons

$$\mathrm{P}(M_S \geq u) \leq P_R(u) = \overline{\Phi}(u) + \frac{\mathrm{OMC}(S)}{2\sqrt{2\pi}} \varphi(u),$$

which is too large. This shows that the bound P_R is not always super exponentially sharp.

2.3 The record method in dimension 3

For example, with the direct method, some difficulties arise in dimension 3 because we need to compute

$$\mathrm{E}|\det(X''(t)|,$$

under some conditional law. This can be conducted only in the isotropic case using random matrices theory, see [5] and even in this case the result is complicated. In dimension 2, the record method is a trick that permits to spare a dimension in the size of the determinant that we have to consider because the conditioning implies a factorization. For example in equation (2.5) we have used the fact that

$$\det(Z'(t)) = X''_{11}(t)X'_2(t),$$

under the condition. In this section we will use the same kind of trick to pass from a 3,3 matrix to a 2,2 matrix and then a 2,2 determinant is just a quadratic form so we can use, to compute the expectation of its absolute value, the Fourier method of Berry and Dennis [9] or Li and Wei [31]. This computation is detailed in Section 2.4.
Before stating the main theorem of this section, we recall the following lemma (see Chapter 5 of Prasolov and Sharygin [45])

Lemma 2.2. *Let $Oxyz$ be a trihedral. Denote by a, b and c the plane angles $\widehat{xOy}$, $\widehat{yOz}$ and $\widehat{zOx}$, respectively. Denote by A, B and C the angles between two faces containing the line Oz, Ox and Oy, respectively. Then,*

a. $\sin a : \sin A = \sin b : \sin B = \sin c : sinC.$

b. $\cos a = \cos b \cos c + \sin b \sin c \cos A.$

Our main result is the following

Theorem 2.3. *Let S be a compact and convex subset of $\mathbb{R}^3$ with non-empty interior and let X satisfy Assumption 1. Suppose, in addition that X is isotropic with respect to the first and second coordinate, i.e*

$$\mathrm{Cov}(X(t_1,t_2,t_3); X(s_1,s_2,t_3)) = \rho((t_1-s_1)^2 + (t_2-s_2)^2) \text{ with } \rho \text{ of class } \mathcal{C}^2.$$

Then, for every real u,

$$\mathrm{P}\{M \geq u\} \leq \quad 1-\Phi(u) + \frac{2\lambda(S)}{\sqrt{2\pi}}\varphi(u) + \frac{\sigma_2(S)\varphi(u)}{4\pi}\left[\sqrt{12\rho''(0)-1}\,\varphi\left(\frac{u}{\sqrt{12\rho''(0)-1}}\right) + u\Phi\left(\frac{u}{\sqrt{12\rho''(0}}\right.\right.$$
$$+\frac{\sigma_3(S)\varphi(u)}{(2\pi)^{3/2}}\left[u^2 - 1 + \frac{(8\rho''(0))^{3/2}\exp\left(-u^2.(24\rho''(0)-2)^{-1}\right)}{\sqrt{24\rho''(0)-2}}\right],$$

where λ is the caliper diameter.

Proof. By the same limit argument as in Theorem 2.1, we can assume that $X(t)$ has $\mathcal{C}^\infty$ paths and that S is a convex polyhedron. Let O be the vertex of S that has minimal third coordinate, we can assume also that this vertex is unique. It is clear that if $X(O) < u$ and $M_S > u$ then the level set

$$\mathcal{C}(u) = \{t \in S : \ X(t) = u\}$$

is non empty and there exists at least one point T having minimal third coordinate on this set. Then,

$$\begin{aligned} \mathrm{P}\{M_S \geq u\} &= \mathrm{P}\{X(O) \geq u\} + \mathrm{P}\{X(O) < u, \, M_S \geq u\} \\ &\leq \mathrm{P}\{X(O) \geq u\} + \mathrm{P}\{\exists T \in S : \ X(T) = u, \, T \, \text{has minimal third coordinate on } \mathcal{C}_u\}. \end{aligned} \tag{2.14}$$

Now, we consider three possibilities:

• Firstly, if T is in the interior of S, then by the same arguments as in Theorem 4.1, for all the point $s \in S$ with the third coordinate smaller than the one of T, $X(s) < X(T)$; it means that, at T, $X(t)$ has a local maximum with respect to the first and second coordinates and is non-decreasing with respect to the third coordinate. Therefore, setting

$$A(t) = \begin{pmatrix} X''_{11}(t) & X''_{12}(t) \\ X''_{12}(t) & X''_{22}(t) \end{pmatrix},$$

we have

$$\{X(T) = u, \ X'_1(T) = 0, \ X'_2(T) = 0, \ A(T) \preceq 0, \ X'_3(T) \geq 0\}.$$

Then, apply the Rice formula to the field $Z = (X, X'_1, X'_2)$ and the Markov inequality,

$$\begin{aligned} & \mathrm{P}\{\exists T \in \overset{\circ}{S} : \ X(T) = u, \, T \, \text{has minimal third coordinate on } \mathcal{C}_u\} \\ \leq \ & \mathrm{P}\{\exists t \in \overset{\circ}{S} : \ X(t) = u, \ X'_1(t) = 0, \ X'_2(t) = 0, \ X'_3(t) \geq 0, \ A(t) \preceq 0\} \\ \leq \ & \mathrm{E}\big(\mathrm{card}\{t \in \overset{\circ}{S} : \ X(t) = u, \ X'_1(t) = 0, \ X'_2(t) = 0, \ X'_3(t) \geq 0, \ A(t) \preceq 0\}\big) \\ = \ & \mathrm{E}\big(\mathrm{card}\{t \in \overset{\circ}{S} : \ Z(t) = (u,0,0), \ X'_3(t) \geq 0, \ A(t) \preceq 0\}\big) \\ = \ & \int_{\overset{\circ}{S}} \mathrm{E}\left(|\det(Z'(t))| \mathbb{I}_{X'_3(t) \leq 0} \mathbb{I}_{A(t) \preceq 0} \mid Z(t) = (u,0,0)\right) \times p_{Z(t)}(u,0,0) \, dt. \end{aligned}$$

Under the condition $Z(t) = (u, 0, 0)$, it is clear that $\det(Z'(t)) = X'_3(t) \det(A(t))$. So, we obtain the bound

$$\sigma_3(S) \, \frac{\varphi(u)}{2\pi} \, \mathrm{E}\left(|\det(A(t))|.\mathbb{I}_{A(t) \preceq 0} X'^{+}_3(t) \mid Z(t) = (u,0,0)\right).$$

From Corollary 2.5 of Section 2.4, we know that

$$\mathrm{E}\left(|\det(A(t))|.\mathbb{I}_{A(t) \preceq 0} \mid Z(t) = (u,0,0)\right) \leq u^2 - 1 + \frac{(8\rho''(0))^{3/2} \exp\left(-u^2.(24\rho''(0) - 2)^{-1}\right)}{\sqrt{24\rho''(0) - 2}}.$$

Hence,

$$\begin{aligned} & \mathrm{P}\{\exists T \in \overset{\circ}{S} : X(T) = u, T \,\text{has minimal third coordinate on}\, \mathcal{C}_u\} \\ \leq \ & \frac{\sigma_3(S)\varphi(u)}{(2\pi)^{3/2}} \left[u^2 - 1 + \frac{(8\rho''(0))^{3/2} \exp\left(-u^2.(24\rho''(0)-2)^{-1}\right)}{\sqrt{24\rho''(0)-2}} \right]. \end{aligned} \tag{2.15}$$

• Secondly, if T is in the interior of a face S_1, then, in this face, we choose the base $\overrightarrow{\alpha}$, $\overrightarrow{\beta}$ such that $\overrightarrow{\alpha}$ is in the horizontal plane $0\ t_1t_2$ such that along this vector, the second coordinate is not decreasing. Let us denote vector $\overrightarrow{\gamma}$ in the horizontal plane that is perpendicular to α and goes into S. It is easy to see that

$$\{X(T) = u,\ X'_\alpha(T) = 0,\ X'_\beta(T) \geq 0,\ X'_\gamma(T) \leq 0,\ X''_\alpha(T) \leq 0\}.$$

Apply Markov inequality and Rice formula to the field $Y(t) = (X(t), X'_\alpha(t))$,

$$\begin{aligned} & \mathrm{P}\{\exists T \in \overset{\circ}{S_1} : X(T) = u, T \,\text{has the minimal third ordinate on}\, \mathcal{C}_u\} \\ \leq \ & \mathrm{P}\{\exists t \in \overset{\circ}{S_1} : X(t) = u,\ X'_\alpha(t) = 0,\ X'_\beta(t) \geq 0,\ X'_\gamma(t) \leq 0,\ X''_\alpha \leq 0\} \\ \leq \ & \mathrm{E}(\mathrm{card}\{t \in \overset{\circ}{S_1} : X(t) = u,\ X'_\alpha(t) = 0,\ X'_\beta(t) \geq 0,\ X'_\gamma(t) \leq 0,\ X''_\alpha \leq 0\}) \\ = \ & \int_{\overset{\circ}{S_1}} \mathrm{E}\left(|\det(Y'(t))| \mathbb{I}_{X'_\beta(t)\geq 0} \mathbb{I}_{X'_\gamma(t)\leq 0} \mathbb{I}_{X''_\alpha(t)\leq 0} \mid Y(t) = (u,0) \right) p_{Y(t)}(u,0)\, dt \\ = \ & \frac{\sigma_2(S_1)\varphi(u)}{\sqrt{2\pi}} \mathrm{E}\left(|X''^{-}_\alpha(t)| X'^{+}_\beta(t) \mathbb{I}_{X'_\gamma(t)\leq 0} \mid Y(t) = (u,0) \right). \end{aligned}$$

As in Theorem 4.1, it is clear that

$$\begin{aligned} \mathrm{E}\big(|X''^{-}_\alpha(t)| \mid Y(t) = (u,0)\big) &= \sqrt{12\rho''(0)-1}\,\varphi\left(\frac{u}{\sqrt{12\rho''(0)-1}}\right) + u\Phi\left(\frac{u}{\sqrt{12\rho''(0)-1}}\right), \\ \mathrm{E}\big(X'^{+}_\beta(t)\mathbb{I}_{X'_\gamma(t)\leq 0} \mid Y(t) = (u,0)\big) &= \frac{1-\cos(\beta,\gamma)}{2\sqrt{2\pi}}. \end{aligned}$$

Observe that the angle between β and γ is the angle θ_1 between the face S_1 and the horizontal plane, then the probability that there exists one point with minimal third coordinate on the level set and in the interior of the face S_1 is at most equal to

$$\frac{\sigma_2(S_1)\varphi(u)(1-\cos\theta_1)}{4\pi} \left[\sqrt{12\rho''(0)-1}\,\varphi\left(\frac{u}{\sqrt{12\rho''(0)-1}}\right) + u\Phi\left(\frac{u}{\sqrt{12\rho''(0)-1}}\right) \right].$$

Taking the sum of all the bounds at each faces, observing that

$$\sum_{i=1}^{n} \sigma_2(S_i)\cos\theta_i = 0,$$

we have the following upper bound for the probability of having a point T with minimal third coordinate on the level set and belonging to a face:

$$\frac{\sigma_2(S)\varphi(u)}{4\pi} \left[\sqrt{12\rho''(0)-1}\,\varphi\left(\frac{u}{\sqrt{12\rho''(0)-1}}\right) + u\Phi\left(\frac{u}{\sqrt{12\rho''(0)-1}}\right) \right]. \tag{2.16}$$

• Thirdly, when T belongs to one edge, for example F_1. Let us define $\overrightarrow{\eta}$ is the upward direction on this edge, i.e such that along this vector, the third coordinate is not decreasing, and $\overrightarrow{\alpha}$ and $\overrightarrow{\beta}$ are the two horizontal directions that go inside two faces containing the edge. Then,

$$\{X(T) = u,\ X'_\eta(T) \geq 0,\ X'_\alpha(T) \leq 0,\ X'_\beta(T) \leq 0\}.$$

By Rice formula, the expectation of the number of the points in F_1 satisfying this condition is

$$\begin{aligned} &\int_{F_1} \mathrm{E}\left(X'^{+}_\eta(t)\mathbb{I}_{X'_\alpha(t)\leq 0}\mathbb{I}_{X'_\beta(t)\leq 0} \mid X(t) = u\right) \times p_{X(t)}(u)\, dt \\ = &\ \sigma_1(F_1)\ \varphi(u)\ \mathrm{E}\left(X'^{+}_\eta(t)\mathbb{I}_{X'_\alpha(t)\leq 0}\mathbb{I}_{X'_\beta(t)\leq 0}\right). \end{aligned}$$

Let $\overrightarrow{a}$ and $\overrightarrow{b}$ be two vectors in two faces containing the edge F_1 and perpendicular to $\overrightarrow{\eta}$; θ_1 be the angle between $\overrightarrow{\alpha}$ and $\overrightarrow{\eta}$; θ_2 be the angle between $\overrightarrow{\beta}$ and $\overrightarrow{\eta}$. It is clear that

$$X'_\alpha(t) = \cos\theta_1\, X'_\eta(t) + \sin\theta_1\, X'_a(t),$$

$$X'_\beta(t) = \cos\theta_2\, X'_\eta(t) + \sin\theta_2\, X'_b(t),$$

and $\mathrm{cov}(X'_a(t), X'_b(t)) = \cos\theta_3$, where θ_3 is the angle between two faces containing the edge F_1. Then,

$$\begin{aligned} &\mathrm{E}\left(X'^{+}_\eta\mathbb{I}_{X'_\alpha(t)\leq 0}\mathbb{I}_{X'_\beta(t)\leq 0}\right) \\ =&\mathrm{E}\left(X'^{+}_\eta\mathbb{I}_{\{\cos\theta_1\, X'_\eta(t)+\sin\theta_1\, X'_a(t)\leq 0\}}\mathbb{I}_{\{\cos\theta_2\, X'_\eta(t)+\sin\theta_2\, X'_b(t)\leq 0\}}\right) \\ =&\int_0^\infty x\,\varphi(x)\,F(x)\,dx, \end{aligned}$$

where

$$\begin{aligned} F(x) =&\mathrm{E}\left(\mathbb{I}_{\{\cos\theta_1\, X'_\eta(t)+\sin\theta_1\, X'_a(t)\leq 0\}}\mathbb{I}_{\{\cos\theta_2\, X'_\eta(t)+\sin\theta_2\, X'_b(t)\leq 0\}} \mid X'_\eta(t) = x\right) \\ =&\int_{-\infty}^{-\cot\theta_1.x} \varphi(y)\,\Phi\left(\frac{-\cot\theta_2.x - \cos\theta_3.y}{\sin\theta_3}\right)\,dy. \end{aligned}$$

So,

$$\begin{aligned} F'(x) =& -\cot\theta_2\ \varphi(-\cot\theta_2\,.x)\ \Phi\left(\frac{-\cot\theta_1\,.x + \cos\theta_3\ \cot\theta_2\,.x}{\sin\theta_3}\right) \\ & -\cot\theta_1\ \varphi(-\cot(\theta_1\,x))\ \Phi\left(\frac{-\cot\theta_2\,.x + \cos\theta_3\ \cot\theta_1\,.x}{\sin\theta_3}\right). \end{aligned}$$

By integration by parts,

$$\begin{aligned} \int_0^\infty x\,\varphi(x)\,F(x)\,dx &= -\int_0^\infty F(x)\,d(\varphi(x)) \\ &= F(0)\varphi(0) + \int_0^\infty \varphi(x)\,F'(x)\,dx. \end{aligned}$$

It is easy to check that

$$\int_0^\infty \varphi(x)\,\Phi(mx)\,dx = \frac{1}{4} + \frac{\arctan(m)}{2\pi},$$

$$\int_{-\infty}^{0} \varphi(x)\,\Phi(mx)\,dx = \frac{1}{4} - \frac{\arctan(m)}{2\pi}.$$

From the above results, we have

$$\begin{aligned} F(0)\varphi(0) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \varphi(y)\Phi\left(\frac{-\cos\theta_3}{\sin\theta_3} y\right) dy \\ &= \frac{1}{(2\pi)^{3/2}} \left(\pi - \theta_3\right). \end{aligned}$$

and

$$\begin{aligned} \int_0^{\infty} \varphi(x)\,F'(x)\,dx &= -\cot\theta_2 \int_0^{\infty} \varphi(x)\varphi(-\cot\theta_2\,.x)\,\Phi\left(\frac{-\cot\theta_1\,.x + \cos\theta_3\,\cot\theta_2\,.x}{\sin\theta_3}\right) dx \\ &\quad - \cot\theta_1 \int_0^{\infty} \varphi(x)\varphi(-\cot\theta_1\,.x)\,\Phi\left(\frac{-\cot\theta_2\,.x + \cos\theta_3\,\cot\theta_1\,.x}{\sin\theta_3}\right) dx \\ &= \frac{-\cos\theta_2}{\sqrt{2\pi}}\left(\frac{1}{4} + \frac{1}{2\pi}\arctan\left(\frac{-\sin\theta_2.\cot\theta_1 + \cos\theta_3\cos\theta_2}{\sin\theta_3}\right)\right) \\ &\quad + \frac{-\cos\theta_1}{\sqrt{2\pi}}\left(\frac{1}{4} + \frac{1}{2\pi}\arctan\left(\frac{-\sin\theta_1\cot\theta_2 + \cos\theta_3\cos\theta_1}{\sin\theta_3}\right)\right). \end{aligned}$$

Therefore, the probability that there exists one point with minimal third coordinate on the level set $\mathcal{C}(u)$ and belonging to F_1 is at most equal to

$$\begin{aligned} \frac{\sigma_1(F_1)(\pi-\theta_3)\varphi(u)}{(2\pi)^{3/2}} + \sigma_1(F_1) \quad &\left[\frac{-\cos\theta_2}{\sqrt{2\pi}}\left(\frac{1}{4} + \frac{1}{2\pi}\arctan\left(\frac{-\sin\theta_2\cot\theta_1 + \cos\theta_3\cos\theta_2}{\sin\theta_3}\right)\right)\right. \\ &\left. + \frac{-\cos\theta_1}{\sqrt{2\pi}}\left(\frac{1}{4} + \frac{1}{2\pi}\arctan\left(\frac{-\sin\theta_1\cot\theta_2 + \cos\theta_3\cos\theta_1}{\sin\theta_3}\right)\right)\right]. \end{aligned}$$

Summing up all the terms at all the edges, we obtain the bound

$$\begin{aligned} \varphi(u)\sum_{i=1}^{n} \frac{\sigma_1(F_i)(\pi-\theta_{3i})}{(2\pi)^{3/2}} + \varphi(u)\sum_{i=1}^{n}\sigma_1(F_i)&\left[\frac{-\cos\theta_{2i}}{\sqrt{2\pi}}\left(\frac{1}{4} + \frac{1}{2\pi}\arctan\left(\frac{-\sin\theta_{2i}\cot\theta_{1i} + \cos\theta_{3i}\cos\theta_{2i}}{\sin\theta_{3i}}\right)\right)\right. \\ &\left. + \frac{-\cos\theta_{1i}}{\sqrt{2\pi}}\left(\frac{1}{4} + \frac{1}{2\pi}\arctan\left(\frac{-\sin\theta_{1i}\cot\theta_{2i} + \cos\theta_{3i}\cos\theta_{1i}}{\sin\theta_{3i}}\right)\right)\right]. \end{aligned}$$

By definition,

$$\sum_{i=1}^{n} \sigma_1(F_i)(\pi - \theta_{3i}) = 4\pi\lambda(S).$$

Now, we prove

$$\begin{aligned} I = \sum_{i=1}^{n} l_i &\left[\frac{\cos\theta_{2i}}{\sqrt{2\pi}}\left(\frac{1}{4} + \frac{1}{2\pi}\arctan\left(\frac{-\sin\theta_{2i}\cot\theta_{1i} + \cos\theta_{3i}\cos\theta_{2i}}{\sin\theta_{3i}}\right)\right)\right. \\ &\left. + \frac{\cos\theta_{1i}}{\sqrt{2\pi}}\left(\frac{1}{4} + \frac{1}{2\pi}\arctan\left(\frac{-\sin\theta_{1i}\cot\theta_{2i} + \cos\theta_{3i}\cos\theta_{1i}}{\sin\theta_{3i}}\right)\right)\right] = 0. \end{aligned}$$

Indeed, from Lemma 2.2, we have

$$\frac{-\sin\theta_1\cot\theta_2 + \cos\theta_3\cos\theta_1}{\sin\theta_3} = \frac{-\cos h}{\sin h},$$

where h is the dihedral angle at $\overrightarrow{\alpha}$, i.e, the angle between the horizontal plane and the face containing $\overrightarrow{\alpha}$ and $\overrightarrow{\eta}$. Since h is constant for each face,

$$I = \sum_{S \in \{S_1,\ldots,S_k\}} \sum_{l \subset S} l \cos\theta_1 \left(\frac{1}{4} + \frac{1}{2\pi} \arctan\left(\frac{-\cos h}{\sin h}\right)\right) = 0.$$

Therefore, we have the following upper bound for the probability of having a point T with minimal third coordinate on the level set and belonging to an edge:

$$\frac{2\lambda(S)\varphi(u)}{(2\pi)^{1/2}}. \tag{2.17}$$

From (2.15), (2.16), (2.17) and the fact that $\mathrm{P}\{X(O) > u\} = \overline{\Phi}(u)$, the result follows. □

2.4 Computation of the absolute value of the determinant of the Hessian matrices

As we see in the proof of Theorem 2.3, we deal with the following

$$\mathrm{E}(|\det(X''(t))|\mathbb{I}_{X''(t)\preceq 0} \mid X(t) = u,\ X_1'(t) = 0,\ X_2'(t) = 0).$$

To evaluate this quantity, we have the following statement:

Theorem 2.4. *Let X be a standard stationary isotropic centered two-dimensional Gaussian field. One has*

$$\mathrm{E}\left(|\det(X''(t))| \mid (X, X_1', X_2')(t) = (u,0,0)\right) = u^2 - 1 + 2\frac{(8\rho''(0))^{3/2}\exp(-u^2.(24\rho''(0)-2)^{-1})}{\sqrt{24\rho''(0)-2}}. \tag{2.18}$$

Proof. Under the condition, the vector $(X_{11}'', X_{12}'', X_{22}'')$ has the same distribution with $(Y_1, Y_2, Y_3) + (-u, 0, -u)$, where (Y_1, Y_2, Y_3) is a centered Gaussian vector with the covariance matrix:

$$\Sigma = \begin{pmatrix} 12\rho''(0)-1 & 0 & 4\rho''(0)-1 \\ 0 & 4\rho''(0) & 0 \\ 4\rho''(0)-1 & 0 & 12\rho''(0)-1 \end{pmatrix}.$$

Then, the LHS in (2.18) can be written as

$$\begin{aligned} & \mathrm{E}(|X_{11}''(t)X_{22}''(t) - X_{12}''(t)^2| \mid (X, X_1', X_2')(t) = (u,0,0)) \\ = \ & \mathrm{E}(|(Y_1-u)(Y_3-u) - Y_2^2|) \\ = \ & \mathrm{E}(|Y_1Y_3 - Y_2^2 - u(Y_1+Y_3) + u^2|) \\ = \ & \mathrm{E}(| < Y, AY > + < b, Y > + u^2|), \end{aligned}$$

where $A = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} -u \\ 0 \\ -u \end{pmatrix}$.
Here, from Theorem 2.1 of [31], the expectation is equal to

$$\mathrm{E}(| < Y, AY > + < b, Y > +u^2|) = \frac{2}{\pi} \int_0^{\infty} t^{-2}(1 - F(t) - \overline{F}(t))dt,$$

where

$$F(t) = \frac{\exp(itu^2 - 2^{-1}t^2 < b, (I_3 - 2it\Sigma A)^{-1}\Sigma b >)}{2\det(I_3 - 2it\Sigma A)^{1/2}}.$$

It is clear that

$$F(t) = \frac{\exp(itu^2[1 - it(16\rho''(0) - 2)]^{-1})}{2(1 + 8it\rho''(0))[1 - it(16\rho''(0) - 2)]^{1/2}},$$

and

$$\bar{F}(t) = \frac{\exp(-itu^2[1 + it(16\rho''(0) - 2)]^{-1})}{2(1 - 8it\rho''(0))[1 + it(16\rho''(0) - 2)]^{1/2}} = F(-t).$$

So, the expectation is equal to

$$\begin{aligned} \frac{2}{\pi}\int_0^{\infty} \frac{1}{t^2}(1 - F(t) - \overline{F}(t))dt = & \ \mathrm{Re}(\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{t^2}(1 - 2.F(t))dt) \\ = & \ \mathrm{Re}(\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{t^2}(1 - \frac{\exp(itu^2[1 - it(16\rho''(0) - 2)]^{-1})}{(1 + 8it\rho''(0))[1 - it(16\rho''(0) - 2)]^{1/2}})dt). \end{aligned}$$

Here, we apply the residue theorem to compute

$$\begin{aligned} & \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{t^2}(1 - \frac{\exp(itu^2[1 - it(16\rho''(0) - 2)]^{-1})}{(1 + 8it\rho''(0))[1 - it(16\rho''(0) - 2)]^{1/2}})dt \\ = & \ 2i.\,(\text{sum of residues in upper half plane}) + i.\,(\text{sum of residues on x-axis})\,. \end{aligned}$$

The residues come from two poles at $i.(8\rho''(0))^{-1}$ and 0 and we see that:
The residue at 0 is equal to

$$\frac{d}{dt}\left(1 - \frac{\exp(itu^2[1 - it(16\rho''(0) - 2)]^{-1})}{(1 + 8it\rho''(0))[1 - it(16\rho''(0) - 2)]^{1/2}}\right)\Bigg|_{t=0} = -i.u^2 + i.$$

And the residue at $i.(8\rho''(0))^{-1}$ is equal to

$$\begin{aligned} & \frac{(1 + 8it\rho''(0))[1 - it(16\rho''(0) - 2)] - \exp(itu^2[1 - it(16\rho''(0) - 2)]^{-1})}{t^2.8i\rho''(0).[1 - it(16\rho''(0) - 2)]^{1/2}}\Bigg|_{t=i.(8\rho''(0))^{-1}} \\ = & \ \frac{(8\rho''(0))^{3/2}\exp(-u^2.(24\rho''(0) - 2)^{-1})}{\sqrt{24\rho''(0) - 2}.i}. \end{aligned}$$

These two residues imply the result. □

We have the corollary

Corollary 2.5. *Let X be a standard stationary isotropic centered Gaussian field. One has*

$$\mathrm{E}(|\det(X''(t))|.\mathbb{I}_{X''(t)\preceq 0} \mid (X, X'_1, X'_2)(t) = (u, 0, 0)) \leq u^2 - 1 + \frac{(8\rho''(0))^{3/2}\exp(-u^2.(24\rho''(0)-2)^{-1})}{\sqrt{24\rho''(0)-2}}.$$

Proof. The result follows from two observations

- $|\det(X''(t))|.\mathbb{I}_{X''(t)\preceq 0} \leq \dfrac{|\det(X''(t))| + \det(X''(t))}{2}$.
- $\mathrm{E}(\det(X''(t)) \mid (X, X'_1, X'_2)(t) = (u, 0, 0)) \ = \ u^2 - 1$.

□

2.5 Numerical comparison

In this section, we compare the upper bounds given by the direct method and record method with the approximation given by the EC method. For simplicity we limit our attention to the case where S is the square $[0, T]^2$ and X is a standard stationary isotropic centered Gaussian field with covariance function $\rho(\|s - t\|^2)$. Note that only $\rho''(0)$ plays a role, the exact form of ρ does not need to be specified. More precisely, we consider

1. the approximation given by the EC method
$$\mathrm{P}_E(u) = \overline{\Phi}(u) + \frac{2T}{\sqrt{2\pi}}\varphi(u) + \frac{T^2}{2\pi}u\varphi(u);$$
2. and the upper bound given by the direct method
$$\begin{aligned}\mathrm{P}_M(u) = \quad & \overline{\Phi}(u) + \frac{2T}{\sqrt{2\pi}}\int_u^\infty \left[c\varphi(x/c) + x\Phi(x/c)\right]\varphi(x)dx \\ & + \frac{T^2}{2\pi}\int_u^\infty \left[x^2 - 1 + \left(\frac{2(c^2+1)}{3}\right)^{3/2}\sqrt{\pi}\frac{\varphi(x/c)}{c}\right]\varphi(x)dx,\end{aligned}$$
where $c = \sqrt{12\rho''(0) - 1}$,
3. and the one given by the record method
$$\mathrm{P}_R(u) = \overline{\Phi}(u) + \frac{2T}{\sqrt{2\pi}}\varphi(u) + \frac{T^2}{2\pi}\left[c\varphi(u/c) + u\Phi(u/c)\right]\varphi(u).$$

It is easy to see that P_E is always less than P_R and P_M. We will prove that $\mathrm{P}_R(u)$ is smaller than $\mathrm{P}_M(u)$ as u is large. Indeed, if we compare the "dimension 1 terms" (corresponding to $\sigma_1(\partial S)$), we have

$$\begin{aligned} & \textstyle\int_u^\infty \left[c\varphi(x/c) + x\Phi(x/c)\right]\varphi(x)dx - \varphi(u) \\ = \ & \textstyle\int_u^\infty \left[c\varphi(x/c) + x\Phi(x/c)\right]\varphi(x)dx - \int_u^\infty x\varphi(x)dx \\ = \ & \textstyle\int_u^\infty \left[c\varphi(x/c) - x\overline{\Phi}(x/c)\right]\varphi(x)dx \geq 0,\end{aligned}$$

since when $x \geq 0$,

$$\frac{\varphi(x)}{x} \geq \overline{\Phi}(x).$$

So the term in the direct method is always larger when $u \geq 0$.

Let us consider now the two terms corresponding to $\sigma_2(S)$:

- $A_d = u\varphi(u) + \int_u^\infty \left[\left(\frac{2(c^2+1)}{3} \right)^{3/2} \sqrt{\pi} \frac{\varphi(x/c)}{c} \right] \varphi(x)dx = u\varphi(u) + \overline{A}_d.$
- $A_r = [c\varphi(u/c) + u\Phi(u/\mathrm{c})]\, \varphi(u) = u\varphi(u) + \overline{A}_r.$

It is easy to show that , as $u \to +\infty$,

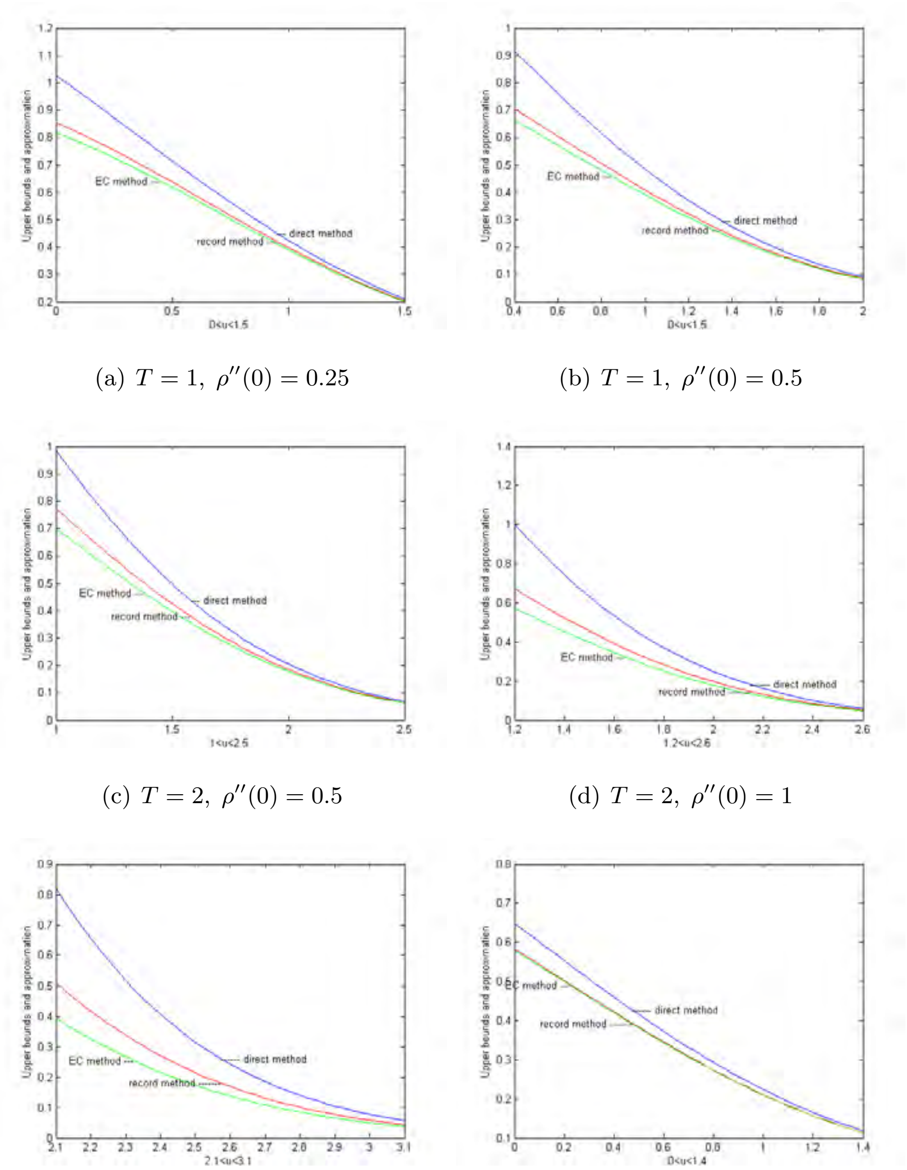
$$\overline{A}_d = \int_u^\infty \varphi\left(\frac{x}{c}\right)\varphi(x)dx = (const)\overline{\Phi}\left(u\sqrt{\frac{1+c^2}{c^2}}\right) \simeq (const)u^{-1}\varphi\left(u\sqrt{\frac{1+c^2}{c^2}}\right).$$

and that

$$\overline{A}_r \simeq (const)u^{-2}\varphi\left(u\sqrt{\frac{1+c^2}{c^2}}\right).$$

This shows that for u sufficiently large A_r is smaller than A_d.

The numerical comparison is performed in Figure 2.4 for six different situations. It shows that the record method is always better than the direct method. EC method and record method are very close, but it is not possible to identify the better among those two since P_E can be smaller than the true value.



(a) $T = 1,\ \rho''(0) = 0.25$
(b) $T = 1,\ \rho''(0) = 0.5$
(c) $T = 2,\ \rho''(0) = 0.5$
(d) $T = 2,\ \rho''(0) = 1$
(e) $T = 4,\ \rho''(0) = 2$
(f) $T = 0.25,\ \rho''(0) = 0.5$

Figure 2.4: Comparison of the two bounds P_R and P_M and the approximation P_E for several values of $\rho''(0)$ and T.

Chapter 3

Asymptotic formula for non locally convex sets

3.1 Introduction

The Euler characteristic method of Adler and Taylor [1] or the direct method of Azaïs and Wschebor [5] give the super exponentially precise expansion for the tail of the maximum of a sufficiently regular random field $X(t)$ defined on a sufficiently regular set S.

An important example of such sets are the convex bodies in $\mathbb{R}^2$ (compact, convex with non-empty interior). If S has a finite number of irregular points, it is proved in [1] that if $X(t)$ is a centered Gaussian field with variance 1 defined on a neighborhood of S and if

$$M_S = \max_{t \in S} X(t),$$

then

$$\mathrm{P}(M_S \geq u) = \overline{\Phi}(u) + \frac{\sigma_1(\partial S)}{2\sqrt{2\pi}}\varphi(u) + \frac{\sigma_2(S)}{2\pi}u\varphi(u) + o(u^{-1}\varphi(u)), \tag{3.1}$$

where $\overline{\Phi}(u)$ and $\varphi(u)$ are the tail distribution and the density of a standard normal variable and σ_i is the Hausdorff measure of dimension i. It is well-known that a formula of the form (3.1) can be extended to a much wider class of sets.

Basically, Adler and Taylor use the local convexity that can be defined as the fact that for every point $t \in S$, the contact cone $\mathcal{C}_t$ generated by the set of directions

$$\left\{\sigma \in \mathbb{R}^2 : \ \|\sigma\| = 1, \exists s_n \in S \text{ such that } s_n \to t \text{ and } \frac{s_n - t}{\|s_n - t\|} \to \sigma \right\},$$

is convex, plus some regularity conditions (see, for example [1]).

Azaïs and Wschebor [6, p. 231] use the condition

$$\kappa(S) = \sup_{t\in S} \sup_{s\in S,\, s\neq t} \frac{\operatorname{dist}(s-t, \mathcal{C}_t)}{\|s-t\|^2} < \infty$$

plus some additional ones.

But none of these methods is able, for example, to deal with the very simple case of S being "the angle" that is the union of two segments with the angle $\beta \in (0,\pi)$, see Figure 3.1,

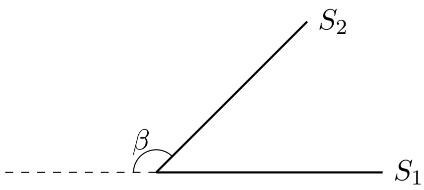


Figure 3.1: The angle- an example of non-local convexity.

which is presented in [1] as a kind of benchmark (see Subsection 3.3.1).

The aim of this chapter is to consider sets as "the angle" and to give a full expansion of the tail in dimension 2, see Theorem 3.3. Additionally, we give an expansion with three terms in dimension 3, see Subsection 3.3.7.

Our main tools are the Steiner formula that gives the volume of the tube around S, and a result of Azaïs and Wschebor that shows that except some negligible events, the excursion set is close to a ball.

Our result extends then the result of Theorem 2 in [7] since it gives an extra term.

Another main result is given by the examples of Section 3.3 that are all new and give rather unexpected results. In particular, it appears that in dimension 2, the coefficient of $\overline{\Phi}(u)$ in (3.1) is not always the Euler characteristic of the parameter set.

Notation and hypotheses

We use the following notation.

- B is a great ball in $\mathbb{R}^n$ containing S such that $\operatorname{dist}(S, \partial B) > 0$.
- $S^{+\epsilon}$ is the tube of S defined as

$$S^{+\epsilon} = \{t \in \mathbb{R}^n :\ \operatorname{dist}(t,S) \leq \epsilon\}.$$

Assumption A: $X(t)$ is a random field defined on a neighborhood NS of $S \subset \mathbb{R}^n$ satisfying

i. $X(t)$ is a centered stationary Gaussian field.

ii. $\text{Var}(X(t)) = 1$ and $\text{Var}(X'(t)) = I_n$.

iii. The paths of $X(t)$ are of class $\mathcal{C}^3$.

iv. For all $s \neq t \in B$, the distribution of $(X(s), X(t), X'(s), X'(t))$ does not degenerate.

v. For all $t \in B$, $\sigma \in \mathcal{S}^{n-1}$, the distribution of $(X(t), X'(t), X''(t)\sigma)$ does not degenerate.

3.2 Main results

Our main tool is the following lemma.

Lemma 3.1. *Let $S_1, \dots, S_m$ be m subsets of S. Assume that there exist two constants $C > 0$ and $d \geq 0$ such that*

$$\sigma_n(S_1^{+\epsilon} \cap \dots \cap S_m^{+\epsilon}) = (C + o(1))\epsilon^{n-d} \text{ as } \epsilon \to 0.$$

Then, as $u \to +\infty$,

$$\mathrm{P}\left(\min_i \{M_{S_i}\} \geq u\right) = u^{d-1}\varphi(u)\left(\frac{C}{2^{d/2}(\pi)^{n/2}}\Gamma(1 + (n-d)/2) + o(1)\right), \tag{3.2}$$

where Γ is the Gamma function.

Remark. We observe that the order of the main term in (3.2) is $u^{d-1}\varphi(u)$ for non negative d . Moreover, a classical result shows that the order of $\overline{\Phi}(u)$ is $u^{-1}\varphi(u)$. Then, in this chapter, an event is said to be "negligible" if its probability is $o(u^{-1}\varphi(u))$.

Proof. Here we essentially follow in Azaïs and Wschebor [7] where it is proven that:

Except some "negligible events", there exists only one local maximum t inside B with value in the interval $[u, u+1]$; and the excursion set K_u above u satisfies

$$B(t, \underline{r}) \subset K_u \subset B(t, \overline{r}),$$

where

$$\underline{r} = \sqrt{2\frac{X(t) - u}{X(t) + u^\alpha}}, \quad \overline{r} = \sqrt{2\frac{X(t) - u}{u - u^\alpha}},$$

with α is a constant $0 < \alpha < 1$ that can be chosen close to zero.

From the fact

$$\mathrm{P}\left(\min_i \{M_{S_i}\} \geq u\right) = \mathrm{P}\left(\{\exists t \in B,\ X(.) \text{ has a local maximum at } t,\ X(t) \geq u\} \cap \{\forall i = 1 \dots m : K_u \cap S_i\right.$$
$$+o(u^{-1}\varphi(u))$$

and the above observations, we have the upper bound

$$\begin{aligned}
&\mathrm{P}\left(\min_i\{M_{S_i}\}\geq u\right)\\
\leq\;& o(u^{-1}\varphi(u))+\mathrm{P}\left(\exists t\in \overset{\circ}{B},\ X(.) \text{ has a local maximum at } t,\ u\leq X(t)\leq u+1,\ t\in \bigcap_{i=1}^{m} S_i^{+\overline{r}}\right)\\
\leq\;& o(u^{-1}\varphi(u))+\mathrm{E}\left(\sharp\{\exists t\in \overset{\circ}{B},\ X(.) \text{ has a local maximum at } t,\ u\leq X(t)\leq u+1,\ t\in \bigcap_{i=1}^{m} S_i^{+\overline{r}}\}\right).
\end{aligned}$$

To compute the expectation, we use the Rice formula to get

$$\begin{aligned}
E :=\;& \mathrm{E}\left(\sharp\{\exists t\in \overset{\circ}{B},\ X(.) \text{ has a local maximum at } t,\ u< X(t)< u+1,\ t\in \bigcap_{i=1}^{m} S_i^{+\overline{r}}\}\right)\\
=\;& \int_u^{u+1} dx \int_{\overset{\circ}{B}} \mathrm{E}\left(|X''(t)|\mathbb{I}_{\{X''(t)\preceq 0\}}\mathbb{I}_{\{t\in \bigcap_{i=1}^{m} S_i^{+\overline{r}}\}}\mid X(t)=x,\ X'(t)=0\right)p_{X(t),X'(t)}(x,0)\sigma_n(dt)\\
=\;& \frac{1}{(2\pi)^{n/2}}\int_u^{u+1}\sigma_n\left(\bigcap_{i=1}^{m} S_i^{+\overline{r}^*}\right)\mathrm{E}\left(|X''(0)|\mathbb{I}_{\{X''(0)\preceq 0\}}\mid X(0)=x, X'(0)=0\right)\varphi(x)\,dx,
\end{aligned}$$

where $\overline{r}^*$ is the value of $\overline{r}$ when $X(t)=x$. Here we use the stationary property of the field and the fact that $X(t)$ and $X'(t)$ are two independent Gaussian vectors.

Using the well-known result (see Delmas [19])

$$\mathrm{E}\left(|X''(0)|\mathbb{I}_{\{X''(0)\preceq 0\}}\mid X(0)=x, X'(0)=0\right)=x^n+O(x^{n-2}) \text{ as } x\to\infty,$$

and the hypothesis

$$\sigma_n(S_1^{+\epsilon}\cap\ldots\cap S_m^{+\epsilon})\simeq C\epsilon^{n-d}\ as\ \epsilon\to 0,$$

we have

$$\begin{aligned}
E =\;& \frac{1}{(2\pi)^{n/2}}\int_u^{u+1} x^n\varphi(x)C\left[\frac{x-u}{u-u^{\alpha}}\right]^{(n-d)/2}dx+o(u^{d-1}\varphi(u))\\
=\;& \frac{Cu^{(n+d)/2}}{2^{d/2}(\pi)^{n/2}}\int_u^{u+1}\varphi(x)(x-u)^{(n-d)/2}dx+o(u^{d-1}\varphi(u)).
\end{aligned}$$

By the change of variable $x=u+y/u$, we obtain

$$\begin{aligned}
E =\;& \frac{C}{2^{d/2}(\pi)^{n/2}}u^{d-1}\varphi(u)\int_0^u \exp\left(-y-\frac{y^2}{2u^2}\right)y^{(n-d)/2}dy+o(u^{d-1}\varphi(u))\\
=\;& u^{d-1}\varphi(u)\left(\frac{C}{2^{d/2}(\pi)^{n/2}}\Gamma(1+(n-d)/2)+o(1)\right).
\end{aligned}$$

For the lower bound, we have

$$\begin{aligned}
&\mathrm{P}\left(\min_i\{M_{S_i}\}\geq u\right)\\
\geq\;& o(u^{-1}\varphi(u))+\mathrm{P}\left(\exists t\in \overset{\circ}{B},\ X(.) \text{ has a local maximum at } t,\ u< X(t)< u+1,\ t\in \bigcap_{i=1}^{m} S_i^{+\underline{r}}\right).
\end{aligned}$$

Denote

$$M_{\underline{r}} = \sharp\{\exists t \in \overset{\circ}{B},\ X(.) \text{ has a local maximum at } t,\ u < X(t) < u+1,\ t \in \bigcap_{i=1}^{m} S_i^{+\underline{r}}\}.$$

It is clear that (see [1], [6])

$$0 \leq \mathrm{E}(M_{\underline{r}}) - \mathrm{P}(M_{\underline{r}} \geq 1) \leq \mathrm{E}(M_{\underline{r}}(M_{\underline{r}}-1))/2 \leq \mathrm{E}(M_u(M_u-1))/2 = o(u^{-1}\varphi(u)),$$

where

$$M_u = \sharp\{\exists t \in \overset{\circ}{B},\ X(.) \text{ has a local maximum at } t,\ X(t) \geq u\}.$$

Then

$$\mathrm{P}\left(\min_i\{M_{S_i}\} \geq u\right) \geq o(u^{-1}\varphi(u)) + \mathrm{E}(M_{\underline{r}}).$$

Here, using again the Rice formula and by the same arguments, we obtain that the upper and lower bounds have the same equivalent formula and the result follows. □

The main object that we consider is the collection of the subsets of $\mathbb{R}^2$ that satisfy the Steiner formula heuristic defined as follows.

Definition 3.1. A compact subset $S \subset \mathbb{R}^2$ is said to satisfy the Steiner formula heuristic (SFH) if it satisfies the following conditions

- As ϵ tends to 0,

$$\sigma_2(S^{+\epsilon}) = \sigma_2(S) + \epsilon L_1(S) + \pi\epsilon^2 L_0(S) + o(\epsilon^2). \tag{3.3}$$

- For all processes $X(t)$ satisfying Assumption A,

$$\mathrm{P}(M_S \geq u) = L_0(S)\overline{\Phi}(u) + L_1(S)\frac{\varphi(u)}{2\sqrt{2\pi}} + \sigma_2(S)\frac{u\varphi(u)}{2\pi} + o(u^{-1}\varphi(u)). \tag{3.4}$$

Remark.

1. If S is a convex body then (3.3) becomes the Steiner formula

$$\sigma_2(S^{+\epsilon}) = \sigma_2(S) + \epsilon L_1(S) + \pi\epsilon^2 L_0(S), \tag{3.5}$$

   that holds true for all $\epsilon \geq 0$. $L_1(S)$ is just the Hausdorff measure of the boundary of S ($\sigma_1(\partial S)$) and $L_0(S)$ is the Euler characteristic of S which is equal to 1.

   If in addition, the number of irregular points of S is finite, then from the result of Adler and Taylor, we have (3.1).

2. If S has a positive reach in the sense that there exists a positive constant r such that for all $t \in S^r$, there is only one projection of t on S, then (3.5) is true for all $\epsilon < r$ (see [2], [20]). Moreovers, if S is simply connected, has piecewise-$\mathcal{C}^3$ boundary and satisfies $\kappa(S) < \infty$, then (3.1) still holds true (see Appendix).

3. In the most general cases, the constant $L_1(S)$ is the outer Minkowski content of S (OMC(S)), for more details see [2], which is defined by

$$\sigma_2(S^{+\epsilon}) = \sigma_2(S) + \epsilon \mathrm{OMC}(S) + o(\epsilon).$$

It can differ from $\sigma_1(\partial S)$, for example in the case of "the square with whiskers", see Figure 3.2,

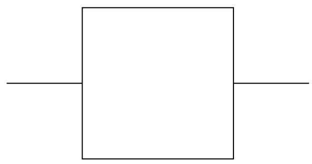


Figure 3.2: The square with whiskers.

In this case, $\sigma_1(\partial S)$ is equal to the perimeter of the square plus the length of the whiskers, while $L_1(S)$ is equal to the perimeter of the square plus two times the length of the whiskers. In addition it should be noticed that $L_0(S)$ is not equal to the Euler characteristic, see Subsection 3.3.4.

Domains with piecewise-$\mathcal{C}^2$ boundary

We assume that the boundary of S consists of a finite union of $\mathcal{C}^2$ curves that will be called "edges". The edge E_i of length L_i can be parametrized on $[0, L_i]$ in a $\mathcal{C}^2$ manner by its arc length. To introduce the case of angle in the plane or the case of whiskers, we consider two kinds of edges:

- Edges that are included in $\overline{\overset{\circ}{S}}$: non isolated edges.
- Edges such that the intersection with $\overline{\overset{\circ}{S}}$ is at most a point: this is the case of whiskers or of the angle.

To limit the number of configurations to consider, we exclude more complicated cases.

Irregular points are the points where the parametrization is no more $\mathcal{C}^2$. We assume that these points belong to four categories:

- Convex binary points: the intersection of two non isolated edges and the contact cone is convex.
- Concave binary points: as above but the contact cone is not convex. Denote $\beta \in [0, \pi[$ by the discontinuity of the angle of the tangent at this point when we choose the orientation for the boundary such that the interior is always on the left.

- Angle points: the intersection of two isolated edges. Denote $\beta \in [0, \pi[$ by the the discontinuity of the angle is in Figure 3.1.

- Concave ternary points: the intersection of two non isolated edges E_1, E_2 and one isolated one E_3. In the main result, these points will be considered with multiplicity two. We associate to these points two angles:

  - β_1 which is the discontinuity of the angle of the tangent when we turn from E_1 to E_3.

  - β_2 which is the discontinuity of the angle of the tangent when we turn from E_3 to E_2.

  To calculate explicitly, we only consider the concave trinary point such that $\beta_1 + \beta_2 \leq \pi$ and we exclude more complicated situations.

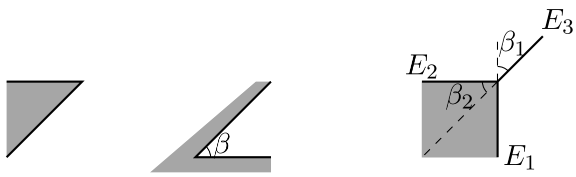


Figure 3.3: Convex, concave binary and concave ternary points, respectively.

Our next lemma shows a way to construct a class of compact subsets of $\mathbb{R}^2$ satisfying the SFH.

Lemma 3.2. *Let S_1, S_2, S_3 and S_4 be four compact sets such that*

1. *For all $i = 1, 2, 3, 4$, S_i has the SFH property.*

2. *$S_1 \cup S_2$, $S_2 \cup S_3$, $S_3 \cup S_4$, and $S_4 \cup S_1$ have the SFH property.*

3. *$S_2 \cap S_4 = \emptyset$ and $S_1 \cap S_3 \cap S_4 = \emptyset$.*

4. *As ϵ tends to 0, there exist two positive constants C_{13} and C_{123} such that*

$$\sigma_2(S_1^{+\epsilon} \cap S_3^{+\epsilon}) \simeq C_{13}\epsilon^2 \text{ and } \sigma_2(S_1^{+\epsilon} \cap S_2^{+\epsilon} \cap S_3^{+\epsilon}) \simeq C_{123}\epsilon^2. \tag{3.6}$$

Then $S = S_1 \cup S_2 \cup S_3 \cup S_4$ also satisfies the SFH and

$$- \quad L_1(S) = L_1(S_1 \cup S_2) + L_1(S_2 \cup S_3) + L_1(S_3 \cup S_4) + L_1(S_4 \cup S_1) - \sum_{i=1}^{4} L_1(S_i),$$

$$- \quad L_0(S) = L_0(S_1 \cup S_2) + L_0(S_2 \cup S_3) + L_0(S_3 \cup S_4) + L_0(S_4 \cup S_1) - \sum_{i=1}^{4} L_0(S_i) + \frac{C_{123} - C_{13}}{\pi}.$$

Proof. • First, we consider the tube formula of S. By the inclusion-exclusion principle,

$$\begin{aligned}
\sigma_2(S^{+\epsilon}) = &\ \sigma_2((S_1\cup S_2\cup S_3\cup S_4)^{+\epsilon})\\
= &\ \sigma_2((S_1\cup S_2)^{+\epsilon})+\sigma_2((S_3\cup S_4)^{+\epsilon})-\sigma_2((S_1\cup S_2)^{+\epsilon}\cap(S_3\cup S_4)^{+\epsilon})\\
= &\ \sigma_2((S_1\cup S_2)^{+\epsilon})+\sigma_2((S_3\cup S_4)^{+\epsilon})-\sigma_2\left(\left((S_1^{+\epsilon}\cup S_2^{+\epsilon})\cap S_3^{+\epsilon}\right)\cup\left((S_1^{+\epsilon}\cup S_2^{+\epsilon})\cap S_4^{+\epsilon}\right)\right)\\
= &\ \sigma_2((S_1\cup S_2)^{+\epsilon})+\sigma_2((S_3\cup S_4)^{+\epsilon})-\sigma_2\left((S_1^{+\epsilon}\cup S_2^{+\epsilon})\cap S_3^{+\epsilon}\right)-\sigma_2\left((S_1^{+\epsilon}\cup S_2^{+\epsilon})\cap S_4^{+\epsilon}\right)\\
&+\sigma_2\left((S_1^{+\epsilon}\cup S_2^{+\epsilon})\cap S_3^{+\epsilon}\cap S_4^{+\epsilon}\right)\\
= &\ \sigma_2((S_1\cup S_2)^{+\epsilon})+\sigma_2((S_3\cup S_4)^{+\epsilon})-\sigma_2(S_1^{+\epsilon}\cap S_3^{+\epsilon})-\sigma_2(S_2^{+\epsilon}\cap S_3^{+\epsilon})+\sigma_2(S_1^{+\epsilon}\cap S_2^{+\epsilon}\cap S_3^{+\epsilon})\\
&-\sigma_2(S_1^{+\epsilon}\cap S_4^{+\epsilon})\\
= &\ \sigma_2((S_1\cup S_2)^{+\epsilon})+\sigma_2((S_2\cup S_3)^{+\epsilon})+\sigma_2((S_3\cup S_4)^{+\epsilon})+\sigma_2((S_4\cup S_1)^{+\epsilon})\\
&-\sigma_2(S_1^{+\epsilon})-\sigma_2(S_2^{+\epsilon})-\sigma_2(S_3^{+\epsilon})-\sigma_2(S_4^{+\epsilon})+\sigma_2(S_1^{+\epsilon}\cap S_2^{+\epsilon}\cap S_3^{+\epsilon})-\sigma_2(S_1^{+\epsilon}\cap S_3^{+\epsilon}).
\end{aligned}$$

Thus we have

$$\sigma_2(S^{+\epsilon})=\sigma_2(S)+\epsilon L_1(S)+\pi\epsilon^2 L_0(S)+o(\epsilon^2),$$

where

- $L_1(S)= L_1(S_1\cup S_2)+L_1(S_2\cup S_3)+L_1(S_3\cup S_4)+L_1(S_4\cup S_1)-\sum_{i=1}^{4}L_1(S_i),$
- $L_0(S)= L_0(S_1\cup S_2)+L_0(S_2\cup S_3)+L_0(S_3\cup S_4)+L_0(S_4\cup S_1)-\sum_{i=1}^{4}L_0(S_i)+\dfrac{C_{123}-C_{13}}{\pi}.$

• For the excursion probability on S, using again the inclusion-exclusion principle,

$$\begin{aligned}
\mathrm{P}(M_S\ge u) = &\ \mathrm{P}(M_{S_1\cup S_2\cup S_3\cup S_4}\ge u)\\
= &\ \sum_{i=1}^{4}\mathrm{P}(M_{S_i}\ge u)-\sum_{1\le i<j\le 4}\mathrm{P}(M_{S_i}\ge u,\,M_{S_j}\ge u)\\
&+\sum_{1\le i<j<k\le 4}\mathrm{P}(M_{S_i}\ge u,\,M_{S_j}\ge u,\,M_{S_k}\ge u)-\mathrm{P}(M_{S_i}\ge u,\,\forall i=1,2,3,4).
\end{aligned}$$

By the Borel-Sudakov-Tsirelson inequality, it is easy to check that $\{M_{S_2}\ge u,\,M_{S_4}\ge u\}$ and $\{M_{S_1}\ge u,\,M_{S_3}\ge u,\,M_{S_4}\ge u\}$ have a negligible probability. Then,

$$\begin{aligned}
\mathrm{P}(M_S\ge u) = &\ \sum_{i=1}^{4}\mathrm{P}(M_{S_i}\ge u)-\mathrm{P}(M_{S_1}\ge u,\,M_{S_2}\ge u)-\mathrm{P}(M_{S_2}\ge u,\,M_{S_3}\ge u)\\
&-\mathrm{P}(M_{S_3}\ge u,\,M_{S_4}\ge u)-\mathrm{P}(M_{S_4}\ge u,\,M_{S_1}\ge u)-\mathrm{P}(M_{S_1}\ge u,\,M_{S_3}\ge u)\\
&+\mathrm{P}(M_{S_1}\ge u,\,M_{S_2}\ge u,\,M_{S_3}\ge u)+o(u^{-1}\varphi(u))\\
= &\ \mathrm{P}(M_{S_1}\ge u,\,M_{S_2}\ge u)+\mathrm{P}(M_{S_2}\ge u,\,M_{S_3}\ge u)+\mathrm{P}(M_{S_3}\ge u,\,M_{S_4}\ge u)\\
&+\mathrm{P}(M_{S_4}\ge u,\,M_{S_1}\ge u)-\sum_{i=1}^{4}\mathrm{P}(M_{S_i}\ge u)-\mathrm{P}(M_{S_1}\ge u,\,M_{S_3}\ge u)\\
&+\mathrm{P}(M_{S_1}\ge u,\,M_{S_2}\ge u,\,M_{S_3}\ge u)+o(u^{-1}\varphi(u)).
\end{aligned}$$

Now, using the SFH property in 1.) and 2.) and applying Lemma 3.1 for two probabilities $P(M_{S_1} \geq u,\ M_{S_3} \geq u)$ and $P(M_{S_1} \geq u,\ M_{S_2} \geq u,\ M_{S_3} \geq u)$, we can deduce that

$$P(M_S \geq u) = L_0(S)\overline{\Phi}(u) + L_1(S)\frac{\varphi(u)}{2\sqrt{2\pi}} + \sigma_2(S)\frac{u\varphi(u)}{2\pi} + o(u^{-1}\varphi(u)).$$

□

An introduction to understand the method

To introduce our method, we consider the simple case of a non-convex polygon as in Figure 3.4. S is decomposed into three polygons S_1, S_2 and S_3 ($S_4 = \emptyset$) as indicated in Figure 3.4.

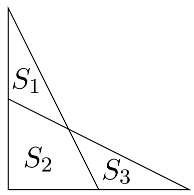


Figure 3.4: Non-convex polygon with concave binary irregular point.

These polygons are convex so they satisfy the SFH.

To apply Lemma 3.2, it remains to compute the area of $(S_1^{+\epsilon} \cap S_3^{+\epsilon})$ and $(S_1^{+\epsilon} \cap S_2^{+\epsilon} \cap S_3^{+\epsilon})$. Elementary geometry shows that $(S_1^{+\epsilon} \cap S_3^{+\epsilon})$ consists of: two sections of disc with angle $(\pi - \beta)$ and two quadrilaterals of area $\epsilon^2 \tan(\beta/2)$; while in $(S_1^{+\epsilon} \cap S_2^{+\epsilon} \cap S_3^{+\epsilon})$ one quadrilateral is replaced by a section of disc of angle β, see Figure 3.5.

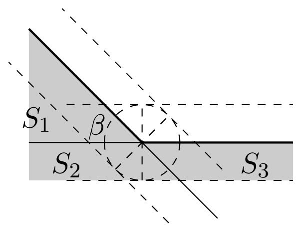


Figure 3.5: Intersection of ϵ-neighborhood sets.

Thus

$$\sigma_2(S_1^{+\epsilon} \cap S_3^{+\epsilon}) = \left[(\pi - \beta) + 2\tan\frac{\beta}{2}\right]\epsilon^2,$$

$$\sigma_2(S_1^{+\epsilon} \cap S_2^{+\epsilon} \cap S_3^{+\epsilon}) = \left[(\pi - \beta) + \frac{\beta}{2} + \tan\frac{\beta}{2}\right]\epsilon^2.$$

As a consequence,

$$C_{123} - C_{13} = \frac{\beta}{2} - \tan\frac{\beta}{2}.$$

This quantity measures the non convexity of the concave binary point. An application of Lemma 3.2 shows that the coefficient of $\overline{\Phi}(u)$ is now $1 + \dfrac{\beta/2 - \tan(\beta/2)}{\pi}$.

Our main result is the following theorem.

Theorem 3.3. *Let S be a compact domain of $\mathbb{R}^2$ with piecewise-$\mathcal{C}^2$ boundary and with concave angles $\beta_1, \ldots, \beta_m$. Let $X(t)$ be a random field satisfying assumption A. Let M_S be the maximum of $X(t)$ on S. Then*

$$\mathrm{P}(M_S \geq u) = \left[\chi(S) + \frac{1}{\pi}\sum_{j=1}^{k}\left(\frac{\beta_i}{2} - \tan\frac{\beta_i}{2}\right)\right]\overline{\Phi}(u) + \frac{\mathrm{OMC}(S)}{2\sqrt{2\pi}}\varphi(u) + \frac{\sigma_2(S)}{2\pi}u\varphi(u) + o(u^{-1}\varphi(u)), \tag{3.7}$$

where $\chi(S)$ is the Euler characteristic of S.

In addition the outer Minkowski content $\mathrm{OMC}(S)$ *is equal to the length of the non isolated edges plus twice the length of the isolated edges.*

Proof. By using the classical inequalities as Borel-Sudakov-Tsirelson Theorem, it is easy to prove that if S consists of several connected components then the tail of these components can be summed with an error which is $o(u^{-1}\varphi(u))$. So we can assume that S is connected.

We will prove by induction on the number of concave points that S satisfies the SFH.

Suppose that S has no concave point. S is whether a $\mathcal{C}^2$ curve in $\mathbb{R}^2$ or $\overline{\overset{\circ}{S}} = S$.

In the first case, using the parametrization of the unique edge, we see that M_S is just the maximum of a smooth random process (with parameter of dimension 1). In that case, the result by Piterbarg, using Rice method for up-crossings see [44] shows that S satisfies the SFH.

In the second case S it has clearly a positive reach in the sense of Federer [20] and in that case,

$$\sigma_2(S^{+\epsilon}) = \chi(S)\pi\epsilon^2 + \mathrm{OMC}(S)\epsilon + \sigma_2(S). \tag{3.8}$$

On the other hand from Theorem 8.12 of Azaïs and Wschebor [6], one can deduce the excursion probability (see Appendix for details).

The induction is based on a "destruction" of the concave points as in the introducing example. Let P be a concave point. There are four possibilities regarding P:

- Concave binary point on the exterior boundary of S. We decompose S into three subsets S_1, S_2 and S_3 as in Figure 3.4. The decomposition is as follows: at P we prolong inward the two tangents and construct to $\mathcal{C}^2$ paths that avoid hole and touch the outside boundary and define S_1, S_2 and S_3 as in Figure 3.6. To apply Lemma 3.2, we set $S_4 = \emptyset$ and remark that to compute $\sigma_2(S_1^{+\epsilon} \cap S_3^{+\epsilon})$ and $\sigma_2(S_1^{+\epsilon} \cap S_2^{+\epsilon} \cap S_3^{+\epsilon})$ we can replace, locally, with an error which is $O(\epsilon^3)$ the two portions of edges starting from P by their tangent. In that case the computation is exactly the same as in the introducing example.

  On the other hand it is easy to see that the numbers of concave points of S_1, S_2, S_3, $S_1 \cup S_2$

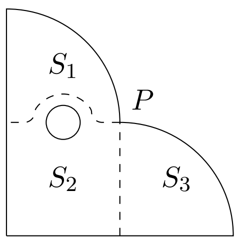


Figure 3.6: Decomposition at a concave point on the exterior boundary.

and $S_2 \cup S_3$ are at most equal to the number of concave points of S minus 1. So they satisfy the SFH by induction. From Lemma 3.2, S satisfies the SFH with the desired constants.

- It is a concave binary point on the boundary of a hole inside S. Using the two curves as above, we decompose S into four subsets as follows: we also choose two regular points on the boundary of the hole, and two corresponding regular points on the exterior boundary of S and construct two smooth curves that connect one regular point on the boundary of the hole with the corresponding one on the exterior boundary, and do not intersect themselves or two curves from the irregular point or additional holes. Then S_1, S_2, S_3, S_4 are constructed as Figure 3.7.

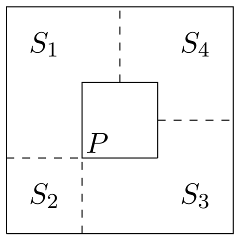


Figure 3.7: Decomposition at a concave point on the interior boundary.

The proof is essentially the same as in the preceding case.

- A concave ternary point. We put S_1 as the isolated edge, S_3 as its complement, $S_2 = P$ and $S_4 = \emptyset$ as in Figure 3.8.

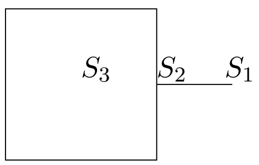


Figure 3.8: Decomposition at a concave ternary point.

- An angle point, we do the same as in the concave ternary point case, see Figure 3.9.

Then the result follows. □

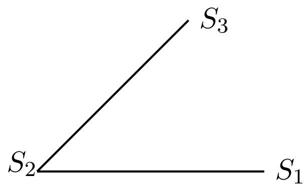


Figure 3.9: At an angle point.

3.3 Examples and remarks

We give examples that are direct applications or direct generalizations of Theorem 3.3. All these results are new.

3.3.1 The angle

Let S be the angle as in Figure 3.1. Then S has the SFH property and

$$\mathrm{P}(M_S \geq u) = \left(1 + \frac{\beta/2 - \tan(\beta/2)}{\pi}\right)\overline{\Phi}(u) + \frac{\sigma_1(S_1) + \sigma_1(S_2)}{\sqrt{2\pi}}\varphi(u) + o(u^{-1}\varphi(u)).$$

3.3.2 The multi-angle

This is an extension of the "angle" case. Let S be a self-avoiding curve that is union of $n+1$ segments with the discontinuity of the angles $\{\beta_1, \ldots, \beta_k\}$. We have

$$\mathrm{P}(M_S \geq u) = \left(1 + \frac{\sum_{i=1}^{k}(\beta_i - 2\tan(\beta_i/2))}{2\pi}\right)\overline{\Phi}(u) + \frac{\sigma_1(S)}{\sqrt{2\pi}}\varphi(u) + o(u^{-1}\varphi(u)).$$

3.3.3 The empty square

Let S be the empty square, i.e. the boundary of a square in $\mathbb{R}^2$, then applying the Lemma 3.2 three times, each time adding one more edge, (3.4) becomes

$$\mathrm{P}(M_S \geq u) = \frac{\pi - 4}{\pi}\overline{\Phi}(u) + \frac{\sigma_1(S)}{\sqrt{2\pi}}\varphi(u) + o(u^{-1}\varphi(u)).$$

In conclusion, when S is a union of some segments in a space of arbitrary dimension, we can give an exact asymptotic expansion with two terms corresponding to $\overline{\Phi}(u)$ and $\varphi(u)$ from the tube formula of S as in the above examples. More general, S can be a union of curves such that if two of them have nonempty intersection, then thay are not tangent.

3.3.4 The full square with whiskers

We consider "the square with whiskers" as in Figure 3.2. In this case, the domain has some concave ternary points. From the Theorem 3.3,

$$\mathrm{P}(M_S \geq u) = \frac{2\pi - 4}{\pi}\overline{\Phi}(u) + \frac{\mathrm{OMC}(S)}{2\sqrt{2\pi}}\varphi(u) + \frac{\sigma_2(S)}{2\pi}u\varphi(u) + o(u^{-1}\varphi(u)).$$

3.3.5 An apparent counter-example

In some strange cases, the condition (3.6) is not satisfied. This can happen when we consider two tangent curves, see Figure 3.10,

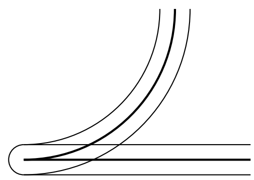


Figure 3.10: Two tangent edges.

Here, S_1 is a subset of a circle of radius R and S_3 is a tangent segment. We see that for ϵ small enough, the area of the intersection between two tubes is

$$\frac{\pi}{2}\epsilon^2 + \frac{(R+\epsilon)^2}{2}\arcsin\frac{2\sqrt{R\epsilon}}{R+\epsilon} - (R-\epsilon)\sqrt{R\epsilon} = \frac{\pi}{2}\epsilon^2 + \frac{8}{3}\sqrt{R}\epsilon^{3/2} + O(\epsilon^{5/2}).$$

In the above equation, we use the fact that as x is small enough,

$$\arcsin x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \dots .$$

It is clear that the order of the area of the intersection is not of 2 as in the condition (3.6), so we can not apply the Lemma 3.2 directly. However, with a careful examination in the proof of Lemma 3.2, we can choose α such that the difference between the upper bound and the lower one of the probability $\mathrm{P}(M_{S_1} \geq u, M_{S_3} \geq u)$ is "negligible". Thus, we have

$$\mathrm{P}(M_{S_1 \cup S_3} \geq u) = \frac{3\overline{\Phi}(u)}{2} - \frac{8\sqrt{R}}{2^{1/4}3\pi}\Gamma(7/4)u^{-1/2}\varphi(u) + \frac{\sigma_1(S_1) + \sigma_1(S_3)}{\sqrt{2\pi}}\varphi(u) + o(u^{-1}\varphi(u)). \quad (3.9)$$

This example is an apparent counter-example to the results of Adler and Taylor. More precisely, S is clearly a piecewise smooth locally convex manifold: it is easy to check that at the intersection point of the circle and the straight line, the contact cone is limited to one dimension thus convex. So if $X(t)$ is sufficiently smooth, it seems that Theorem 14.3.3 of [1] implies the validity of the Euler characteristic and Theorem 12.4.2 gives an expansion of the Euler characteristic function should apply. This would be clearly in contradiction with the term $u^{-1/2}\varphi(u)$ in (3.9).

In fact, there is no contradiction: Theorem 14.3.3 demands also the manifold to be regular in the sense of definition 9.22 of [1] and the present set is not a cone space in the sense of definition 8.3.1 of [1]. This shows that the local convexity itself is not sufficient.

3.3.6 Other domains in dimension 2

The result of Theorem 1 can be extended to more general domains for example domains with ternary points. For $\beta_1 + \beta_2 \geq \pi$ or domains with four intersecting edges but it is difficult to give a general simple formula as (3.7).

3.3.7 Some remarks and examples in dimension 3

The procedure that we have done in dimension 2 can be also used in dimension 3. However, we can not have a full expansion, in fact, the coefficient of $\overline{\Phi}(u)$ is not determined when S is not locally convex. Here we give some examples.

- S is a dihedral that is the union of two non coplanar rectangles S_1 and S_2 with a common edge, see Figure 3.11,

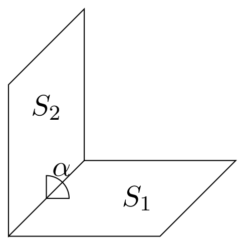


Figure 3.11: Example of concave dihedral.

Then, by using Lemma 3.2,

$$\begin{aligned} \mathrm{P}(M_S \geq u) = & \frac{\sigma_1(\partial S_1) + \sigma_1(\partial S_2) - \sigma_1(S_1 \cap S_2)((\pi+\alpha)/2 + \cot(\alpha/2))/\pi}{2\sqrt{2\pi}} \varphi(u) \\ & + \frac{\sigma_2(S_1) + \sigma_2(S_2)}{2\pi} u\varphi(u) + o(\varphi(u)). \end{aligned}$$

- S has the $L-$shape, see Figure 3.12,

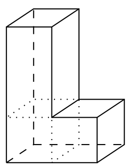


Figure 3.12: L-shape.

Then, by the argument of decomposing S into three subsets S_1, S_2 and S_3, we have

$$\mathrm{P}(M_S \geq u) = \frac{\varphi(u)L_1(S)}{\sqrt{2\pi}} + \frac{L_2(S)u\varphi(u)}{2\pi} + \frac{L_3(S)(u^2-1)\varphi(u)}{(2\pi)^{3/2}} + o(\varphi(u)), \tag{3.10}$$

where the coefficients $\{L_i(S),\ i = 1, \ldots, 3\}$ are given by the Steiner formula.

- In a more complicated case, that is nonconvex trihedral, see Figure 3.13,

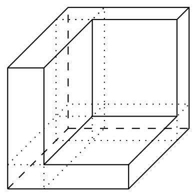


Figure 3.13: Example of nonconvex trihedrel.

We extend the planes of the nonconvex trihedral so that thay decompose S into small subsets with separate interiors. Then, we repeatedly use the inclusion-exclusion principle and Lemma 3.2 to obtain (3.10).

In general, by the same arguments and using the induction, when S is a polytope,

$$\mathrm{P}(M_S \geq u) = \frac{\varphi(u)L_1(S)}{2\sqrt{2\pi}} + \frac{L_2(S)u\varphi(u)}{2\pi} + \frac{L_3(S)(u^2-1)\varphi(u)}{(2\pi)^{3/2}} + o(\varphi(u)),$$

where

- $L_3(S)$ is the volume of S.
- $L_2(S)$ is half of the surface area.
- To calculate $L_1(S)$, we consider two kinds of edge: convex and concave. Denote $\{(\alpha_i, l_{1i}),\ i = 1, \ldots, h\}$ by the set of couples of convex inside angle and the length of the corresponding edge and $\{(\beta_j, l_{2j}),\ j = 1, \ldots, k\}$ by the set of couples of concave inside angle and the length of the corresponding edge. Then,

$$L_1(S) = \sum_{i=1}^{h} \frac{(\pi - \alpha_i)}{2\pi} l_{1i} + \sum_{j=1}^{k} \frac{\cot(\beta_j/2)}{\pi} l_{2j}.$$

Conclusion

In all the examples considered, the Steiner formula for the tube governs the expansion of the tail of the maximum as if the excursion set were exactly a unique ball with random radius. We have found no counter-example to that principle and a conjecture is that the result is true for a much wider class of sets as those considered in this chapter.

3.4 Appendix

We will prove that a compact connected domain in $\mathbb{R}^2$ with piecewise-$\mathcal{C}^2$ boundary and without concave irregular point satisfies the SFH. Firstly the Steiner formula (3.8) has already been established. Now, we consider the excursion probability. We recall the following definitions

- Let S_2 be the interior of S; S_1 by the union of the $\mathcal{C}^2$ edges and S_0 by the union of the convex irregular points.
- For $t \in S_j$, $X'_j(t)$ and $X''_j(t)$ are respectively the first and second derivatives of X along S_j; $X'_{j,N}(t)$ denotes the outward normal derivative.

In our case, it is easy to see that

$$\kappa(S) = \sup_{t\in S} \sup_{s\in S,\, s\neq t} \frac{\operatorname{dist}(s-t, \mathcal{C}_t)}{\|s-t\|^2} < \infty.$$

In order to apply Theorem 8.12 and Corollary 8.13 of Azaïs and Wschebor[6], we have to check the conditions (A1) to (A5), page 185 in [6]. The first three ones are easy. Note that since the edges are of dimension 1, a direct proof of Rice formula can be performed without assuming that they are of class $\mathcal{C}^3$ as in (A1).

- The condition (A4) states that the maximum is attained at a single point. It can be deduced from the Bulinskaya lemma (Proposition 6.11 in [6]) since for $s \neq t$, $(X(s), X(t), X'(s), X'(t))$ has a non-degenerate distribution.
- The condition (A5) that states that almost surely there is no point $t \in S$ such that $X'(t) = 0$ and $\det(X''(t)) = 0$, can be deduced from Proposition 6.5 in [6] applied to the process $X'(t)$ which is $\mathcal{C}^2$.

Since all the required conditions are met, we have

$$\liminf_{u\to+\infty} -2u^{-2}\log\Big[\int_u^\infty p^E(x)dx - \mathrm{P}\{M_S \geq u\}\Big] \geq 1 + \inf_{t\in S}\frac{1}{\sigma_t^2+\kappa_t^2} > 1, \qquad (3.11)$$

where

- $p^E(x)$ is the approximation of the density of the maximum given by the Euler Characteristic method. More precisely

$$\begin{aligned} p^E(x) = & \sum_{t\in S_0} \mathrm{E}\left(\mathbb{I}_{X'_0(t)\in\widehat{\mathcal{C}}_{t,0}} \mid X(t) = x\right)\varphi(x) \\ & + \sum_{j=1}^{2}(-1)^j \int_{S_j} \mathrm{E}\left(\det\left(X''_j(t)\right)\mathbb{I}_{X'_{j,N}(t)\in\widehat{\mathcal{C}}_{t,j}} \mid X(t) = x,\, X'_j(t) = 0\right)\frac{\varphi(x)}{(2\pi)^{j/2}}dt, \end{aligned} \qquad (3.12)$$

with $\widehat{C}_{t,j}$ is the dual cone of the contact cone C_t,

$$\widehat{C}_{t,j} = \{z \in \mathbb{R}^2 : \langle z, x\rangle \geq 0, \ \forall\, x \in C_t\}.$$

- $$\sigma_t^2 = \sup_{s\in S\setminus\{t\}} \frac{\mathrm{Var}\left(X(s) \mid X(t), X'(t)\right)}{(1 - \mathrm{Cov}(X(s), X(t)))^2}.$$

- $$\kappa_t = \sup_{s\in S\setminus\{t\}} \frac{\mathrm{dist}\left(\frac{\partial}{\partial t}\mathrm{Cov}(X(s), X(t)), C_t\right)}{1 - \mathrm{Cov}(X(s), X(t))}.$$

We compute $p^E(x)$ as follows:

- When $j = 2$, there is no normal space and $X'_{2,N}(t)$ makes no sense. It is easy to see that (see Azaïs and Wschebor [6, p. 244])

$$\int_{S_2} \mathrm{E}\left(\det\left(X_2''(t)\right) \mid X(t) = x,\ X_2'(t) = 0\right) dt = \sigma_2(S)(x^2 - 1).$$

- When $j = 0$, $X'_{0,N}(t) = X'(t)$ and

$$\mathrm{E}\left(\mathbb{I}_{X'(t)\in\widehat{C}_{t,0}} \mid X(t) = x\right) = \frac{\mathcal{A}(\widehat{C}_t)}{2\pi};$$

where $\mathcal{A}(\widehat{C}_{t,0})$ is the angle of the cone that is equal to the discontinuity of the angle of the tangent at the irregular point t.

- When $j = 1$, we consider a point t on an edge L on the exterior boundary. At this point, the second derivative along this curve can be expressed as

$$X_1''(t) = X_T''(t) + C(t)X'_{1,N}(t),$$

where X_T'' is the tangent projection and $C(t)$ is the signed curvature at the point t.

It is easy to check that the covariance function of the vector $(X_T'', X'_{1,N}, X, X_1')$ is

$$\begin{pmatrix} \mathrm{Var}(X_T'') & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, for such edge L,

$$\begin{aligned} & \mathrm{E}\left(X_1''(t)\mathbb{I}_{X'_{1,N}(t)\in\widehat{C}_{t,1}} \mid X(t) = x,\ X_1'(t) = 0\right) \\ = \ & \mathrm{E}\left(\left(-x + C(t)X'_{1,N}(t)\right)\mathbb{I}_{X'_{1,N}(t)\in\widehat{C}_{t,1}}\right) \\ = \ & \frac{-x}{2} + \frac{C(t)}{\sqrt{2\pi}}, \end{aligned}$$

and

$$-\int_L \mathrm{E}\left(X_1''(t)\mathbb{I}_{X_{1,N}'(t)\in\widehat{C}_{t,1}} \mid X(t)=x,\, X_1'(t)=0\right)\frac{\varphi(x)}{\sqrt{2\pi}}dt = \frac{\sigma_1(L)x}{2\sqrt{2\pi}}\varphi(x) - \frac{\varphi(x)}{2\pi}\int_L C(t)dt.$$

the quantity $-\int_L C(t)dt$ can be viewed as the variation of the angle of the tangent from the beginning to the end of this edge.

Since we complete a whole turn in the positive orientation:

$$\sum_{\text{irregular points of the exterior bounadry}} \mathcal{A}(\widehat{C}_t) + \sum_{\text{edges of the exterior boundary}} -\int_{L_i} C(t)dt = 2\pi.$$

For a point t on an edge L_i of the interior boundary (holes), the interpretation of the second derivative changes into

$$X_1''(t) = X_T''(t) - C(t)X_{1,N}'(t).$$

Therefore,

$$-\int_{L_i} \mathrm{E}\left(X_1''(t)\mathbb{I}_{X_{1,N}'(t)\in\widehat{C}_{t,1}} \mid X(t)=x,\, X_1'(t)=0\right)\frac{\varphi(x)}{\sqrt{2\pi}}dt = \frac{\sigma_1(L_i)x}{2\sqrt{2\pi}}\varphi(x) + \frac{\varphi(x)}{2\pi}\int_{L_i} C(t)dt.$$

For the boundary of a hole inside S,

$$\sum_{\text{irregular points}} \mathcal{A}(\widehat{C}_t) + \sum_{\text{edges}} \int_{L_i} C(t)dt = -2\pi.$$

In conclusion, substituting into (3.12),

$$p^E(x) = \chi(S)\varphi(x) + \frac{\sigma_1(\partial S)}{2\sqrt{2\pi}}x\varphi(x) + \frac{\sigma_2(S)}{2\pi}(x^2-1)\varphi(x),$$

and we obtain the asymptotic expansion

$$\mathrm{P}(M_S \geq u) = \chi(S)\overline{\Phi}(u) + \frac{\sigma_1(\partial S)}{2\sqrt{2\pi}}\varphi(u) + \frac{\sigma_2(S)}{2\pi}u\varphi(u) + Rest,$$

where $Rest$ is super exponentially smaller in the sense of (3.11).

That implies the correspondence between the asymptotic expansion and the Steiner formula.

Chapter 4

Rate of convergence of CLT for sojourn time

4.1 Introduction

Let $X = \{X(t), t \in \mathbb{R}^d\}$ be a stationary centered Gaussian field and T be a measurable subset of $\mathbb{R}^d$. The sojourn time (or the volume of the excursion set) of X above the level u_T in T is defined as

$$\int_T \mathbb{I}(X(t) \geq u_T)dt.$$

The origin of this subject is the intersection between the study of the geometric properties of random surfaces and the one of the non-linear functionals of Gaussian fields. Moreover, it has many applications in statistics of random processes (see, for example, Spodarev and Timmermann[14]).

The case of a fixed level: $u_T = u = const$ has been addressed in dimension 1 by of Sun [50], Chambers and Slud [16], Major [32], and Giraitis and Surgailis [22]. Later on, some multidimensional versions were proved by Breuer and Major [13], Arcones [3], Ivanov and Leonenko [24] and Bulinski, Spodarev and Timmermann [14]. Their works are based on the following assumption

(B). $\{X(t) : t \in \mathbb{R}^d\}$ is a stationary centered Gaussian field with unit variance and covariance function $\rho(t)$ such that

$$\int_{\mathbb{R}^d} |\rho(t)|dt < \infty,$$

and can be presented in the following statement.

Theorem 4.1. *Let $\{X(t) : t \in \mathbb{R}^d\}$ be a random field satisfying the condition (B). For a fixed real-valued u, define the sojourn time as*

$$S_T = \int_{[0,T]^d} \mathbb{I}(X(t) \geq u)dt. \tag{4.1}$$

Then, as T tends to infinity,

$$\frac{S_T - T^d\overline{\Phi}(u)}{\sqrt{T^d}} \xrightarrow{d} \mathcal{N}(0,\sigma^2),$$

where

$$0 < \sigma^2 = \sum_{n=1}^{\infty} \frac{\varphi^2(u)H_{n-1}^2(u)}{n!} \int_{\mathbb{R}^d} \rho^n(t)dt < \infty, \tag{4.2}$$

φ is the density function of the standard Gaussian law and $\overline{\Phi}$ is the tail of its distribution.

Berman [8] considered the problem for a Gaussian process in the case when the level depends on T. When the covariance function is not integrable, he assumed that the main component of the sojourn time is the first chaos in the Wiener chaos expansion. Else, his arguments were based on the spectral representation

$$\rho(t) = \int_R b(t+s)b(s)ds,$$

with the mixing condition $b \in L^1 \cap L^2$ and the m-dependent method. More precisely, he approximated the function b by a sequence of functions with compact support obtaining a family of m-dependent processes converges to the original one, and then he could use the central limit theorems that had been proved for this kind of process. His method can be applied in the multivariate case.

However, the above works do not give us much information about the rate of convergence for the central limit theorems. Then, in this chapter, we aim to control the speed in both cases: the fixed and the moving level. Our approaches come from the recent techniques, developed by Nualart, Peccati and Nourdin ([36],[38],[41], etc.), that are the combination between the Malliavin calculus and the Stein's method. Here, we consider the Wasserstein distance for two integrable variables

$$d(X,Y) = \sup_{h\in \mathrm{Lip}(1)} |\mathrm{E}(h(X)) - \mathrm{E}(h(Y))|,$$

where Lip(1) is the collection of all Lipschitz functions with Lipschitz constant ≤ 1. Our main results are the following:

Theorem 4.2 (Fixed level). *Let $\{X(t) : t \in \mathbb{R}^d\}$ be a random field satisfying the condition (B). Assume that the covariance function ρ satisfies*

$$\int_{\mathbb{R}^d\setminus[-a,a]^d} |\rho(t)|dt \leq (const)(\log a)^{-1}, \text{ for } a \to \infty. \tag{4.3}$$

Let S_T be defined by (4.1). Then,

$$d\left(\frac{S_T - \mathrm{E}(S_T)}{\sqrt{T^d}}, \mathcal{N}(0,\sigma^2)\right) \leq C(\log T)^{-1/4},$$

where C is a constant depending on the field and the level, and σ^2 satisfies (4.2).

Note that the condition (4.3) is weak, for example if

$$\rho(t) \cong (const)\|t\|^{-\alpha}, \quad t \to +\infty$$

for some positive $\alpha > d$, then it is met. Here and in the following, the notation $f(x) \cong g(x),\ x \to a$ means that $\lim_{x\to a} \frac{f(x)}{g(x)} = 1$.

Theorem 4.3 **(Moving level).** *Let* $\{X(t) : t \in \mathbb{R}^d\}$ *be a random field satisfying the condition (B). Suppose that there exists a positive constant* $\alpha \in]0; 2]$ *such that in a neighborhood of* 0*, the covariance function* ρ *satisfies*

$$1 - \rho(t) \cong (const)\|t\|^{\alpha} \text{ for } t \to 0.$$

Let u_T *be a function that tends to infinity. One defines the sojourn time as*

$$S_T = \int_{[0,T]^d} \mathbb{I}(X(t) \geq u_T)dt.$$

Then, for every $\beta \in (0; d/2)$*, there exists a constant* C_β *depending on the field such that*

$$d\left(\frac{S_T - \mathrm{E}(S_T)}{\sqrt{\mathrm{Var}(S_T)}}, \mathcal{N}(0,1)\right) \leq C_\beta \left[\sqrt{\frac{u_T^{\frac{2+\alpha}{\alpha}}}{(\log T)^{1/6}}} + \frac{1}{T^\beta \varphi(u_T) u_T}\right].$$

In Nourdin et al [38], the authors consider a very general case of Theorem 4.2 in the discrete time and obtain the bound under the form of an optimization problem. Here, in our particular case, we deal with a continuous time field and give an explicit bound.

4.2 Preliminaries

We use some notations that come from the Malliavin calculus introduced as follows.

- *Isonormal Gaussian process*
  Let $\mathfrak{H}$ be a real separable Hilbert space. Denote by $X = \{X(h) : h \in \mathfrak{H}\}$ an isonormal Gaussian process over $\mathfrak{H}$, that is a centered Gaussian family, defined on some probability space $(\Omega, \mathcal{F}, \mathrm{P})$, and $\mathrm{E}(X(h)X(g)) = \langle h, g\rangle_{\mathfrak{H}}$ for every $h,\ g \in \mathfrak{H}$. We assume that $\mathcal{F}$ is generated by X.

- *Wiener chaos expansion*
  The n-th Hermite polynomial is

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{\frac{-x^2}{2}}.$$

For every $n \geq 1$, the n-th Wiener chaos $\mathcal{H}_n$ is defined as the closed linear subspace of $L^2(\Omega, \mathcal{F}, \mathrm{P})$ generated by the random variables of the type $H_n(X(h))$, where $h \in \mathfrak{H}$ is such that $\|h\|_{\mathfrak{H}} = 1$. Then, every square-integrable random variable $Z \in (\Omega, \mathcal{F}, \mathrm{P})$ has the Wiener chaos expansion

$$Z = \sum_{n=0}^{\infty} J_n(Z), \tag{4.4}$$

where $J_0(Z) = \mathrm{E}(Z)$ and $J_n(Z)$ is the projection of Z on $\mathcal{H}_n$. Besides, for any $n \geq 1$ and $h \in \mathfrak{H}$, $\|h\|_{\mathfrak{H}} = 1$, the application

$$I_n(h^{\otimes n}) = H_n(X(h)),$$

can be extended to a linear isometry between the symmetric tensor product $\mathfrak{H}^{\odot n}$ equipped with the norm $\sqrt{n!}\|.\|_{\mathfrak{H}^{\otimes n}}$ and the n-th Wiener chaos $\mathcal{H}_n$. So, Z can be also decomposed in the form

$$Z = \sum_{n=0}^{\infty} I_n(f_n),$$

where $I_0(c) = c$ for all real c, $f_0 = \mathrm{E}(Z)$ and $f_n \in \mathfrak{H}^{\odot n}$, $n \geq 1$, are uniquely determined.

- *Contraction and multiplication*

  Let $\{e_k,\ k \geq 1\}$ be a complete orthonormal system in $\mathfrak{H}$. Given $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, then for every $r = 0, 1, \ldots, p \wedge q$, the contraction of f and g of order r is the element of $\mathfrak{H}^{\otimes(p+q-2r)}$ defined by

$$f \otimes_r g = \sum_{i_1,\ldots,i_r=1}^{\infty} \langle f,\, e_{i_1} \otimes \ldots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g,\, e_{i_1} \otimes \ldots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}}.$$

  Then,

$$I_p(f) I_q(g) = \sum_{r=0}^{p\wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \widetilde{\otimes}_r g),$$

  where $f \widetilde{\otimes}_r g \in \mathfrak{H}^{\odot(p+q-2r)}$ is the symmetrization of $f \otimes_r g$.

- *Malliavin derivatives*

  Let Z be a random variable of the smooth form

$$Z = g(X(h_1), \ldots, X(h_n)),$$

  where $n \geq 1$, $g : \mathbb{R}^n \to \mathbb{R}$ is an infinitely differentiable function with compact support and $h_i \in \mathfrak{H}$. Then, the Malliavin derivative of Z is the element of $L^2(\Omega, \mathfrak{H})$ defined as

$$DZ = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(X(h_1), \ldots, X(h_n)) h_i.$$

- *Ornstein-Uhlenbeck operators*
  The operator L is defined as $L = \sum_{n=0}^{\infty} -nJ_n$. The domain of L is

$$DomL = \{Z \in L^2(\Omega) : \sum_{n=1}^{\infty} n^2 \|J_n(Z)\|_2^2 < \infty\},$$

where $\|J_n(Z)\|_2 = \|J_n(Z)\|_{L^2(\Omega)}$. Define the operator L^{-1}, called the pseudo-inverse of L, as $L^{-1}(Z) = \sum_{n=1}^{\infty} -\frac{1}{n} J_n(Z)$ for all $Z \in L^2(\Omega)$.

4.3 The fixed level case

Lemma 4.4. *For every $n \geq 2$, let F_n be*

$$F_n = \frac{1}{\sqrt{T^d}} \int_{[0,T]^d} H_n(X(t))dt.$$

Then,

$$\mathrm{Var}(\|DF_n\|_{\mathfrak{H}}^2) \leq \frac{n^4}{T^d} \sum_{r=0}^{n-2} (r!)^2 \binom{n-1}{r}^4 (2n-2-2r)! \left(\int_{\mathbb{R}^d} |\rho(t)| dt \right)^3.$$

Proof. The Malliavin derivative of F_n is

$$DF_n = \frac{1}{\sqrt{T^d}} \int_{[0,T]^d} nH_{n-1}(X(t))\psi_t dt,$$

where ψ_t is the element in $\mathfrak{H}$ corresponding to $X(t)$, i.e, $X(t) = \overline{X}(\psi(t))$. And,

$$\|DF_n\|_{\mathfrak{H}}^2 = \frac{1}{T^d} n^2 \int_{[0,T]^d \times [0,T]^d} \rho(t-s) H_{n-1}(X(t)) H_{n-1}(X(s)) dtds.$$

From the Mehler's formula, it is clear that

$$\mathrm{E}[\|DF_n\|_{\mathfrak{H}}^2] = \frac{1}{T^d} n^2 \int_{[0,T]^d \times [0,T]^d} (n-1)! \rho^n(t-s) \, dt \, ds = n\mathrm{Var}(F_n).$$

Using the fact that

$$H_{n-1}(X(t)) = I_{n-1}(\psi_t^{\otimes n-1}),$$

and

$$\begin{aligned} I_{n-1}(\psi_t^{\otimes n-1}) I_{n-1}(\psi_s^{\otimes n-1}) &= \sum_{r=0}^{n-1} r! \binom{n-1}{r}^2 I_{2n-2-2r}(\psi_t^{\otimes n-1} \widetilde{\otimes}_r \psi_s^{\otimes n-1}) \\ &= \sum_{r=0}^{n-1} r! \binom{n-1}{r}^2 \rho^r(t-s) I_{2n-2-2r}(\psi_t^{\otimes n-1-r} \widetilde{\otimes} \psi_s^{\otimes n-1-r}), \end{aligned}$$

$\|DF_T\|_{\mathfrak{H}}^2$ can be expressed as

$$\|DF_T\|_{\mathfrak{H}}^2 = \frac{1}{T^d} \sum_{r=0}^{n-1} \int_{[0,T]^d \times [0,T]^d} n^2 r! \binom{n-1}{r}^2 \rho^{r+1}(t-s) I_{2n-2-2r}(\psi_t^{\otimes n-1-r} \widetilde{\otimes} \psi_s^{\otimes n-1-r}) \, dt \, ds.$$

And, from the orthogonality of the chaos, the variance of $\|DF_T\|^2_{\mathfrak{H}}$ is equal to

$$\frac{n^4}{T^{2d}}\sum_{r=0}^{n-2}\int_{[0,T]^{d\times 4}} (r!)^2\tbinom{n-1}{r}^4\rho^{r+1}(t-s)\rho^{r+1}(t'-s') \times(2n-2-2r)!\langle\psi_t^{\otimes n-1-r}\widetilde{\otimes}\psi_s^{\otimes n-1-r},\psi_{t'}^{\otimes n-1-r}\widetilde{\otimes}\psi_{s'}^{\otimes n-1-r}\rangle\, dtdsdt'ds'.$$

Each element of the scalar product has the form

$$\rho^{n-1-r-i}(t-t')\rho^{n-1-r-i}(s-s')\rho^i(t-s')\rho^i(s-t'),$$

for some $i\in[0;n-1-r]$. And

$$\int_{[0,T]^{d\times 4}}\rho^{r+1}(t-s)\rho^{r+1}(t'-s')\rho^{n-1-r-i}(t-t')\rho^{n-1-r-i}(s-s')\rho^i(t-s')\rho^i(s-t')\, dtdsdt'ds'$$

is at most equal to

$$\int_{[0,T]^{d\times 4}}|\rho(t-s)\rho(t'-s')\rho(t-t')\rho(s-s')|\, dtdsdt'ds',$$

or

$$\int_{[0,T]^{d\times 4}}|\rho(t-s)\rho(t'-s')\rho(t-s')\rho(s-t')|\, dtdsdt'ds'.$$

With the change of variable $y=(t-s,t'-s',t-t',s')$,

$$\int_{[0,T]^{d\times 4}}|\rho(t-s)\rho(t'-s')\rho(t-t')\rho(s-s')|\, dtdsdt'ds'$$

can be written as

$$\int_{[0,T]^d}dy_4\int_{A_{y_4}}|\rho(y_1)\rho(y_2)\rho(y_3)\rho(y_2+y_3-y_1)|\, dy_1dy_2dy_3,$$

where A_{y_4} is some domain in $\mathbb{R}^3$ that depends on y_4. It is at most equal to

$$\int_{[0,T]^d}dy_4\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d}|\rho(y_1)\rho(y_2)\rho(y_3)|\, dy_1dy_2dy_3=T^d\left(\int_{\mathbb{R}^d}|\rho(t)|dt\right)^3.$$

The same bound is obtained for the others. So, the variance of $\|DF_T\|^2_{\mathfrak{H}}$ is at most equal to

$$\frac{n^4}{T^d}\sum_{r=0}^{n-2}(r!)^2\binom{n-1}{r}^4(2n-2-2r)!\left(\int_{\mathbb{R}^d}|\rho(t)|dt\right)^3.$$

□

We use some facts about Hermite polynomials (see Szegö [52]).

Lemma 4.5. • *For a fixed point u, there exists a constant C_u such that*

$$e^{-u^2/4}|H_n(u)|\le C_u(n/e)^{n/2}\ \forall\, n\in\mathbb{N}. \tag{4.5}$$

- *There exists a constant K such that, for all u, n,*

$$\frac{\varphi(u)|H_n(u)|}{\sqrt{n!}} < K. \tag{4.6}$$

- *As n tends to infinity,*

$$\max_{x\in\mathbb{R}} e^{-x^2/4}|H_n(x)| \cong (const)\sqrt{n!}\, n^{-1/12}. \tag{4.7}$$

Proof of Theorem 4.2. It is clear that

$$d\left(\frac{S_T - \mathrm{E}(S_T)}{\sqrt{T^d}}, \mathcal{N}(0,\sigma^2)\right) \le d\left(\frac{S_T - \mathrm{E}(S_T)}{\sqrt{T^d}}, \frac{S_{T,N_T} - \mathrm{E}(S_{T,N_T})}{\sqrt{T}}\right) + d\left(\frac{S_{T,N_T} - \mathrm{E}(S_{T,N_T})}{\sqrt{T^d}}, \mathcal{N}(0,\sigma^2_{N_T})\right) + d\left(\mathcal{N}(0,\sigma^2_{N_T}), \mathcal{N}(0,\sigma^2)\right) = d_1 + d_2 + d_3$$

where S_{T,N_T} is the truncation of S_T at position N_T in the Wiener chaos expansion. N_T will be chosen later on.

i) **(Bound for d_1)** It is easy to show that

$$\begin{aligned} & d\left(\frac{S_T - \mathrm{E}(S_T)}{\sqrt{T^d}}, \frac{S_{T,N_T} - \mathrm{E}(S_{T,N_T})}{\sqrt{T^d}}\right) \\ \le & \left\|\frac{S_T - S_{T,N_T}}{\sqrt{T^d}}\right\|_2 \\ = & \sqrt{\sum_{n=N_T+1}^{\infty} \frac{\varphi^2(u)H^2_{n-1}(u)}{n!T^d} \int_{[-T,T]^d} \rho^n(t) \prod_{j=1}^{d}(T - |t_j|)dt}. \end{aligned}$$

Here, from (4.5) and the Stirling formula

$$n! \sim \sqrt{2\pi n}(n/e)^n,$$

we obtain the bound for d_1

$$d_1 \le C_u\sqrt{\varphi(u)}\sqrt{\int_{\mathbb{R}^d}|\rho(t)|dt}\sqrt{\sum_{n=N_T+1}^{\infty} n^{-(1+\frac{1}{2})}} \le (const)N_T^{-1/4}. \tag{4.8}$$

ii) **(Bound for d_2)** From Theorem 3.1 of [38], it is clear that

$$\begin{aligned} & d\left(\frac{S_{T,N_T} - \mathrm{E}(S_{T,N_T})}{\sqrt{T^d}}, \mathcal{N}(0,\sigma^2_{N_T})\right) \\ \le & \left\|\sigma^2_{N_T} - \left\langle D\frac{S_T - \mathrm{E}(S_T)}{\sqrt{T^d}}, -DL^{-1}\frac{S_T - \mathrm{E}(S_T)}{\sqrt{T^d}}\right\rangle_{\mathfrak{H}}\right\|_2 \\ \le & \sum_{p,q=1}^{N_T}\left\|\delta_{pq}\sigma^2_{T,p} - q^{-1}\left\langle DJ_p\left(\frac{S_T - \mathrm{E}(S_T)}{\sqrt{T^d}}\right), DJ_q\left(\frac{S_T - \mathrm{E}(S_T)}{\sqrt{T^d}}\right)\right\rangle_{\mathfrak{H}}\right\|_2, \end{aligned}$$

where J_p is the component in the p-th chaos defined in (4.4) and $\sigma^2_{T,p}$ is the variance of $J_p(\frac{S_T - \mathrm{E}(S_T)}{\sqrt{T^d}})$.

– If $p = q = 1$,

$$\left\| \sigma^2_{T,1} - \left\langle DJ_1\left(\frac{S_T - \mathrm{E}(S_T)}{\sqrt{T^d}}\right), DJ_1\left(\frac{S_T - \mathrm{E}(S_T)}{\sqrt{T^d}}\right)\right\rangle_{\mathfrak{H}} \right\|_2 = 0.$$

– If $p = q > 1$,

$$\begin{aligned} &\left\| \sigma^2_{T,p} - p^{-1}\left\langle DJ_p\left(\frac{S_T - \mathrm{E}(S_T)}{\sqrt{T^d}}\right), DJ_p\left(\frac{S_T - \mathrm{E}(S_T)}{\sqrt{T^d}}\right)\right\rangle_{\mathfrak{H}} \right\|_2 \\ =& p^{-1}\sqrt{\mathrm{Var}\left(DJ_p\left(\frac{S_T - \mathrm{E}(S_T)}{\sqrt{T^d}}\right)\right)} \\ =& \frac{\varphi^2(u)H^2_{p-1}(u)}{(p!)^2}p^{-1}\sqrt{\mathrm{Var}\left(\left\|D\left(\frac{1}{\sqrt{T^d}}\int_0^T H_p(X(t))dt\right)\right\|^2_{\mathfrak{H}}\right)}. \end{aligned}$$

Then, from Lemma 4.4, it is at most equal to

$$\frac{\varphi^2(u)H^2_{p-1}(u)}{(p!)^2}\frac{p}{\sqrt{T^d}}\sqrt{\left(\int_{\mathbb{R}^d}|\rho(t)|dt\right)^3\left(\sum_{r=0}^{p-2}(r!)^2\binom{p-1}{r}^4(2p-2-2r)!\right)}.$$

– If $p > 1$ and $q = 1$, then

$$\begin{aligned} &\left\langle D\left(\frac{1}{\sqrt{T^d}}\int_{[0,T]^d} H_p(X(t))dt\right), D\left(\frac{1}{\sqrt{T^d}}\int_{[0,T]^d} H_1(X(t))dt\right)\right\rangle_{\mathfrak{H}} \\ =& \frac{p}{T^d}\int_{[0,T]^d\times[0,T]^d}\rho(t-s)H_{p-1}(X(t))dtds. \end{aligned}$$

So, its variance is

$$\frac{1}{T^{2d}}p^2\int_{[0,T]^{d\times 4}}(p-1)!\rho(t-s)\rho(t'-s')\rho^{p-1}(t-t')dtdsdt'ds' \le \frac{p^2(p-1)!}{T^d}\left(\int_{\mathbb{R}^d}|\rho(t)|dt\right)^3.$$

– If $p, q > 1$ and $p \neq q$, then

$$\begin{aligned} &\left\langle D\left(\frac{1}{\sqrt{T^d}}\int_0^T H_p(X(t))dt\right), D\left(\frac{1}{\sqrt{T^d}}\int_0^T H_q(X(t))dt\right)\right\rangle_{\mathfrak{H}} \\ =& \frac{pq}{T^d}\int_{[0,T]^d\times[0,T]^d}\rho(t-s)H_{p-1}(X(t))H_{q-1}(X(s))dtds \\ =& \frac{pq}{T^d}\sum_{r=0}^{p\wedge q-1}\int_{[0,T]^d\times[0,T]^d} r!\binom{p-1}{r}\binom{q-1}{r}\rho(t-s)I_{p+q-2-2r}((\psi_t^{\otimes p-1}\widetilde{\otimes}_r\psi_s^{\otimes q-1})_s)dtds. \end{aligned}$$

So, its variance is at most equal to

$$\le \frac{(pq)^2}{T^d}\left(\int_{\mathbb{R}^d}|\rho(t)|dt\right)^3\left(\sum_{r=0}^{p\wedge q-1}(r!)^2\binom{p-1}{r}^2\binom{q-1}{r}^2(p+q-2-2r)!\right).$$

We obtain the bound for d_2

$$\sqrt{\left(\int_{\mathbb{R}^d}|\rho(t)|dt\right)^3}\left[\sum_{p=2}^{N_T}\frac{\varphi^2(u)H_{p-1}^2(u)}{(p!)^2}\frac{p}{\sqrt{T^d}}\sqrt{\sum_{r=0}^{p-2}(r!)^2\binom{p-1}{r}^4(2p-2-2r)!}\right.$$

$$\left.+\sum_{p,q=1;p\neq q}^{N_T}\left(\frac{1}{p}+\frac{1}{q}\right)\frac{\varphi^2(u)|H_{p-1}(u)H_{q-1}(u)|}{p!q!}\frac{pq}{\sqrt{T^d}}\sqrt{\sum_{r=0}^{p\wedge q-1}(r!)^2\binom{p-1}{r}^2\binom{q-1}{r}^2(p+q-2-2r)!}\right].$$

So,

$$d_2\leq(const)\frac{3^{N_T}}{\sqrt{T^d}}. \tag{4.9}$$

Indeed, from

$$\begin{aligned}
&\frac{1}{((p-1)!)^2}\sum_{r=0}^{p-2}(r!)^2\binom{p-1}{r}^4(2p-2-2r)!\\
=&\sum_{r=0}^{p-2}\binom{p-1}{r}^2\binom{2p-2-2r}{p-1-r}\\
\leq&\sum_{r=0}^{p-2}\binom{p-1}{r}^2 2^{2p-2-2r}\\
\leq&\ 2^{2p-2}\left(\sum_{r=0}^{p-2}\binom{p-1}{r}2^{-r}\right)^2\\
\leq&\ 2^{2p-2}(1+1/2)^{2p-2}=9^{p-1},
\end{aligned}$$

and (4.6), the first term is at most equal to

$$(const)\frac{1}{\sqrt{T^d}}\sum_{p=2}^{N_T}\frac{3^{p-1}}{p};$$

and the same for the second term.

iii) **(Bound for d_3)** It is easy to show that

$$\begin{aligned}
d^2(\mathcal{N}(0,\sigma_{N_T}^2),\mathcal{N}(0,\sigma^2))\leq\ &(const)(\sigma^2-\sigma_{N_T}^2)\\
=\ &\left[\sum_{n=1}^{N_T}\frac{\varphi^2(u)H_{n-1}^2(u)}{n!}\int_{[-T,T]^d}\rho^n(t)\frac{T^d-\prod_{j=1}^{d}(T-|t_j|)}{T^d}dt\right.\\
&+\sum_{n=1}^{N_T}\frac{\varphi^2(u)H_{n-1}^2(u)}{n!}\int_{\mathbb{R}^d\setminus[-T,T]^d}\rho^n(t)dt\\
&\left.+\sum_{n=N_T+1}^{\infty}\frac{\varphi^2(u)H_{n-1}^2(u)}{n!}\int_{\mathbb{R}^d}\rho^n(t)dt\right].
\end{aligned}$$

From part i), the third term is at most equal to $(const)N_T^{-1/2}$. For the first term, it is equal to

$$\sum_{n=1}^{N_T} \frac{\varphi^2(u)H_{n-1}^2(u)}{n!} \int_{[-\sqrt{T},\sqrt{T}]^d} \rho^n(t) \frac{T^d - \prod_{j=1}^{d}(T-|t_j|)}{T^d} dt$$
$$+ \sum_{n=1}^{N_T} \frac{\varphi^2(u)H_{n-1}^2(u)}{n!} \int_{[-T,T]^d \setminus [-\sqrt{T},\sqrt{T}]^d} \rho^n(t) \frac{T^d - \prod_{j=1}^{d}(T-|t_j|)}{T^d} dt,$$

which is at most equal to

$$\sum_{n=1}^{N_T} \frac{\varphi^2(u)H_{n-1}^2(u)}{n!\sqrt{T}} \int_{[-\sqrt{T},\sqrt{T}]^d} |\rho^n(t)|dt$$
$$+ \sum_{n=1}^{N_T} \frac{\varphi^2(u)H_{n-1}^2(u)}{n!} \int_{[-T,T]^d \setminus [-\sqrt{T},\sqrt{T}]^d} |\rho^n(t)|dt.$$

The first part is at most equal to $\dfrac{(const)}{\sqrt{T}}$. The sum of the second part and the second term is

$$\sum_{n=1}^{N_T} \frac{\varphi^2(u)H_{n-1}^2(u)}{n!} \int_{\mathbb{R}^d \setminus [-\sqrt{T},\sqrt{T}]^d} |\rho^n(t)|dt,$$

and at most equal to $(const)(\log T)^{-1}$ (from (4.3)). So,

$$d_3 \leq (const)(N_T^{-1/4} + T^{-1/4} + (\log T)^{-1/2}). \tag{4.10}$$

Summing up three bounds (4.8), (4.9) and (4.10), by choosing $N_T = (\log T)/4$, we have the result. □

4.4 The moving level case

In this section, we assume that the level depends on T and we denote by u_T. Then the sojourn time

$$S_T = \int_{[0,T]^d} \mathbb{I}(X(t) \geq u_T)dt$$

has

$$\mathrm{E}(S_T) = T^d \overline{\Phi}(u_T)$$

and

$$\mathrm{Var}(S_T) = \int_{[-T,T]^d} \prod_{j=1}^{d}(T-|t_j|)dt \int_0^{\rho(t)} \varphi(u_T, u_T, y)dy,$$

where

$$\varphi(u_T, u_T, y) = \frac{1}{2\pi\sqrt{1-y^2}} \exp\left(\frac{-u_T^2}{1+y}\right)$$

is the density of the bivariate normal vector

$$\mathcal{N}\left(0, \begin{bmatrix} 1 & y \\ y & 1 \end{bmatrix}\right).$$

When u_T tends to infinity,

$$\frac{\mathrm{Var}(S_T)}{T^d} \to 0,$$

then the Theorem 4.1 and 4.2 no longer hold. So, at first, we generalize the results of Berman [8] (chapter 8) to estimate the variance of S_T (the detailed proofs are given in the Appendix).

Lemma 4.6. *If the covariance function ρ satisfies the conditions in Theorem 4.3, then, for every $\epsilon > 0$,*

$$\int_{[-T,T]^d} \prod_{j=1}^{d} (T - |t_j|) dt \int_0^{\rho(t)} \varphi(u_T, u_T, y) dy \cong T^d \int_{[-\epsilon,\epsilon]^d} \int_0^{\rho(t)} \varphi(u_T, u_T, y) dy dt,$$

for $T, u_T \to \infty$.

So, let $B(u)$ be some function that satisfies

$$B(u) \cong \int_{[-\epsilon,\epsilon]^d} \int_0^{\rho(t)} \varphi(u, u, y) dy dt, \text{ for } u \to \infty.$$

Then,

$$\mathrm{Var}(S_T) \cong T^d B(u_T),$$

for $T, u_T \to \infty$.

Lemma 4.7. *If the covariance function ρ satisfies the conditions in Theorem 4.3, then,*

$$B(u) \cong (const) \frac{\varphi(u)}{u^{\frac{2+\alpha}{\alpha}}}, \text{ for } u \to \infty.$$

Proof of Theorem 4.3. The distance between $\frac{S_T - \mathrm{E}(S_T)}{\sqrt{\mathrm{Var}(S_T)}}$ and the standard Gaussian variable is at most equal to

$$d\left(\frac{S_T - \mathrm{E}(S_T)}{\sqrt{\mathrm{Var}(S_T)}}, \frac{S_{T,N_T} - \mathrm{E}(S_{T,N_T})}{\sqrt{\mathrm{Var}(S_{T,N_T})}}\right) + d\left(\frac{S_{T,N_T} - \mathrm{E}(S_{T,N_T})}{\sqrt{\mathrm{Var}(S_{T,N_T})}}, \mathcal{N}(0,1)\right),$$

where S_{T,N_T} is the truncate variable of S_T at position N_T in the Wiener chaos expansion. N_T will be chosen later on.

- The first term is at most equal to (up to some multiplicative constants)

$$\begin{aligned}
&\sqrt{\frac{\mathrm{Var}(S_T - S_{N_T})}{\mathrm{Var}(S_T)}} \\
= \ & (const)\sqrt{\frac{\sum_{n=N_T+1}^{\infty} \frac{\varphi^2(u_T)H_{n-1}^2(u_T)}{n!} \int_{[-T,T]^d} \rho^n(t) \prod_{j=1}^{d}(T-|t_j|)dt}{\mathrm{Var}(S_T)}} \\
\cong \ & (const)\sqrt{u_T^{\frac{2+\alpha}{\alpha}} \sum_{n=N_T+1}^{\infty} \frac{\varphi(u_T)H_{n-1}^2(u_T)}{n!} \int_{[-T,T]^d} \rho^n(t) \frac{\prod_{j=1}^{d}(T-|t_j|)}{T^d} dt} \\
\leq \ & (const)\sqrt{u_T^{\frac{2+\alpha}{\alpha}} \int_{\mathbb{R}^d} |\rho(t)|dt \sum_{n=N_T+1}^{\infty} \frac{\varphi(u_T)H_{n-1}^2(u_T)}{n!}} \\
\leq \ & (const)\sqrt{u_T^{\frac{2+\alpha}{\alpha}} \sum_{n=N_T+1}^{\infty} n^{-(1+\frac{1}{6})}} \quad ,
\end{aligned}$$

where in the third line, we use the approximation

$$\mathrm{Var}(S_T) \cong T^d B(u_T) \cong (const) T^d \frac{\varphi(u_T)}{u_T^{\frac{2+\alpha}{\alpha}}},$$

and in the last one, the fact (4.7) is used. Then, we have the bound

$$(const)\sqrt{\frac{\varphi(u_T)\int_{\mathbb{R}^d}|\rho(t)|dt \sum_{n=N_T+1}^{\infty} n^{-(1+\frac{1}{6})}}{B(u_T)}} \leq (const)\sqrt{\frac{u_T^{\frac{2+\alpha}{\alpha}}}{N_T^{1/6}}}. \qquad (4.11)$$

- For the second term, as the same argument in part ii) in the proof of Theorem 4.2, we have the bound

$$(const)\frac{3^{N_T}}{\sqrt{T^d}\sqrt{\sum_{n=1}^{N_T} \frac{\varphi^2(u_T)H_{n-1}^2(u_T)}{n!} 2\int_{[-T,T]^d} \rho^n(t) \frac{\prod_{j=1}^{d}(T-|t_j|)}{T^d} dt}},$$

which is at most equal to

$$\frac{3^{N_T}}{T^{d/2}\varphi(u_T)u_T}. \qquad (4.12)$$

Summing up (4.11) and (4.12), by choosing N_T such that $3^{N_T} = T^{-\beta+d/2}$, the result follows. □

We have the following corollary

Corollary 4.8. *Let* $\{X(t) : t \in \mathbb{R}^d\}$ *be a random field satisfying the condition (B). Suppose that there exists a positive constant* $\alpha \in]0;2]$ *such that in a neighborhood of* 0*, the covariance function* ρ *satisfies*

$$1 - \rho(t) \cong (const)\|t\|^{\alpha} \text{ for } t \to 0.$$

One defines the sojourn time

$$S_T = \int_{[0,T]^d} \mathbb{I}(X(t) \geq u_T) dt.$$

Let u_T be a function that tends to infinity. Then, if

$$(\log T)^{-1/6} u_T^{\frac{2+\alpha}{\alpha}} \to 0,$$

one has

$$\frac{S_T - \mathrm{E}(S_T)}{\sqrt{\mathrm{Var}(S_T)}} \xrightarrow{d} \mathcal{N}(0,1).$$

Proof. Since $(\log T)^{-1/6} u_T^{\frac{2+\alpha}{\alpha}} \to 0$, it is easy to see that

$$\frac{1}{T^\beta \varphi(u_T) u_T} \to 0,$$

for all $\beta \in (0; d/2)$ .From Theorem 4.3, the result follows. □

This extends, under the stronger hypothesis on u_T, the results of Berman to Gaussian fields in $\mathbb{R}^d$ with $d > 1$.

Appendix: Proofs of the Lemmas 4.6-4.7

In this Appendix, we prove the Lemmas 4.6-4.7 analogously to the similar ones in [8] with some minor changes.

Proof of Lemma 4.6. It suffices to show that

$$\frac{\int_{[-T,T]^d \setminus [-\epsilon,\epsilon]^d} \prod_{j=1}^{d} (T - |t_j|) \int_0^{\rho(t)} \phi(u_T, u_T, y)\, dydt}{\int_{[-\epsilon,\epsilon]^d} \prod_{j=1}^{d} (T - |t_j|) \int_0^{\rho(t)} \phi(u_T, u_T, y) dydt} \tag{4.13}$$

tends to 0 for $u_T, T \to \infty$. In fact, denote

$$\eta = 1 - \max(|\rho(s)| \ : \ s \notin (-\epsilon, \epsilon)^d).$$

If $\eta = 0$ then there exists $x \neq 0$ such that $|\rho(x)| = 1$, then the field is x- or $2x$- periodic and the integral $\int_{\mathbb{R}^d} |\rho(t)| dt$ can not converge. Therefore, η is strictly positive. Since the function $\varphi(u_T, u_T, y)$ is increasing with respect to y, the numerator in (4.13) is at most equal to

$$T^d \varphi(u_T, u_T, 1 - \eta) \int_{[-T,T]^d \setminus [-\epsilon,\epsilon]^d} |\rho(t)| dt. \tag{4.14}$$

The denominator in (4.13) can be decomposed as

$$\int_{[-\epsilon,\epsilon]^d} \prod_{j=1}^{d}(T-|t_j|)\int_0^{\rho(t)^+}\varphi(u_T,u_T,y)dydt - \int_{[-\epsilon,\epsilon]^d} \prod_{j=1}^{d}(T-|t_j|)\int_{-\rho(t)^-}^{0}\varphi(u_T,u_T,y)dydt. \tag{4.15}$$

There exists a positive constant $c<1$, such that $\rho(t)^- \leq c$, $\forall t$, then the second term in (4.15) is at most equal to

$$(2\epsilon)^d T^d \frac{1}{\sqrt{1-c^2}}\varphi^2(u_T). \tag{4.16}$$

Choose $\delta < \eta$ and $\epsilon' < \epsilon$ such that

$$\min(\rho(t) \ : \ t \in [-\epsilon',\epsilon']^d) \geq 1-\delta,$$

then the first term in (4.15) is lower-bounded by

$$\begin{aligned} & \int_{[-\epsilon',\epsilon']^d} \prod_{j=1}^{d}(T-|t_j|)\int_0^{\rho(t)}\varphi(u_T,u_T,y)dydt \\ \geq \ & (T-\epsilon')^d \int_{[-\epsilon',\epsilon']^d}\int_{1-\delta}^{\rho(t)}\varphi(u_T,u_T,y)dydt. \end{aligned}$$

and it has the lower bound

$$(T-\epsilon')^d\varphi(u_T,u_T,1-\delta)\int_{[-\epsilon',\epsilon']^d}(\rho(t)-1+\delta)dt. \tag{4.17}$$

It is clear that (4.14) and (4.16) are negligible with respect to (4.17) when u_T and T tend to infinity. it implies the result. □

To prove the lemma 4.7, we need the following two results:

Lemma 4.9. *For every $\theta > 1$, there exists a constant $K(\theta) > 0$, such that, asymptotically*

$$B(u) \geq K(\theta)\exp(-u^2\theta/2).$$

Proof. It suffices to prove the lemma for θ in a neighborhood of 1. In such case, using (4.17), we can choose δ such that

$$\exp(-u^2\theta/2) = \varphi(u,u,1-\delta),$$

and we are done. □

Lemma 4.10. *For every $\delta \in (0,1)$, one has*

$$\limsup_{u\to\infty} \frac{B(u)}{2(\frac{2}{2-\delta})^{1/2}\frac{\varphi(u)}{u}\int_{[-\epsilon,\epsilon]^d}\overline{\Phi}\left(u\left[\frac{1-\rho(t)}{2}\right]^{1/2}\right)dt} \leq 1,$$

and

$$\liminf_{u\to\infty} \frac{B(u)}{[2(2-\delta)]^{1/2}\frac{\varphi(u)}{u}\int_{[-\epsilon,\epsilon]^d}\overline{\Phi}\left(u\left[\frac{1-\rho(t)}{2-\delta}\right]^{1/2}\right)dt} \geq 1.$$

Proof. For any $\delta \in (0,1)$, there exists $\epsilon > 0$ such that $1-\rho(s) < \delta$, $\forall s \in [-\epsilon,\epsilon]^d$. Then,

$$\int_{[-\epsilon,\epsilon]^d}\int_0^{1-\delta} \varphi(u,u,y)dydt = (2\epsilon)^d \int_0^{1-\delta} \varphi(u,u,y)dy \leq (2\epsilon)^d(1-\delta)\varphi(u,u,1-\delta).$$

Since

$$\varphi(u,u,1-\delta) = \frac{1}{2\pi\sqrt{1-(1-\delta)^2}} \exp\left(\frac{-u^2}{2-\delta}\right),$$

and from Lemma 4.9, $B(u)$ is asymptotically greater than $K(\theta)\exp(-u^2\theta/2)$ for every $\theta > 1$, then by choosing

$$1 < \theta < \frac{2}{2-\delta},$$

$\int_{[-\epsilon,\epsilon]^d}\int_0^{1-\delta} \varphi(u,u,y)dydt$ is negligible with respect to $B(u)$ when u tends to infinity. Hence, $B(u)$ is asymptotically equal to

$$\begin{aligned}&\int_{[-\epsilon,\epsilon]^d}\int_{1-\delta}^{\rho(t)} \varphi(u,u,y)dydt\\ =&\varphi(u)\int_{[-\epsilon,\epsilon]^d}\int_{1-\delta}^{\rho(t)} \frac{1}{\sqrt{1-y^2}}\varphi\left(u\left[\frac{1-y}{1+y}\right]^{1/2}\right)dydt,\end{aligned}$$

which is equal to, by the change of variable $z = u^2(1-y)$,

$$\frac{\varphi(u)}{u}\int_{[-\epsilon,\epsilon]^d}\int_{u^2(1-\rho(t))}^{u^2\delta} \frac{1}{\sqrt{z(2-z/u^2)}}\varphi\left(\left[\frac{z}{2-z/u^2}\right]^{1/2}\right)dzdt. \tag{4.18}$$

An upper bound of (4.18) is

$$\frac{1}{\sqrt{2-\delta}}\frac{\varphi(u)}{u}\int_{[-\epsilon,\epsilon]^d}\int_{u^2(1-\rho(t))}^{\infty} \varphi(\sqrt{z/2})\frac{dz}{\sqrt{z}}dt,$$

which is equal to, by the change of variable $x = \sqrt{z/2}$,

$$\begin{aligned}&\frac{2\sqrt{2}}{\sqrt{2-\delta}}\frac{\varphi(u)}{u}\int_{[-\epsilon,\epsilon]^d}\int_{u^2(1-\rho(t))}^{\infty} \varphi(x)dxdt\\ =&\frac{2\sqrt{2}}{\sqrt{2-\delta}}\frac{\varphi(u)}{u}\int_{[-\epsilon,\epsilon]^d}\overline{\Phi}\left(u\left[\frac{1-\rho(t)}{2}\right]^{1/2}\right)dt.\end{aligned}$$

A lower bound of (4.18) is

$$\frac{\varphi(u)}{u}\int_{[-\epsilon,\epsilon]^d}\int_{u^2(1-\rho(t))}^{u^2\delta} \varphi\left(\left[\frac{z}{2-\delta}\right]^{1/2}\right)\frac{dz}{\sqrt{2z}}dt,$$

which is equal to, by the change of variable $x = \sqrt{z/(2-\delta)}$,

$$\begin{aligned}&\sqrt{2(2-\delta)}\frac{\varphi(u)}{u}\int_{[-\epsilon,\epsilon]^d}\int_{u(\frac{1-\rho(t)}{2-\delta})^{1/2}}^{u(\delta/(2-\delta))^{1/2}} \varphi(x)dxdt\\ =&\sqrt{2(2-\delta)}\frac{\varphi(u)}{u}\int_{[-\epsilon,\epsilon]^d}\left[\overline{\Phi}\left(u\left[\frac{1-\rho(t)}{2-\delta}\right]^{1/2}\right) - \overline{\Phi}\left(u\left[\frac{\delta}{2-\delta}\right]^{1/2}\right)\right]dt.\end{aligned}$$

Since

$$\lim_{u\to\infty} \sup_{s\in[-\epsilon,\epsilon]^d} \frac{\overline{\Phi}\left(u\left[\frac{1-\rho(t)}{2-\delta}\right]^{1/2}\right)}{\overline{\Phi}\left(u\left[\frac{\delta}{2-\delta}\right]^{1/2}\right)} = 0,$$

the lower bound is asymptotically equal to

$$[2(2-\delta)]^{1/2}\frac{\varphi(u)}{u}\int_{[-\epsilon,\epsilon]^d}\overline{\Phi}\left(u\left[\frac{1-\rho(t)}{2-\delta}\right]^{1/2}\right)dt.$$

□

Proof of Lemma 4.7. By change of variable $t = z/u^{2/\alpha}$, the asymptotically upper bound in Lemma 4.10 is equal to

$$2\left(\frac{2}{2-\delta}\right)^{1/2}\frac{\varphi(u)}{u^{\frac{2+\alpha}{\alpha}}}\int_{[-u^{2/\alpha}\epsilon,u^{2/\alpha}\epsilon]^d}\overline{\Phi}\left(u\left[\frac{1-\rho(z/u^{2/\alpha})}{2}\right]^{1/2}\right)dz.$$

It is clear that

$$u^2(1-\rho(z/u^{2/\alpha})) \to C\|z\|^\alpha \text{ for } u\to\infty,$$

then by dominated convergence, this upper bound is asymptotically equal to

$$2\left(\frac{2}{2-\delta}\right)^{1/2}\frac{\varphi(u)}{u^{\frac{2+\alpha}{\alpha}}}\int_{\mathbb{R}^d}\overline{\Phi}(C\|z\|^\alpha)dz.$$

By the same argument, the lower one in Lemma 4.10 is asymptotically equal to

$$[2(2-\delta)]^{1/2}\frac{\varphi(u)}{u^{\frac{2+\alpha}{\alpha}}}\int_{\mathbb{R}^d}\overline{\Phi}(C\|z\|^\alpha)dz.$$

Let δ tend to 0, we obtain the result. □

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Résumé

Dans cette thèse, nous étudions les propriétés de la surface d'un champ alétoire. Plus précisément, nous nous intéressons à la loi du maximum d'un champ gaussien centré stationnaire et au volume de l'ensemble d'excursion (le temps de séjour). Nous améliorons la "méthode des records" en dimension 2 et la prolongeons à dimension 3 pour donner des bornes supérieures pour la queue de la distribution du maximum. Nous donnons aussi la formule asymptotique de cette queue en dimension 2. Il y a une correspondance entre la formule asymptotique et les coefficients de la formule de Steiner du domaine considéré. Il s'agit d'une prolongation du résultat de Adler. Nous étudions la vitesse de convergence dans le théorème de la limite centrale pour le temps de séjour dans deux cas: à niveau fixe et à niveau variable.

Mots-clefs. Formule de Rice, loi du maximum, champ gaussien, temps de séjour.

Abstract

In this thesis, we study the properties of the paths of random fields. More precisely, we are interested in the distribution of the maximum of stationary centered Gaussian field and the volume of the excursion set (sojourn time). We extend slightly the "record method" in dimension 2 and developpe it in dimension 3 to give an upper bound for the tail of the distribution of the maximum. We also give an asymptotic formula for this tail in dimension 2. There is a correspondence between the asymptotic formula and the coefficients of the Steiner formula of the domain considered. This can be viewed as an extension of some results of Adler. We study the rate of convergence of the central limit theorems of the sojourn time in both cases: fixed and moving level.

Keywords. Rice formula, distribution of the maximum, Gaussian field, sojourn time.

Classifications. 60G15, 60G60, 60G70.