Université Fédérale
Toulouse Midi-Pyrénées

THÈSE

En vue de l'obtention du

DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE

Délivré par :

Université Toulouse 3 Paul Sabatier (UT3 Paul Sabatier)

Si vous êtes en cotutelle internationale, remplissez ce champs en notant : Cotutelle internationale avec "nom de l'établissement", sinon effacer ce texte pour qu'il n'apparaisse pas à l'impression

---

Présentée et soutenue par :
Boubacar DIALLO

le 21 novembre 2014

Titre :

Métriques prescrites sur le bord du coeur convexe d'une variété anti-de Sitter globalement hyperbolique maximale compacte de dimension trois

---

École doctorale et discipline ou spécialité :

ED MITT : Domaine Mathématiques : Mathématiques fondamentales

Unité de recherche :

UMR 5219

Directeur/trice(s) de Thèse :

Jean-Marc SCHLENKER

Jury :

M. Thierry BARBOT (Rapporteur), Professeur à l'Université d'Avignon et des Pays du Vaucluse
M. Francesco BONSANTE (Examinateur), Ricercatore, Università degli Studi di Pavia
M. Xavier BUFF (Examinateur), Professeur à l'Université Toulouse 3 Paul Sabatier
M. Jean-Pierre OTAL (Examinateur), Professeur à l'Université Toulouse 3 Paul Sabatier
M. Richard A. WENTWORTH (Examinateur), Professor at University of Maryland
M. Jean-Marc SCHLENKER (Directeur de thèse), Professeur à l'Université du Luxembourg
M. Cyril LECUIRE (Invité), Maître de Conférences à l'Université Toulouse 3 Paul Sabatier

Prescribing metrics on the boundary of convex cores of globally hyperbolic maximal compact AdS manifolds

Boubacar DIALLO

20th November 2014

Contents

Remerciements

Cette thèse n'aurait jamais vue le jour sans la patience, la persévérance et les précieux conseils de Jean-Marc Schlenker qui a accepté d'en être le directeur. Je l'en remercie chaleureusement, autant pour ses qualités de soutien moral, d'investissement humain, sa disponibilité à répondre à mes questions sans fin, que pour ses innombrables connaissances en géométrie et les différents géomètres que j'ai pu rencontrer par son intermédiaire. Je le remercie aussi et surtout pour m'avoir permis de venir à Toulouse faire de la recherche.

Je remercie également Thierry Barbot et Francois Labourie d'avoir accepté d'être les rapporteurs de mon travail de thèse, l'éclairage de Thierry a été capital pour améliorer et clarifier mon manuscrit. Je les remercie d'autant plus que Thierry m'a accueilli une année entière à Avignon dans une atmosphère excellente, avec des collègues de qualité, et que Francois m'a suivi durant ma derniére année de Master à Orsay, qu'il m'a donné goût à la géométrie hyperbolique par le biais de son cours spécialisé et du mémoire de recherche qu'il m'a aidé à réaliser sur le livre de Jean-Pierre Otal et pour m'avoir orienté vers Jean-Marc pour mon doctorat. Merci évidemment à Jean-Pierre pour avoir rédigé son livre de la manière la plus accessible possible à un novice et qui m'a ainsi ouvert les portes de la géométrie de dimension trois (ô combien riche et difficile!). Je lui dois indirectement le choix de mon sujet de thèse.

Je remercie les chercheurs qui se sont particulièrement intéressés à mes travaux, outre mon directeur de thèse et les rapporteurs, notamment Francois Béguin, Francesco Bonsante, Jeff Danciger, Francois Fillastre, Charles Frances, Francois Guéritaud, Fanny Kassel, Steve Kerckhoff, Kirill Krasnov, Cyril Lecuire, Grégoire Montcouquiol, Caroline Series, Graham Smith, Tan Ser Peow, Abdelghani Zeghib. J'ai eu des discussions très fructueuses avec chacun d'eux.

Je remercie également les membres de mon jury et tout particulièrement Richard A. Wentworth et Xavier Buff pour leurs présences.

Je remercie tous les collègues (anciens ou présents) de l'IMT, pour leur accueil, en particulier Javier Aramayona, Jérôme Bertrand, Michel Boileau, Arnaud Chéritat, Francois Dahmani, Yohann Genzmer, Vincent Guirardel, Marc Reversat, Pascale Roesch, Greg McShane. Merci à Denis-Charles Cisinski, Guillaume Constant, Anne Cumenge, Philippe Monnier, Tien Zung Nguyen, Julio Rebelo et Jean-Michel Roquejoffre avec qui j'ai partagé des groupes de cours-TD. Merci également aux secrétaires Sylvie Crabos, Martine Labruyère, Jocelyne Picard et Agnès Requis pour leur disponibilité.

Merci aux collègues d'Avignon Marie-Claude Arnaud-Delabrière, Philippe Bolle, Marc Ciligot-Travain, Erwann Delay, Frédéric Naud, Andréa Venturelli pour leur accueil. J'ai particulièrement apprécié mon passage chez eux.

Je remercie également tous les doctorants (anciens ou présents) que j'ai croisés durant mes études, entre autres Medhi Belraouti, Alain Couvreur, Alexandre Dezotti, Tony Ezome, Jonah Gaster, Thomas Gauthier, Anne Granier, Chloé Grégoire, Brice Loustau, Sara Maloni, Florian Marty, Alexandre Paiva, Cécile Poirier, Frédéric Protin, Claire Renard, Clara Rossi, Landry Salle, Jean-Christophe San Saturnino, Dmitry Slutsky, Jérémy Toulisse, James Wallbridge.

Je remercie enfin mes amis personnels Alex, Antoine d'Hossegor, Eddy, Marc, Nico de Rennes, Rachel, Raf, Ray, Sylvain 'Popiet', Thibault, Thierry et les membres de ma famille qui ont été présents malgré les difficultés, mon frère Papus, ma soeur Kadi et son mari Guillaume, mon oncle Ismael et sa femme Monique, ma tante Aïda, ma cousine Céline et mes deux parents.

Je garde une pensée émue pour ma grand-mère paternelle ainsi que pour mon oncle Abdou Doumbia qui auraient été très heureux de savoir que j'ai pu mené mon doctorat à son terme. Je leur dédie ce présent manuscrit.

Résumé de la thèse

Le but de cette thèse est d'apporter une réponse partielle positive à l'une des conjectures de G.Mess, datant des années 90, sur la géométrie du bord du coeur convexe d'une variété anti-de Sitter globalement hyperbolique maximale compacte de dimension 3.

Plus précisément, nous montrons que chaque couple de métriques hyperboliques sur une surface fermée S de genre au moins 2 s'obtient (d'au moins une facon) comme couple de métriques du bord supérieur (respectivement inférieur) du coeur convexe d'une variété anti-de Sitter M globalement hyperbolique maximale compacte admettant une surface de Cauchy homéomorphe S.

Nous relions ce théorème aux divers résultats déjà obtenus dans les contextes hyperbolique et anti-de Sitter, respectivement, en dimension 3 concernant les problèmes de prescriptions de métriques et de laminations de plissage (du bord du coeur convexe). Nous évoquons le problème d'unicité de la prescription, notamment au voisinage du lieu Fuchsien de l'espace des structures hyperbolique et anti-de Sitter, respectivement.

Notre travail nous permet d'aborder diverses questions intéressantes en géométrie: problème des immersions isométriques, études des actions de groupes discrets sur des espaces symétriques lorentziens, géométrie lorentzienne globale en dimension 2+1 et ses applications à la physique.

Notons que la géométrie anti-de Sitter en dimension 3 fournit un cadre idéal à l'étude des tremblements de terre en théorie de Teichmüller hyperbolique, de même que les variétés hyperboliques quasifuchsiennes sont utiles à la compréhension de la version quasiconforme de cette théorie, ainsi qu'à l'étude des structures projectives complexes sur une surface de genre au moins 2. Ainsi notre résultat s'exprime purement en terme de géométrie hyperbolique des surfaces.

Par ailleurs la thèse met en lumière les analogies existantes entre la théorie anti-de Sitter globalement hyperbolique d'une part et la théorie hyperbolique quasifuchsienne d'autre part, tout en développant de nouvelles techniques d'approche de notre question principale, les méthodes connues dans le cadre hyperbolique ne s'appliquant pas du côté lorentzien.

Summary of the thesis

The aim of this thesis is to give a partial positive answer to a conjecture of G.Mess, from the '90s, about the geometry of the boundary of the convex core of a globally hyperbolic maximal compact three dimensional anti-de Sitter manifold.

More precisely, we prove that each pair of hyperbolic metrics on a closed surface S of genus at least 2 can be obtained as the pair of metrics of the upper (resp. lower) boundary component of the convex core of a (possibly non unique) globally hyperbolic maximal compact anti-de Sitter manifold whose Cauchy surfaces are homeomorphic to S.

We relate our theorem to various results already achieved in the hyperbolic and the anti-de Sitter settings, respectively, in dimension 3, regarding the issues of prescribing the metrics and the pleating laminations (of the boundary of the convex core). We tackle the uniqueness issue, mainly near the Fuchsian locus of the space of hyperbolic (resp. anti-de Sitter) structures.

Our work allows us to approach different interesting topics in geometry: isometric immersions, discrete group actions on lorentzian symmetric spaces, global lorentzian geometry in dimension 2+1 with applications to Physics.

Note that anti-de Sitter geometry in dimension 3 is a natural framework to study earthquakes in hyperbolic Teichmuller theory, the same way quasi-fuchsian hyperbolic manifolds help us understand the conformal aspect of the theory as well as complex projective structures on surfaces of genus at least 2. Thus we can formulate our result purely in terms of hyperbolic geometry on surfaces.

Moreover our thesis highlights existing analogies between the theory of anti-de Sitter globally hyperbolic manifolds on the one hand, and on the theory of quasifuchsian hyperbolic manifolds on the other hand, involving new methods for our main achievement, since those already known in the hyperbolic case do not apply to the lorentzian one.

Chapter 1

Motivations

1.1 First mathematical motivations : a tool for Teichmüller theory

In his pioneering work on three dimensional manifolds and hyperbolic geometry [31], Thurston introduced new tools in Teichmüller theory and Kleinian group theory. Some of those tools later proved to be essential for Mess in his study of Lorentzian manifolds of constant curvature in dimension three [24]. Let us review part of their works.

Let S be a closed surface of negative Euler characteristic. It admits metrics of constant curvature -1.

Definition 1.1.1 (Teichmüller space). *The Teichmüller space of S, $\mathcal{T}(S)$, is the space of deformations of hyperbolic metrics on surfaces of negative Euler characteristic up to isotopy (i.e. two hyperbolic metrics define the same point in $\mathcal{T}(S)$ if and only if there exists a diffeomorphism of S, isotopic to the identity, which is an isometry between them).*

It admits a natural (quotient) topology and an action of the modular group of S, i.e. the group of isotopy classes of diffeomorphisms of S.

Thurston introduced the space $\mathcal{ML}(S)$ of measured laminations on S.

Definition 1.1.2 (Measured geodesic lamination). *Let g be a hyperbolic metric on S. A measured geodesic lamination on (S, g) is a disjoint union of simple complete geodesics, together with a (countably additive) measure on each arc transverse to λ. This transverse measure has to be the same for two* *transverse arcs which are images of one another by a homotopy respecting transversality to* λ*.*

If g and g' are two hyperbolic metrics on S, two measured geodesic laminations λ and λ' for g and g' respectively, are said to be equivalent if the total weight is the same for any closed curve on S. This defines a notion of measured lamination on S, defined as an equivalence class of measured geodesic laminations under this identification.

The space of measured laminations on S is denoted by $\mathcal{ML}(S)$. The quotient space defining $\mathcal{ML}(S)$ admits a natural topology (the weak topology). Indeed, measured laminations, can be identified with real-valued functions on the set of isotopy classes of essential simple closed curves.

For any m in $\mathcal{T}(S)$ and any $\overline{\lambda}$ in $\mathcal{ML}(S)$, $\overline{\lambda}$ can be uniquely realized as a measured geodesic lamination, i.e. a closed subset of S which is the union of disjoint complete (infinite or closed) simple geodesics, without self intersection, with a transverse measure (see for instance [7]).

If a is a positive real number and γ a simple closed curve, one can define a measured lamination on S by letting, for each arc c transverse to γ, the lamination $a\gamma$ be a times the infimum of geometric intersection numbers of c with curves γ' freely isotopic to γ. Call such a measured lamination a weighted curve or a rational lamination.

It defines a map from the set $\mathcal{S}$ of isotopy classes of disjoint non trivial weighted simple closed curves to $\mathbb{R}$. $\mathcal{M}L(S)$ is a completion of the space of laminations defined by elements of $\mathcal{S}$ (with the weak topology), which is thus a dense subset of $\mathcal{M}L(S)$.

Proposition 1.1.3. *Let g is the genus of S. Then $\mathcal{M}L(S)$ and $\mathcal{T}(S)$ are topological manifolds of the same dimensions, both homeomorphic to $\mathbb{R}^n$, where $n = 6g - 6$.*

The reader can find a proof and more details in [17].

The quotient space $\mathcal{PML}(S)$ of non-zero measured laminations by the action of $\mathbb{R}_{>0}$ on transverse measures (by multiplication) is also a space of great interest. The following statement is proved in [17].

Proposition 1.1.4 (Compactification of Teichmüller space). *$\mathcal{P}ML(S)$ is a topological sphere of dimension $6g - 7$ which compactifies $\mathcal{T}(S)$ in a such a way that the action of the mapping-class group of S on $\mathcal{T}(S)$ extends continuously to the entire closed ball.*

This compactification has many applications, in particular in the classification of elements of the mapping-class group [17] and the hyperbolization of three manifolds which fiber over the circle [27].

Another main result of Thurston using measured laminations is the earthquake theorem.

Definition 1.1.5 (Left earthquake). *A left earthquake along a rational lamination c (a simple closed curve with the intersection number as transverse measure) is a map from $\mathcal{T}(S)$ to itself. It takes a hyperbolic metric m, cuts the surface S along the geodesic representative γ of c and glues back the two pieces of $S-\gamma$ by shifting the right piece to the left of its original position by an amount equal to the transverse measure of c.*

This gives a well defined continuous map $\mathcal{E}^l_c$ on $\mathcal{T}(S)$, which extends to any measured lamination, see [32]. Thurston proved that earthquakes are homeomorphisms (the inverse of a left earthquake is a right earthquake). He also proved the

Theorem 1.1.6 (Earthquake theorem). *For any two points in $\mathcal{T}(S)$, there is a unique λ in $\mathcal{ML}(S)$ such that the left (resp. the right) earthquake along λ sends one point to the other.*

See [20].

1.2 Further mathematical motivations

In this section we recall some basic facts on three-dimensional geometry, Kleinian groups and anti-de Sitter manifolds.

1.2.1 Complex projective structures, quasifuchsian manifolds and hyperbolic ends

Measured laminations also occur in the study of complex projective structures via pleated surfaces.

Definition 1.2.1 (Pleated surface). *A pleated surface $\overline{S}$ with pleating locus λ in $\mathcal{ML}(S)$ is a hyperbolic surface (with hyperbolic metric m) together with an isometry f from (S, m) to a hyperbolic manifold N which is a totally geodesic immersion outside of the realization of λ as a geodesic measured* *lamination $\overline{\lambda}$, and which sends each leaf of $\overline{\lambda}$ to a geodesic in N, bending the surface 'by an amount equal to the transverse measure of λ'.*

For hyperbolic manifolds N which are called hyperbolic (tame) ends, this allows to define a complex projective structure by considering the projective structure induced on the boundary at infinity of N.

Thurston proved that any complex projective structure on S arises in this way. Endowing the space of complex projective structures with a natural topology, Thurston proved more precisely that

Theorem 1.2.2 (Parametrization of the space of complex projective structures by pleated surfaces). *The map which sends a pair of λ in $\mathcal{ML}(S)$ and m in $\mathcal{T}(S)$ to the projective structure associated to the corresponding pleated surface is a homeomorphism.*

Moreover Thurston related quasiconformal deformation theory as in Ahlfors and Bers theory to quasifuchsian hyperbolic manifolds via the simultaneous uniformization theorem [3].

Definition 1.2.3 (Quasifuchsian hyperbolic manifold). *A manifold M of dimension 3, homeomorphic to $S \times (0,1)$, is called quasifuchsian if it is a complete Riemannian manifold of constant curvature -1 which admits a non-empty compact convex subset.*

Note that here we say that $K \subset M$ is convex if any geodesic segment with endpoints in K is contained in K. With this definition, whenever K is a non-empty convex subset of M, M retracts on K.

The last condition of the definition is equivalent to M being convex cocompact, which in our case is equivalent to the usual definition of quasifuchsian manifolds via limit sets of action of surface groups.

A quasifuchsian structure on M is a quasifuchsian metric up to isotopy.

Theorem 1.2.4 (Simultaneous uniformization theorem [3]). *The map sending a quasifuchsian structure on a manifold to the pair of conformal structures on the two boundary surfaces at infinity (reversing the orientation of lower infinity surface) is a homeomorphism.*

1.2.2 Globally hyperbolic anti-de Sitter structures as analogs of quasifuchsian structures

Mess established a deep analogy in the Lorentzian setting between Thurston's theory of quasifuchsian manifolds and Lorentzian manifolds of constant curvature equal to -1 which satisfy a condition which reminds of their hyperbolic counterpart, namely being globally hyperbolic.

In both hyperbolic and Lorentzian cases, the geometry of the manifolds under consideration are described by external data (via a simultaneous uniformation-like theorem) and by internal data (pleated surfaces). It is a nice and particularly simple instance of the holography principle which, roughly speaking, relates structures of a manifold to structures on its boundary.

Moreover, Mess recovered Thurston's second proof of his earthquake theorem in terms of globally hyperbolic negatively curved Lorentzian manifolds. (See [32] for this second proof without the use of AdS geometry.)

1.2.3 Geometric structures in dimension three

Thurston defined (X, G) structures on manifolds [31], X being a model space (which we may suppose to be simply connected) and G being a Lie group acting analytically on X, i.e only the identity element can fix a non empty connected open subset of X. (X, G) are required to be maximal with respect to these properties.

(X, G) manifolds (with G and X as above) have a well defined holonomy representation, up to conjugation, and a developing map taking values in X (up to equivalence).

Definition 1.2.5 (Geometric structure). *An (X, G) structure on a manifold N is a maximal atlas of charts (U_i, f_i) where U_i is an open subset of N, f_i a homeomorphism from U_i to an open subset V_i of X, the charts satisfying a compatibility relation, namely the transition maps are restriction of elements of G on each connected open subsets of X where they are defined.*

Thurston showed that eight geometries are needed to classify three manifolds [31]. One of Thurston's eight geometries, namely SOL geometry, is modelled on the universal cover $\widetilde{SL_2\mathbb{R}}$ of the Lie group $SL_2\mathbb{R}$ with the action of the group $\widetilde{SL_2\mathbb{R}} \times 1$. It is thus a particular case of $\widetilde{AdS_3}$ geometry.

1.2.4 Relation between surface geometry and three dimensional manifolds

Hyperbolic geometry in dimension three seems to be useful to understand (complex) quasiconformal Teichmüller theory as well as complex projective structures while Lorentzian geometry of curvature -1 (which is called anti-de Sitter geometry) seems to be the perfect place to understand (metric) Teichmüller theory as well as earthquakes on surfaces of negative Euler characteristic, or even minimal lagrangian diffeomorphisms of hyperbolic surfaces (see [11] for definitions and further details).

Both theories allow for geometric descriptions of the cotangent bundle of Teichm̈uller space, either by means of quadratic (Hopf) differentials or infinitesimal earthquake deformations along measured geodesic laminations. See [22] for further details.

Thurston defined the length of an arbitrary measured lamination on a hyperbolic surface as a generalization of the product measure of the arc length by the transverse measure. In case of a weighted curve (a, γ) it is just the product of the weight a by the length of the geodesic in the free homotopy class of γ. Using convexity of length functions along earthquake paths, the Nielsen Realization conjecture was proved by Kerckhoff [20].

Later Bonahon defined transverse Hölder distributions [6] as a tool to study the piecewise linear structure of the space of measured laminations. He obtained a formula for the differential of length function and of other quantities related to earthquakes and bending maps [5].

Pleated surfaces occur both in hyperbolic quasifuchsian manifolds and in globally hyperbolic (maximal) anti-de Sitter manifolds. In particular, in both Riemannian and Lorentzian settings, the corresponding (i.e. either quasifuchsian or globally hyperbolic maximal anti-de Sitter) three manifolds M, homeomorphic to a product $S \times \mathbb{R}$ (where S is a closed surface) contain either a totally geodesic surface homeomorphic to S, in which case they are called Fuchsian, or can be retracted on their minimal non-empty convex subset, their convex core, which turn out to be a topological three manifold whose boundary consists of two pleated surfaces S^+ and S^-.

The restriction to S^+ and S^- of the ambient path metric on M turn out to be a pair of smooth hyperbolic metrics h^+ and h^-, projecting to points m^+ and m^- in $\mathcal{T}(S)$. Furthermore, S^+ and S^-, are pleated surfaces, bent along two laminations λ^+ and λ^-. It can be shown that those two laminations fill up S : any non homotopically trivial closed curve on S has a positive geometric intersection number with either λ^+ and λ^-, and any (isotopy class of) simple closed curve in either of the two laminations has a transverse measure less than or equal to π.

Thurston asked if any pair of conformal structures on S (resp. any two laminations on S which fill up the surface) can be obtained as the boundary conformal structures m^+ and m^- (resp. as the bending laminations λ^+ and λ^-) of a unique quasifuchsian manifold M homeomorphic to $S \times \mathbb{R}$.

As far as prescribing the bending laminations, Bonahon and Otal proved in [9] that any pair of filling laminations on S can be realized as the bending measured laminations of a quasifuchsian manifold structure on M. Moreover, they proved the uniqueness of the quasifuchsian structure for any pair of rational laminations λ^+ and λ^-. Bonahon proved in [8] that any pair of filling laminations on S which are sufficiently close to the zero lamination can be uniquely obtained as the pair of bending laminations of a quasifuchsian structure on M. If one takes as S the (once) punctured torus, Series proved uniqueness of the quasifuchsian structure corresponding to a pair of bending laminations in [25]. No other uniqueness results are known.

As far as the pair of boundary conformal (i.e hyperbolic) structures is concerned, existence of a quasifuchsian structure with given hyperbolic structures on its convex core is known from work of Epstein and Marden [15], and Labourie [23], independently. No uniqueness result is known.

Mess asked the analog questions in the case of globally hyperbolic anti-de Sitter structures, i.e. can one uniquely prescribe either the pair of boundary hyperpolic structures or the pair of bending laminations of the convex core of a globally hyperbolic maximal anti-de Sitter manifold M. Thanks to Bonsante and Schlenker [12], the analog of Bonahon's unique existence result on almost Fuchsian manifolds holds as well as the existence part of Bonahon and Otal's theorem which prescribes the bending laminations.

As far as the prescription of the hyperbolic structures on the boundary of the convex core of a globally hyperbolic maximal anti-de Sitter M is concerned, our present thesis brings a positive answer to the existence part of Mess' question.

1.3 Physical motivations : relativity

1.3.1 Relativity

Constant curvature Lorentzian manifolds satisfying good notions of causality (for instance being globally hyperbolic) provide interesting examples of spacetimes for relativity theory in physics.

Globally hyperbolic manifolds admit several good time functions. In particular the boundary surfaces of the convex core of a globally hyperbolic maximal anti-de Sitter space time are level sets of the cosmological time function (in fact, one surface is the $\pi/2$ level set of the cosmological time and the other is the same level set when time orientation is reversed).

$2+1$ spacetimes are easier to understand and are good toy models in the search for quantization of gravity. Witten showed in [33] that $2+1$ spacetimes can indeed be quantized.

1.3.2 Solutions to Einstein's equation

Globally hyperbolic anti-de Sitter manifolds are solutions of the Einstein equation with negative cosmological constant. They are then a subject of interest, easier to understand in dimension 3 rather than the more realistic dimension 4 since such solutions are necessarily of constant sectional curvature, see [4].

Chapter 2

Mathematical background

2.1 Lorentzian geometry

We recall here some basic notions of Lorentzian geometry which will be useful below.

2.1.1 Anti-de Sitter space

Anti de Sitter space (AdS) is the model space of Lorentzian manifolds of constant sectional curvature equal to -1. Its $n-$dimensional avatar can be defined as the -1 level set

$$AdS_n = \{x \in \mathbb{R}^{n+1} \mid \langle x, x\rangle = -1\}$$

of the standard nondegenerate quadratic form of index 2 in $\mathbb{R}^{n+1}$,

$$\langle x, x\rangle = -x_1^2 - x_2^2 + x_3^2 + \cdots + x_{n+1}^2$$

with the induced bilinear form on each affine tangent subspace (it has index one on such subspaces).

Another model for anti-de Sitter space in dimension 3 is just the lie group $SL_2\mathbb{R}$ with its induced killing metric. This allows to identify the (identity component of the) isometry group of AdS_3 with $(SL_2\mathbb{R} \times SL_2\mathbb{R})/J$, acting by left and right matrix multiplication, where J is the group of order 2 generated by $(-I_2, -I_2)$.

An anti de Sitter 3-manifold is thus a (smooth, connected) manifold M endowed with a symmetric bilinear covariant 2-tensor of index one, everywhere nondegenerate, whose sectional curvatures are all equal to -1. By classical results (see [4]), such an M is locally isometric to the model space AdS_3, anti de Sitter space of dimension three. If we restrict ourselves to oriented and time oriented manifolds, it is therefore endowed with a (X, G) structure, where X is AdS_3 and G the identity component of its isometry group. Both definition are equivalent in that case. Let's call such objects AdS_3 spacetimes.

2.1.2 Causality relations

An AdS_3 spacetime is said to be globally hyperbolic if it admits a Cauchy hypersurface: a spacelike surface which intersects every inextendable timelike curve exactly once. If the spacetime has a compact Cauchy surface, then every Cauchy surface is compact, see e.g. [26]. Moreover if the spacetime cannot be isometrically embedded in a stricty larger spacetime by an isometry sending a Cauchy surface to another one, then it is called globally hyperbolic maximal compact (GHMC).

2.1.3 Constraint equations

A GHMC anti-de Sitter spacetime is well defined if we know a Cauchy surface with enough data to allow us to reconstruct the whole manifold. Typically, those data are the induced metric and the second fundamental form of the Cauchy surface, but they must satisfy some relations. Those are called the constraint equations in physics. They correspond to the Gauss and Codazzi equations relating the first and second fundamental forms (or equivalently the first fundamental form and the shape operator of the isometric immersion) of a hypersurface in an Einstein (pseudo-)Riemannian manifold. See [4].

2.1.4 Cauchy development

Given S and two tensors I and II of the right covariance (with I being a metric tensor) satisfying the constraint equations, the Cauchy development of S equipped with this data is just the maximal globally hyperbolic spacetime which admits S as a Cauchy surface with I as induced metric and II as second fundamental form.

This space-time exists and is unique if I and II satisfy the constraint equations, see [14]. In dimension 3, this basically reduces to the fundamental theorem of surface theory — any couple (I, II) satisfying the Gauss and

Codazzi equations can be uniquely realized as the induced metric and second fundamental form of an equivariant surface in AdS_3 — but in higher dimension one needs to solve a non-linear hyperbolic PDE.

2.2 Hyperbolic surfaces and holonomy representations

A (closed) hyperbolic surface can be defined either by a Riemannian metric of constant sectional curvature -1 or (thanks to the uniformization theorem) by a quotient of hyperbolic plane by a discrete group of isometries, isomorphic to the fundamental group of the initial surface, acting properly discontinuously on hyperbolic plane. This allows for a description of Teichmüller space in terms of Fuchsian representations of the fundamental group of S into $PSL(2, \mathbb{R})$, up to conjugations. Indeed the space of holonomies of the induced (X, G)-structure, where X is hyperbolic space and G its group of orientation preserving isometries, is just Teichmüller space.

2.2.1 Riemann surfaces and hyperbolic surfaces

Suppose that S is a closed oriented surface of genus at least 2.

Then choosing a complex structure on S is equivalent to choosing a conformal class of metrics on S, because, on a surface, any Riemannian metric locally admits isothermic coordinates.

Moreover, conformal classes of metrics on S are in one-to-one correspondence with the Teichmüller space of S, because of the following essential result.

Theorem 2.2.1 (Riemann Uniformization Theorem). *Any conformal class on S contains a unique hyperbolic metric.*

2.2.2 Holonomy representations

Recall that we defined the Teichmüller space of a surface S, as a space of (equivalence class of) metrics of constant curvature (equal to -1) on S. Each such (equivalence class of) metrics gives rise to a unique (equivalence class of) (X, G) structure on S, where X is the hyperbolic plane and G the group of its orientation preserving isometries. For such a structure, one can associate an equivalence class of pairs (dev, hol), where dev is called a developing map and hol a holonomy representation, for the equivalence relation identifying both isotopic developing maps and holonomy representations in the same conjugacy class by elements of G.

Thanks to the holonomy map, we get an identification of Teichmüller space with the space of discrete, faithful and cocompact representations of the fundamental group of S (modulo conjugation), since holonomy representations coming from constant curvature metrics are of this type (we call them Fuchsian) and conversely any (class of) Fuchsian representation determines a unique (class of) (X, G) endowing S with a (class of) hyperbolic metrics.

More generally, any representation of the fundamental group of S to $PSL_2\mathbb{R}$ has a well defined Euler number, which is an integer. A theorem of Goldman [18] asserts that the absolute value of this Euler number is bounded by the absolute value of S's Euler characteristic, and equality between the two quantities occur if and only if the representation is Fuchsian (possibly up to conjugation by an orientation-reversing isometry of hyperbolic plane).

Recall that the identity component of the isometry group of AdS_3 identifies (up to index 2) with $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$. Thus holonomies of (X, G) spacetimes M (with X being anti-de Sitter space and G the identity component of its isometry group) project on the set of ordered pairs of representations from the fundamental group of M to $PSL_2(\mathbb{R})$. In the case where M is globally hyperbolic maximal with a surface S (of negative Euler characteristic) as a Cauchy surface, one of Mess's main theorem ([24, proposition 19], see also [1]) asserts that such a pair (ρ_l, ρ_r) is a pair of Fuchsian representations. Mess proved this theorem by showing that such left and right representations have maximal Euler number, and then using Goldman's theorem he obtained the desired result.

Conversely Mess proved that to such a pair of Fuchsian representations is the holonomy pair of a unique GHMC AdS_3 manifold M homeomorphic to $S \times \mathbb{R}$ (in particular, $\pi_1(x, M)$ and $\pi_1(x, S)$, with suitable choice of basepoint x, are isomorphic). This is the exact analog of Bers theorem for quasifuchsian hyperbolic 3-manifold. This gives a natural homeomorphism between the space of GHMC AdS_3 structures (with orientation and time orientation) and $\mathcal{T}(S) \times \mathcal{T}(S)$.

2.3 Teichmüller space and 3-dimensional manifolds

2.3.1 Quasifuchsian hyperbolic manifolds

Recall that we defined quasifuchsian hyperbolic manifolds above. Since such manifolds have constant curvature, they can be equivalently defined by a (X, G) structure on $S \times \mathbb{R}$, where X is hyperbolic three-space and G its group of orientation preserving isometries, satisfying some conditions about the limit set.

The limit set of a hyperbolic manifold is the set of accumulation points, on the boundary sphere at infinity, of orbits, under the action of the holonomy representation, of points hyperbolic three-space. It is a closed subset of the sphere at infinity, and a three-manifold with a (X, G) structure (where X is hyperbolic three-space and G its group of orientation preserving isometries) is quasifuchsian if and only if its limit set is a topological circle, and each connected component of the complement of this circle in the sphere at infinity is preserved by the action of the fundamental group.

The domain of discontinuity is the complement of the limit set in the sphere at infinity.

It is a classical fact that the action of the holonomy representation on hyperbolic space extends to a free and properly discontinuous action on the domain of discontinuity by Möbius transformations. The quotient space thus obtained is a pair of surfaces each equipped with a complex structure. They compactify the initial quasifuchsian manifold and are called the Riemann surfaces at infinity.

Bers' simultaneous uniformization theorem asserts that for each pair (h_∞, h'_∞) of conformal structures on S, there is a unique quasifuchsian manifold, homeomorphic to $S \times \mathbb{R}$ whose Riemann surfaces at infinity are S equipped with h_∞ and S equipped with h'_∞ and the reverse orientation.

Note that Bers' theorem and Mess' theorem on holonomies of anti-de Sitter spacetimes are quite similar.

Both in the lorentzian and quasifuchsian settings, one can define limit sets. Taking the quotient by holonomy representations of the convex hull of the limit set (respectively in anti-de Sitter space or hyperbolic space) gives the convex core of the (respectively GHMC or quasifuchsian) manifold.

Except for the degenerate case where the convex core is a single totally geodesic surface, the boundary of the convex core consists of two surfaces, whose induced path metric is hyperbolic, but which are bent along geodesic laminations.

2.3.2 Laminations and pleated surfaces

Recall that we have defined the convex core of a GHMC manifold, which is the analog of the definition of the convex core of a quasifuchsian manifold.

Mess related the boundary data of the convex core of a GHMC manifold to the left and right holonomy pair by means of earthquakes.

Let S be a closed surface of negative Euler characteristic, M a globally hyperbolic maximal AdS_3 spacetime with S as a Cauchy surface, (ρ_l,ρ_r) the holonomy of M. Let λ_+, λ_- the upper and lower pleating laminations of the convex core of M, m_+ and m_- the corresponding boundary hyperbolic structures.

Then

$$\begin{aligned}\rho_l &= E^l_{\lambda_+/2}(m_+)\\ \rho_r &= E^r_{\lambda_+/2}(m_+)\\ \rho_l &= E^r_{\lambda_-/2}(m_-)\\ \rho_r &= E^l_{\lambda_-/2}(m_-)\ .\end{aligned}$$

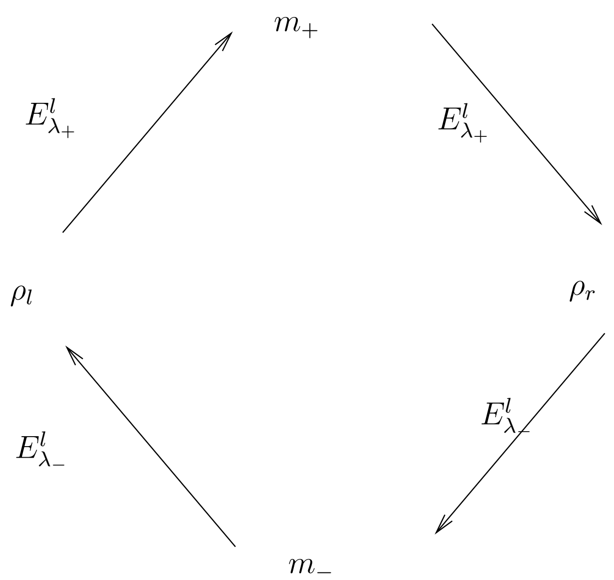

Figure 2.1: Mess diagram

Chapter 3

Main question and tools involved

In the remaining chapters of this thesis, S is a closed surface of negative Euler characteristic and $\mathbb{H}^2$ is the hyperbolic plane.

3.1 Analogies with quasifuchsian hyperbolic three-manifolds

Mess noted that his theorem on holonomies of $GHMC$ anti de Sitter spacetimes of dimension 3 is the analog of the simultaneous uniformization theorem of Bers for quasifuchsian hyperbolic three manifolds [3]. Thurston asked whether one could uniquely prescribe the two hyperbolic metrics on the boundary of the convex core of such a manifold [31], [13]. So far, only existence has been proved, thanks to works of Epstein and Marden [15] on Thurston and Sullivan's $K = 2$ conjecture (which happens to be false, see [16]), and Labourie [23], independently.

In his work [24], Mess established further analogies between quasifuchsian hyperbolic 3-manifolds and $GHMC$ AdS_3 spacetimes. Indeed such spacetimes M have a well defined convex core, which as in the hyperbolic setting is the minimal non-empty closed convex subset. Except for the Fuchsian case where both upper and lower boundary metrics are equal, the convex core has two boundary components which are pleated hyperbolic surfaces. Both are thus equipped with hyperbolic metrics and measured bending laminations. In particular this defines a map Φ from the space of $GHMC$ AdS

structures (with Cauchy surface S of fixed topological type), identified with $\mathcal{T}(S)\times\mathcal{T}(S)$, to $\mathcal{T}(S)\times\mathcal{T}(S)$, which sends a structure to the ordered pair of upper and lower boundary hyperbolic metrics. In the quasifuchsian setting, the analogous map is then onto.

3.2 The Mess conjecture

Mess asked whether the map Φ is one-to-one and onto, that is, whether an ordered pair of hyperbolic metrics on S can be uniquely realized as the upper and lower boundary metrics of a $GHMC$ AdS_3 spacetime M. This is the analog of Thurston's conjecture for quasifuchsian manifolds. Uniqueness is still an open question. The present thesis gives a positive answer to the existence part of this conjecture of Mess.

Main Theorem 3.2.1 (Prescribing the boundary metrics of the convex core of a 3-dimensional adS GHMC spacetime). *For any points m_1 and m_2 in $\mathcal{T}(S)$, there exists a globally hyperbolic maximal compact adS spacetime M such that m_1 and m_2 are the induced metrics on the upper (resp. lower) boundary component of the convex core of M.*

That is to say: Φ is a surjective map.

Note that our statement, in the anti de Sitter setting, cannot be proved by methods of Epstein and Marden. Indeed, the K-quasiconformal constant in their theorem cannot exist in our context, because it would contradict the earthquake theorem (see next section). There's no restriction on the bending measures of our spacetimes, as opposed to the hyperbolic setting. Moreover, the analog of Labourie's theorem [23, théorème 1] or of Schlenker's theorem [29] remains unknown in the AdS setting.

3.3 Relation to Teichmüller theory and earthquakes

Recall that earthquakes were defined by Thurston in [32] by extension to measured laminations of the case of simple closed curve. Let E^l_λ and E^r_λ be the left and right earthquakes along a measured lamination λ. Thurston proved that those two maps from $\mathcal{T}(S)$ to itself are (continuous and) bijective and in fact are inverse to each other. His earthquake theorem asserts that for any m and m' in $\mathcal{T}(S)$, there is a unique measured lamination λ (resp. λ') such that the left (resp. right) earthquake along λ (resp. λ') sends m to m'.

Recall also that Mess rephrased this earthquake theorem in the context of pleated surfaces in AdS_3 geometry.

Let S be a closed surface of negative Euler characteristic, M a globally hyperbolic maximal adS_3 spacetime with S as a Cauchy surface, (ρ_l,ρ_r) the holonomy of M. Let λ_+, λ_- the upper and lower pleating laminations of the convex core of M, m_+ and m_- the corresponding boundary hyperbolic structures.

Then

$$\begin{aligned}\rho_l &= E^l_{\lambda_+/2}(m_+)\\ \rho_r &= E^r_{\lambda_+/2}(m_+)\\ \rho_l &= E^r_{\lambda_-/2}(m_-)\\ \rho_r &= E^l_{\lambda_-/2}(m_-)\ .\end{aligned}$$

Thanks to Thurston's theorem, our map Φ of the previous section is continuous. Via the earthquake theorem, surjectivity of Φ is thus equivalent to the following statement:

Main Theorem 3.3.1 (Prescribing middle points of 2 intersecting earthquakes paths). *For any two points m_+ and m_- in $\mathcal{T}(S)$, there are left and right earthquakes, joining two points say ρ_l and ρ_r in $\mathcal{T}(S)$, such that m_+ and m_- are the middle points of the corresponding earthquake paths.*

Again we fail to prove uniqueness.

3.4 How to show that Φ is onto

Recall that Φ is the map from $\mathcal{T}(S) \times \mathcal{T}(S)$ to itself which associates to a GHMC structure on a manifold M with $\pi_1(M) \cong \pi_1(S)$ the ordered pair of upper and lower boundary metrics on its convex core. Thanks to the earthquake theorem and to the relations described in Mess diagram (Figure 2.3.2), Φ is continuous. We just need to show that it is a proper map, and since it will have a well defined degree, that it is a map of degree one. It then follows that Φ is onto.

Theorem 3.4.1 (Properness theorem). *With notations as in the previous sections, Φ is a proper map.*

Equivalently, the preimage by Φ of every compact subset of $\mathcal{T}(S) \times \mathcal{T}(S)$ is a compact set.

We may rephrase it with sequences in $\mathcal{T}(S)$ instead of compact sets:

Theorem 3.4.2 (Properness theorem via sequences). *Let $(m_+^n)_{n\geq 0}$, $(m_-^n)_{n\geq 0}$, $(\rho_l^n)_{n\geq 0}$ and $(\rho_r^n)_{n\geq 0}$ be sequences of points in $\mathcal{T}(S)$, linked by the diagram of Mess as in Figure 2.3.2, such that $(\rho_r^n)_{n\geq 0}$ tends to ∞. If $(m_+^n)_{n\geq 0}$ converges to some m_+^∞ in $\mathcal{T}(S)$, then $(m_-^n)_{n\geq 0}$ tends to ∞.*

In our context, we can (equivalently) replace "'$(\rho_r^n)_{n\geq 0}$ tends to ∞" by "$l_{m_+^n}(\lambda_+^n)$ tends to ∞". See next section for further details.

Theorem 3.4.3 (Degree theorem). *Φ is properly homotopic to a homeomorphism.*

Since a homeomorphism has degree one, it then follows that:

Corollary 3.4.4. *The map Φ has degree one.*

In particular, Φ is a surjective map.

Chapter 4

Proof of properness theorem

4.1 Outline of the proof of the properness theorem

Our main result here is Theorem 3.4.2 (properness of Φ).

We prove this theorem with three propositions, which will be proven in the next section.

Suppose first that the support of λ_+ is a simple closed curve c. The first proposition says that for points x of c (on $\partial_+ C$), where there is a large amount of pleating on $\partial_+ C$ on both sides of x, the distance to the opposite boundary component of the convex core is near $\pi/2$. Cutting c by an orthogonal (totally geodesic) plane at such points, the assertion is about 2-dimensional AdS geometry.

In the statement of our proposition, we let C be a convex subset of AdS_2, with spacelike boundary, whose closure in $AdS_2 \cup \partial_\infty AdS_2$ intersects $\partial_\infty AdS_2$ in 2 points, one on each component of $\partial_\infty AdS_2$. Let x be a point of the upper boundary component $\partial_+ C$ of C, Π_x a support line of C at x, $P_{l,x}$ (resp. $P_{r,x}$) a support line of C at the point of $\partial_+ C$ at distance η_0 from the right (resp. the left) of x, for some η_0 which will be defined in the statement. We denote by $\phi_{r,x}$ (resp. $\phi_{l,x}$) the angle between Π_x and $P_{r,x}$ (resp. $P_{l,x}$). Let τ be a time-like geodesic orthogonal at x to Π_x. See Figure 4.1.

Proposition 4.1.1. *For all $\epsilon > 0$, there exist $A > 0$ and $\eta_0 > 0$ such that, for all x in $\partial_+ C$, if $\partial_+ C$ admits support lines $P_{l,x}$ (resp. $P_{r,x}$) at points located at distances less than η_0 from x on $\partial_+ C$, making angles $\phi_{l,x}$ (resp. $\phi_{r,x}$) with*

Π_x *such that*

$$\inf(\phi_{r,x}, \phi_{l,x}) \geq A$$

then

$$l(\tau \cap C) \geq \pi/2 - \epsilon.$$

Note that it is proved in [12] that we always have $l(\tau \cap C) < \pi/2$.

The next one is a slight refinement of Lemma 4.7 of [12].

Let S be a closed surface of negative Euler characteristic, $\alpha_0 > 0$. Let $g \in \mathcal{T}(S)$ and let c a closed geodesic for g. For any point $x \in c$, let $g_x^r(\alpha_0)$ (resp. $g_x^l(\alpha_0)$) be the geodesic segment of length α_0 (for the metric g), orthogonal at x to the right (resp. the left) of c. Let $n_l(x)$ (resp. $n_r(x)$) be the intersection number of $g_x^r(\alpha_0)$ (resp. $g_x^l(\alpha_0)$) with c, including x.

Proposition 4.1.2. *For all $\alpha_0 > 0$ and for all $\delta_0 > 0$, there exists some $\beta_0 > 0$ (depending on α_0, δ_0 and the genus of S) such that:*

$$l_g(\{x \in c : \inf(n_l(x), n_r(x)) \leq \beta_0 l_g(c)\}) \leq \delta_0 l_g(c).$$

What the proposition says is: if the length of c is large enough, for most points x of c, the left (resp. right) going arc orthogonal to c at x, of fixed length α_0 intersects c a lot.

By density of weighted curves in $\mathcal{ML}(S)$ and continuity of length functions [21] and intersection numbers, we get a similar version for more general measured laminations on S, namely:

Corollary 4.1.3. *With the same notations as above, let $\lambda \in \mathcal{ML}(S)$ be a measured lamination. For all $\alpha_0 > 0$ and for all $\delta_0 > 0$, there exists some $\beta_0 > 0$ (depending on α_0, δ_0 and the genus of S) such that:*

$$l_g(\{x \in \lambda : \inf(i(g_x^l(\alpha_0), \lambda), i(g_x^r(\alpha_0), \lambda)) \leq \beta_0 l_g(\lambda)\}) \leq \delta_0 l_g(\lambda).$$

Recall that $l_g(\lambda) = w l_g(c)$ if λ has weight w and support the simple closed curve c(see [21]).

The next and last proposition applied in our situation enables us to compare the length of λ_+ for m_+ and m_-, (see Figure 2.3.2).

Let m_+,m_-, λ_+,λ_- be as in Theorem 3.4.1. They are thus linked by Mess diagram.

Proposition 4.1.4. *Let K be a compact subset of $\mathcal{T}(S)$. For all $\epsilon > 0$, there exists some $A > 0$ such that if m_+, m_- (resp. λ_+, λ_-) are points in $\mathcal{T}(S)$ (resp. $\mathcal{ML}(S)$) linked by Mess diagram as in Figure 2.3.2, such that $m_+ \in K$ and $l_{m_+}(\lambda_+) \geq A$, then $l_{m_-}(\lambda_+) \leq \epsilon l_{m_+}(\lambda_+)$.*

This last proposition extends to arbitrary λ_+ by density of weighted multicurves is $\mathcal{ML}(S)$, by continuity of length functions and continuity of m_- with respect to m_+ and λ_+.
The three propositions are essential tools to prove Theorem 3.4.1. In fact, Theorem 3.4.1 is a consequence of Proposition 4.1.4, which is a result of both Propositions 4.1.3 and 4.1.1.

Suppose there exist sequences $(m_+^n)_{n\geq 0}$, $(m_-^n)_{n\geq 0}$, $(\rho_l^n)_{n\geq 0}$ and $(\rho_r^n)_{n\geq 0}$ in $\mathcal{T}(S)$, linked by Mess diagram, such that m_+^n converges to $m_+^\infty \in \mathcal{T}(S)$. Suppose also that ρ_l^n leaves every compact set in $\mathcal{T}(S)$ (i.e. tends to ∞). Then, by Mess diagram again, the corresponding sequence of upper boundary's pleating lamination λ_+^n leaves any compact set in $\mathcal{ML}(S)$. In particular, its length, measured with respect to m_+^n, tends to infinity.

Let $\epsilon > 0$ and A be as in Proposition 4.1.4. There exists n_0 (depending on ϵ and A) such that for $n \geq n_0$, one has $l_{m_+^n}(\lambda_+^n) \geq A$, hence $l_{m_-^n}(\lambda_+^n) \leq \epsilon l_{m_+^n}(\lambda_+^n)$. Since m_+^n converges in $\mathcal{T}(S)$, this implies that m_-^n leaves every compact subset of $\mathcal{T}(S)$ (i.e tends to infinity), proving the properness of Φ.

4.2 Proof of the main propositions

We begin with the proof of Proposition 4.1.1, we first need a lemma comparing the lengths of two past (resp. future) convex spacelike arcs, with the same endpoints, one being in the past (resp. future) of the other. We define a future (resp. past) convex spacelike curve to be a spacelike curve σ in AdS_2 whose future, i.e the set of endpoints of timelike curves whose starting points lie on σ, (resp. whose past) is a convex subset of AdS_2.

Lemma 4.2.1 (Lengths of convex arcs with fixed endpoints). *Let σ_0 and σ_1 be two future (resp. past) convex spacelike curves with the same endpoints in AdS_2. Suppose that σ_0 lies in the future (resp. past) of σ_1. Then*

$$l(\sigma_0) \geq l(\sigma_1).$$

Proof of the lemma. We first examine the case where σ_0 and σ_1 are piecewise geodesic arcs.

First, suppose σ_0 is a geodesic. Then by induction on the number of geodesic arcs in σ_1, this case is a consequence of Sublemma 4.6 of [12] (the "reverse triangle inequality").

Now suppose σ_0 is a piecewise geodesic arc. Consider the geodesic arc σ_0' joining the endpoints of σ_0. By orthogonal projection of each vertex of σ_1 on

σ'_0, we get timelike arcs joining those vertices to points of σ'_0 and orthogonal to σ'_0. Now we can pull back σ_1 along those arcs (with the normal exponential map) until it reaches (at least) one point of σ_0. We get a piecewise geodesic arc σ'_1, which is longer than σ_1, spacelike and future convex but this time σ'_1 and σ_0 have a common point. We can then apply the induction argument on both sides of this common point on σ_0 and σ'_1.

Then the general case follows by approximation. □

We keep the same notations as in the statement of Proposition 4.1.1 .

Proof of Proposition 4.1.1. Letting x_l (resp. x_r) be the intersection point of Π_x and $P_{l,x}$ (resp. $P_{r,x}$), we note that the (spacelike) distance between x and x_l (resp. x and x_r) is less than α_0, by Lemma 4.2.1.

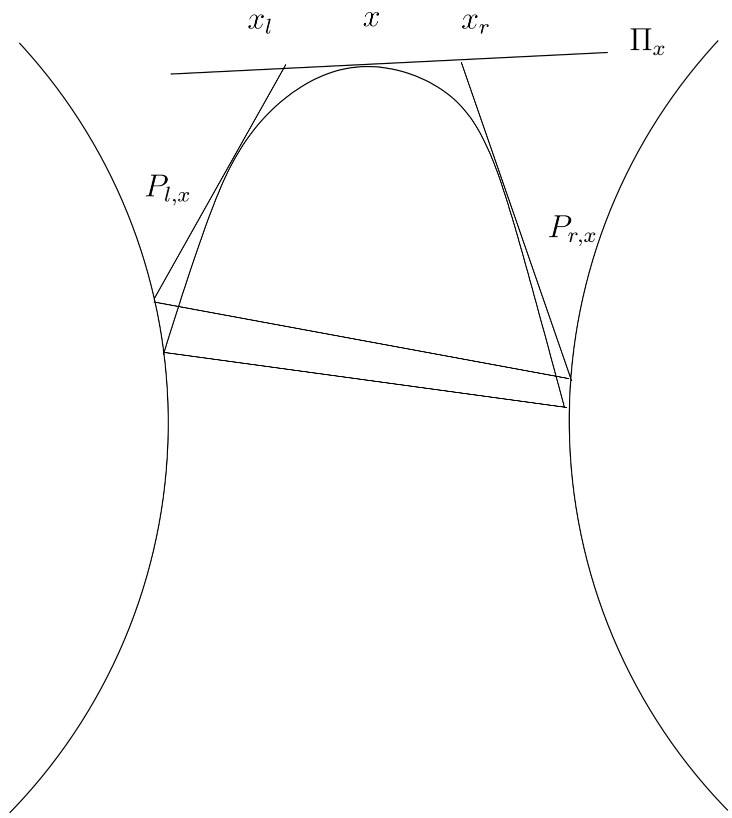

Figure 4.1: Cutting the convex core by a plane orthogonal to a support line

The configuration made by the three spacelike lines and the spacelike line $\Pi^{'}$ joining the two points at infinity of $P_{l,x}$ and $P_{r,x}$ in the past of Π_x admits a well-known limit situation: when α_0 goes to zero, x_l and x_r tend to x, moreover when $\phi_{l,x}$ and $\phi_{r,x}$ tend to infinity, $P_{l,x}$ and $P_{r,x}$ tend to lightlike lines so as to let $\Pi^{'}$ tend to the dual line Π^{*} to x, at (timelike) distance $\pi/2$ from x.

The result follows since by convexity,

$$l(\tau \cap C) \geq l(I^{+}(\Pi^{'}) \cap \tau).$$

□

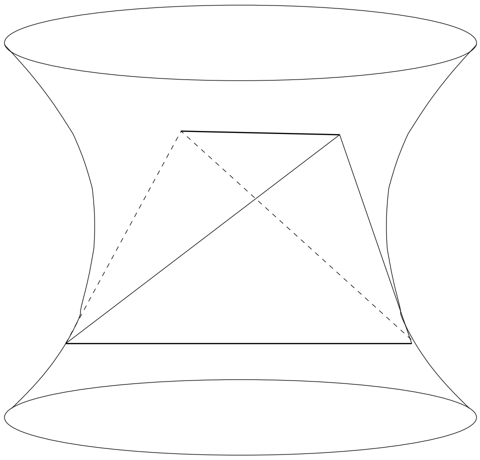

Figure 4.2: Dual lines

Note that in our case, since α_0 tends to 0, having our intersection numbers $n_l(x)$ and $n_r(x)$ tend to ∞ is equivalent to having the angles $\phi_{l,x}$ and $\phi_{r,x}$ tend to ∞ (since the left and right geodesic segments of length α_0 at x meet c with an angle near from $\pi/2$).

We now prove Proposition 4.1.2, we need the following lemma.

Let $g \in \mathcal{T}(S)$ and let $\alpha_0 > 0$. Let c a closed geodesic for g. For any point $x \in c$, let $g_x^r(\alpha_0)$ (resp. $g_x^l(\alpha_0)$) be the geodesic segment of length α_0 (for the metric g), orthogonal at x to the right (resp. the left) of c. Let $n_l(x)$ (resp. $n_r(x)$) be the intersection number of $g_x^r(\alpha_0)$ (resp. $g_x^l(\alpha_0)$) with c, including x.

Lemma 4.2.2. *For all $\alpha_0 > 0$ and for all $\delta_0 > 0$, there exists some $\beta_0 > 0$ (depending on α_0, δ_0 and the genus of S) such that:*

$$l_g(\{x \in c : n_l(x) \leq \beta_0 l_g(c)\}) \leq \delta_0 l_g(c).$$

The similar statement is true for $n_r(x)$ by the same argument. The proposition then follows by combining both statements.

Sublemma 4.2.3. *There exists a positive real number γ_0 (depending on α_0) as follows. Let D_0, D_1 be two disjoint lines in the hyperbolic plane and let x be a point in the connected component of the complement in H^2 of $D_0 \cup D_1$ whose boundary contains those two lines. Suppose that $d(x, D_0) \leq \gamma_0$ and $d(x, D_1) \leq \gamma_0$. Then the geodesic segment of length α_0 starting orthogonally from D_0 and containing x intersects D_1.*

Proof of the sublemma. Let c_0 and c_1 be the geodesic segments joining x to its orthogonal projections on D_0 and D_1, respectively. Then $d(x, D_0) = l(c_0)$ and $d(x, D_1) = l(c_1)$. Let D be a hyperbolic line containing x and disjoint from D_0 and D_1, θ_0 and θ_1 be the respective angles between D and c_0, D and c_1. Then

$$\cosh(l(c_0)) \sin(\theta_0) = \cosh(d(D, D_0))$$

and

$$\cosh(l(c_1)) \sin(\theta_1) = \cosh(d(D, D_1))$$

(See [28, page 88]). Since $d(D, D_0) \leq l(c_0) \leq \gamma_0$ and $d(D, D_1) \leq l(c_1) \leq \gamma_0$, the four quantities $\cosh(l(c_0))$, $\cosh(d(D, D_0))$, $\cosh(l(c_1))$ and $\cosh(d(D, D_1))$ tend to one when γ_0 tends to zero, so both $\sin(\theta_0)$ and $\sin(\theta_1)$ tend to one as well, hence

$$|\pi/2 - \theta_0| \leq \delta_0, |\pi/2 - \theta_1| \leq \delta_0$$

for sufficiently small γ_0 (depending on δ_0).

Let now c_1' be the half-line extending c_1 on the other side of x. The angle between c_0 and c_1' is less than $2\delta_0$. If γ_0 is small enough, then δ_0 is small and c_1' intersects D_0 (which is orthogonal to c_0) at distance less than α_0. This is the required statement. □

Proof of the lemma. Our proof is mostly the same as in [12, Appendix A] Let $\delta_0 > 0$, γ_0 as in Sublemma 4.2.3, and $0 < \beta_0 < \delta_0\gamma_0/(2\pi|\chi(S)|)$. Let

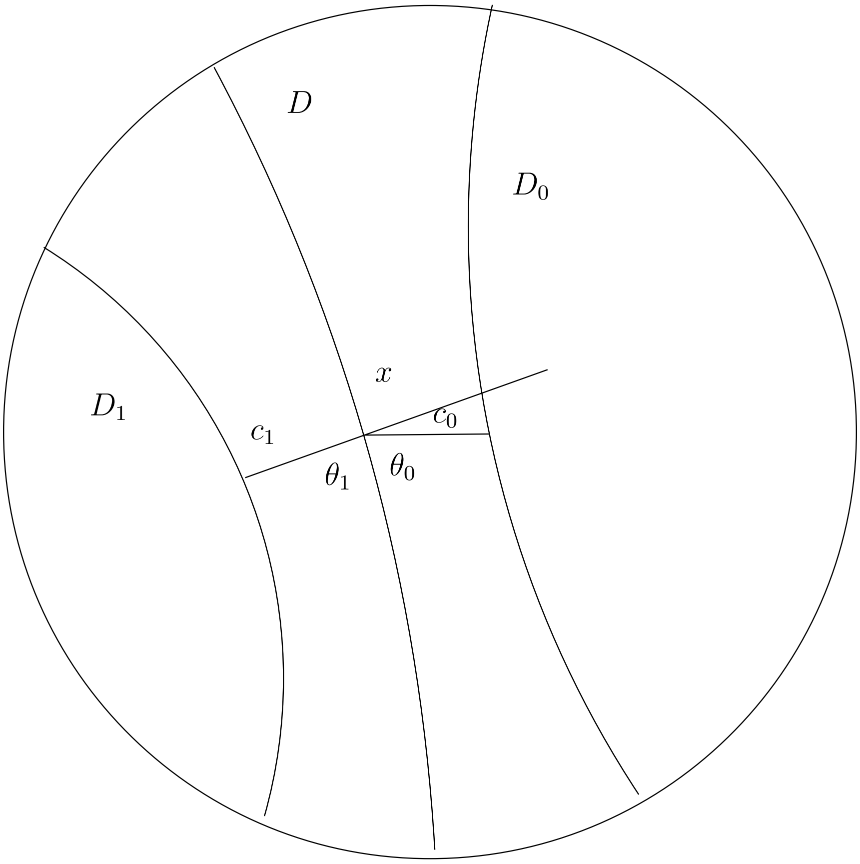

Figure 4.3: Intersections of plane hyperbolic geodesics

$\overline{c} = \{x \in c : n_l(x) \leq \beta_0 l_g(c)\}$. Fix γ_0 as in sublemma 4.2.3 and consider the normal exponential map:

$$exp : \begin{array}{l} \overline{c} \times [0, \gamma_0] \to S \\ (s, r) \mapsto g_s^l(r) \end{array}$$

It is a distance increasing map so it increases areas. Moreover, the sublemma shows that each $x \in S$ has at most n_0 preimages in $\overline{c} \times [0, \gamma_0]$, where n_0 is the integer part of $\beta_0 l_g(c)$. Indeed, suppose that x is the image of (y_1, r_1), ..., (y_k, r_k). Let $\overline{x}$, $\overline{y_1}$, ... , $\overline{y_k}$ be lifts of x, y_1, ..., y_k to the universal cover H^2 of (S, g) chosen so that, for all $j \leq k$, the segment $\exp(\{\overline{y_j}\} \times [0, r])$, which is a lift of $\exp(\{y_j\} \times [0, r'])$, contains both $\overline{x}$ and $\overline{y_j}$. Finally let D_j be the lift of c containing y_j.

The D_j are mutually disjoint lines: indeed since the segment $[\overline{x}, \overline{y_j}]$ is orthogonal to D_j for each j, those lines cannot coincide.

After possibly changing the indices, we may suppose there are half lines P_j, $j = 1, 2$, bounded by D_j, $j = 1, 2$, respectively, that do not meet any other D_j, $j > 2$. We can in particular suppose P_1 does not contain $\overline{x}$ (after possibly exchanging D_1 and D_2). For $j \geq 2$, either D_j disconnects D_1 from $\overline{x}$ or $\overline{x}$ is contained in the region bounded by D_1 and D_j. In the latter case, the sublemma can be applied since from $\overline{x}$ to D_1 and D_j is less than γ_0. In both cases, the segment of length α_0 starting from $\overline{y_1}$ and passing through $\overline{x}$, which lifts $g^r_x(\alpha_0)$, meets D_j. Since $g^r_x(\alpha_0)$ meets c at most n_0 times (including y_1), we conclude that $k \leq n_0$.

Since the area of (S, g) is $2\pi|\chi(S)|$, it follows that

$$\gamma_0 l_g(\overline{c}) \leq 2\pi(n_0)|\chi(S)| \leq 2\pi\beta_0 l_g(c)|\chi(S)| \leq \gamma_0\delta_0 l_g(c).$$

□

It remains to prove Proposition 4.1.4. We keep the same notations as in the statement of this proposition.

Proof of Proposition 4.1.4. Let $\epsilon > 0$, let K be a compact subset of $\mathcal{T}(S)$. We choose $\epsilon' > 0$ (say $\epsilon' < \pi/2$), such that

$$\cos(\pi/2 - \epsilon') \leq \epsilon/2.$$

Proposition 4.1.1 gives the existence of constants A' and η_0 such that if

$$\inf(\phi_{r,x}, \phi_{l,x}) \geq A'$$

then

$$l(\tau \cap C) \geq \pi/2 - \epsilon'$$

(with the notations of Proposition 4.1.1). We choose $\alpha_0 \leq \inf(\eta_0, \epsilon/2)$. Let $\delta_0 = \epsilon/2$ and let β_0 be the constant given by Proposition 4.1.3. We choose $A \geq A'/\beta_0$.

Suppose that the support of λ_+ is a closed curve c. We want to prove that if $l_{m_+}(c) \geq A$, then $l_{m_-}(c) \leq \epsilon l_{m_+}(c)$. Note that by compactness of K, all our constants may be chosen uniformly in m_+, we can then pass to the limit for a general lamination λ_+.

Let $A(c)$ be a totally geodesic timelike annulus orthogonal to a support plane of C along c. Let $c_- = A(c) \cap \partial_- C$ and let c'_- be the curve at distance

$\pi/2-\epsilon'$ in the past of c (lying in $A(c)$), whose length we want to bound from above. Then c'_- is a future convex curve and

$$l(c'_-) = \cos(\pi/2 - \epsilon') l_{m_+}(c),$$

hence

$$l(c'_-) \leq (\epsilon/2) l_{m_+}(c).$$

Consider now the orthogonal projection Pr_- from c_- to c'_-. There are two types of points on c_-, those at distance greater than or equal to $\pi/2-\epsilon'$ from c and the others.

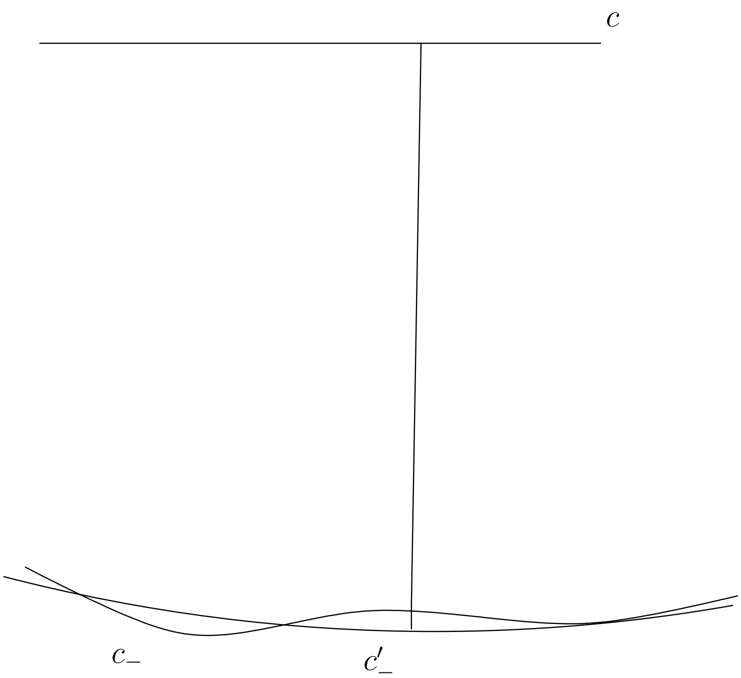

Figure 4.4: Comparing length on top and bottom of the convex core

Let

$$B_- = \{\tau_x \cap \partial_- C, x \in c, \inf(n_l(x), n_r(x)) \geq \beta_0 l(c)\}.$$

Since

$$\beta_0 l(c) \geq A',$$

points of B_- belongs to

$$\{y \in c_-, d(y,c) \geq \pi/2 - \epsilon'\}.$$

Those points (which we called the points of the first type) are in the past of c'_-, so by Lemma 4.2.1,

$$l(B_-) \leq l(Pr_-(B_-)),$$

hence by composition of projections,

$$l(B_-) \leq (\epsilon/2) l(Pr(B_-)),$$

where Pr is the projection from c_- to c.
Let

$$B = \{\tau_x \cap \partial_- C, x \in c, \inf(n_l(x), n_r(x)) \leq \beta_0 l(c)\}.$$

Lemma 4.2.1 gives

$$l(\{x \in c, \inf(n_l(x), n_r(x)) \leq \beta_0 l(c)\}) \geq l(B) \quad .$$

Moreover (using the fact that $l_0 \leq \epsilon l(c)/2$)

$$l(\{x \in c, \inf(n_l(x), n_r(x)) \leq \beta_0 l(c)\}) \leq \sup(\delta_0 l(c), l_0) \leq \epsilon l(c)/2 \quad .$$

Then we obtain

$$\epsilon l(c)/2 \geq l(B) \quad .$$

We get the desired result by addition. □

Chapter 5

Proof of the degree theorem

5.1 Computing degrees

In this section, we investigate the degree of the proper map Φ. Our main aim is Theorem 3.4.3 (Φ is properly homotopic to a homeomorphism).

Recall [19] that the degree of proper maps between manifolds is well defined and that a map whose degree is nonzero is surjective. Thus the last proposition ends the proof of the main theorem of the article.

In [2], the authors prove the existence of a map Φ_{k_+,k_-} from $\mathcal{G}H(S)$ (Three dimensional GHMC AdS structures with Cauchy surface homeomorphic to S, identified with $\mathcal{T}(S) \times \mathcal{T}(S)$) to $\mathcal{T}(S) \times \mathcal{T}(S)$ which associates to a GHMC AdS manifold the pair of conformal structures of the unique past convex (resp. future convex) surfaces with constant sectional curvatures equal to k_+ and k_- respectively (see also [10] where this map is considered).

They also prove that the map φ

$$(k^+, k^-, \rho_l, \rho_r) \mapsto \Phi_{k_+,k_-}(\rho_l, \rho_r) = (\Phi^+_{k_+,k_-}(\rho_l, \rho_r), \Phi^-_{k_+,k_-}(\rho_l, \rho_r))$$

is continuous (Lemma 12.4 from [2]) on $(-\infty, -1)^2 \times \mathcal{T}(S) \times \mathcal{T}(S)$.

The continuity on $(-\infty, -1]^2 \times \mathcal{T}(S) \times \mathcal{T}(S)$ follows from the following lemma:

Lemma 5.1.1 (Convergence of a sequence of hyperbolic metrics). *Let m be a point of $\mathcal{T}(S)$ and C_n be a sequence of real numbers greater than one which converge to* 1.

Then any sequence $m_n \in \mathcal{T}(S)$ satisfying

$$m_n \leq C_n m$$

(in the sense of length spectra) converge to m.

(We say that $m_1 \leq m_2$ in the sense of the length spectrum, i.e. if for any simple closed curve σ having σ_1 and σ_2 as geodecic representatives for m_1 and m_2, we have $l_{m_1}(\sigma_1) \leq l_{m_2}(\sigma_2)$). This is a consequence of a classical fact proved by Thurston [30]. Indeed, by the following lemma, such a sequence admits a convergent subsequence, whose limit m_∞ satisfies $m_\infty \leq m$ (in the sense of length spectra). Then Thurston proved that this gives $m_\infty = m$.

The map φ just defined ($= \Phi_{k_+,k_-}(\rho_l, \rho_r)$), when restricted to any set of the form $[-C, -1]^2 \times \mathcal{T}(S) \times \mathcal{T}(S)$ (for any $C > 1$), is proper thanks to the following lemmas (which are corollaries of results from [30]):

Lemma 5.1.2 (Compactness of sets of metrics first version). *Let m be a point of $\mathcal{T}(S)$ and let $C > 1$.*

Then the set of metrics m' in $\mathcal{T}(S)$ such that: $Cm' \geq m$ (in the sense of length spectra) is compact in $\mathcal{T}(S)$.

The same lemma is true when considering metrics m' such that $m' \leq Cm$.

By a slight extention we get the following lemma:

Lemma 5.1.3 (Compactness of sets of metrics revisited). *Let K be a compact of $\mathcal{T}(S)$, let $C > 1$. Then the set of metrics m' such that $Cm' > m$ for some $m \in K$ (in the sense of length spectra) is compact.*

Consider the slicing of an AdS GHMC manifold given by the map φ ($= \Phi_{k_+,k_-}(\rho_l, \rho_r)$) . For any $k < -1$, let k^* be the curvature of the dual to the surface (see [24] for the definition of duality) of curvature k ($k^* < -1$). Then we have (for instance, see [2])

$$k^* = \frac{-k}{k+1}.$$

Let $C > 1$. Let us then prove the properness of the map φ ($= \Phi_{k_+,k_-}(\rho_l, \rho_r)$), restricted to any set of the form $[-C, -1]^2 \times \mathcal{T}(S) \times \mathcal{T}(S)$. Let (m_n^+) and (m_n^-) be convergent sequences in $\mathcal{T}(S)$ and ρ_n^l, ρ_n^r, k_n^+ and k_n^- be sequences such that $-C \leq k_n^+$, $-C \leq k_n^-$ for all n, and such that the image of

$$(k_n^+, k_n^-, \rho_n^l, \rho_n^r)$$

by φ is $((m_n^+), (m_n^-))$.

By properness of $\Phi_{-1,-1} = \Phi$ (proved in the previous section) and the fact (proved in [10]) that

$$C\Phi^{+}_{k_+,k_-} < \Phi^{+}_{-1,-1}$$

and

$$C\Phi^{-}_{k_+,k_-} < \Phi^{-}_{-1,-1},$$

the sequences k^+_n,k^-_n,ρ^l_n,ρ^r_n stay in a compact set, hence they admit convergent subsequences.

By invariance of the degree of a map under a proper homotopy, all the maps Φ_{k_+,k_-} have the same degree, which is given by this last lemma:

Lemma 5.1.4 (Degree of Φ_{k,k^*}). *The map Φ_{k,k^*} has degree one*

In fact it is a homeomorphism (see [10] for a proof and related results).

Chapter 6

Conjectural statements

6.1 A conjectural overview of another proof

Having proved that our main map Φ of Theorem 3.2.1 (or of its cousin version Theorem 3.3.1) has degree one, it is natural to wonder whether it is possible to prove the uniqueness part of Mess conjecture, which can be expressed as follows:

Conjecture 6.1.1 (Unique existence conjecture). *The map Φ of Theorem 3.2.1 is a homeomorphism from $\mathcal{T}(S) \times \mathcal{T}(S)$ to itself.*

Recall that a homeomorphism is a proper map of degree one but not all such maps are homeomorphisms.

In fact, $\mathcal{T}(S) \times \mathcal{T}(S)$ being simply connected, it would suffice to prove that Φ is a local homeomorphism, since any continuous map which is both proper and a local homeomorphism between manifolds is a covering [19] and any covering of $\mathcal{T}(S) \times \mathcal{T}(S)$ by itself is a homeomorphism.

As far as the analog issue of prescribing pleating laminations is concerned, Bonahon [8] in the hyperbolic setting, on the one hand, and Bonsante and Schlenker [12] in the anti-de Sitter setting, on the other hand, proved that we do have a local homeomorphism near the Fuchsian locus in the space of quasifuchsian (resp. GHMC AdS) structures, that is, near the set of structures where the pair of points at infinity in $\mathcal{T}(S) \times \mathcal{T}(\overline{S})$ (resp. the pair of left and right holonomies in $\mathcal{T}(S) \times \mathcal{T}(S)$) are complex conjugate (resp. the same). Otherwise stated, the Fuchsian locus is the diagonal in the product $\mathcal{T}(S) \times \mathcal{T}(S)$. Let us call it $F(S)$ in our (AdS) case. Note that

Caroline Series proved in [25] that uniqueness of the pair of bending measured laminations holds for quasifuchsian manifolds homeomorphic to the product of an interval by a punctured torus. In the same vein as Bonahon's results (resp. Bonsante and Schlenker's results), we state the following conjecture:

Conjecture 6.1.2 (Homeomorphism near the Fuchsian locus). *There is a neighborhood U (resp. V) of $F(S)$ (resp. the diagonal $\Delta(\mathcal{T}(S))$ in $\mathcal{T}(S) \times \mathcal{T}(S)$) such that the map Φ of Theorem 3.2.1 is a homeomorphism from U to V.*

Thanks to works of Bonahon [5], we already know that Φ is smooth. To prove Theorem 6.1.2, we would like Φ to be a local diffeomorphism from a neighborhood of $F(S)$ to a neighborhood of $\Delta(\mathcal{T}(S))$. Unfortunately, it fails to be so. Indeed, one easily notices that the tangent map of Φ at some point (m, m) is $F(S)$ sends every tangent vector of the form $(u,0)$ (resp. $(0, v)$) to $(u/2, u/2)$ (resp. $(v/2, v/2)$), hence it annihilates any tangent vector of the form $(-u, u)$.

To remedy this, we consider instead the manifolds $\mathcal{T}(S) \times \mathcal{T}(S)$ blown up along $F(S)$ (resp. $\mathcal{T}(S) \times \mathcal{T}(S)$ blown up along the diagonal) and hope to prove that the map $\overline{\Phi}$ thus obtained is a local diffeomorphism near the unit normal bundle of $F(S)$, $N^1F(S)$ (defined as the quotient of the normal bundle of $F(S)$ by the natural action of $\mathbb{R}_{>0}$ by scalar multiplication).

Recall [5] that the blown up manifold

$$Bl_{F(S)}(\mathcal{T}(S) \times \mathcal{T}(S))$$

is defined by suitably glueing $N^1F(S)$ to the complement of $F(S)$ in $\mathcal{T}(S) \times \mathcal{T}(S)$ (the same construction is valid for the blow up of $\mathcal{T}(S) \times \mathcal{T}(S)$ along the diagonal).

Let us take global coordinates for $\mathcal{T}(S) \times \mathcal{T}(S)$. We still denote Φ our map in those coordinates:

$$\Phi(\rho, \rho') = (m_+(\rho, \rho'), m_-(\rho, \rho'))$$

Fix a Riemannian metric in $\mathcal{T}(S)$. For any ρ in $\mathcal{T}(S)$, any unit vector u in $T_\rho\mathcal{T}(S)$ and any germ γ of smooth curve in $\mathcal{T}(S) \times \mathcal{T}(S)$ at 0 such that

$$\gamma(0) = (\rho, \rho)$$

and

$$\dot{\gamma}(0) = (u, -u),$$

we denote

$$\gamma = (\gamma_l, \gamma_r).$$

(γ depends on u and ρ.) We also denote as $m^+(\dot{\gamma}_l) - m^-(\dot{\gamma}_r) mod.\mathbb{R}^+$ the limit of the difference $m^+(\gamma_l(s)) - m^-(\gamma_r(s))$ (s in a neighborhood of zero in $\mathbb{R}$) when s tends to zero, taken modulo $\mathbb{R}^+$. This difference makes sense since we take global coordinates. It can be defined more intrinsically if one considers the difference of vectors in a tubular neighborhood of the diagonal in $\mathcal{T}(S) \times \mathcal{T}(S)$, which is a vector bundle over the diagonal.

Conjecture 6.1.3 (Local diffeomorphism after blowing up). *Keeping the same notations as above, the map*

$$\overline{\Phi}((\rho,\rho),(u,-u) mod.\mathbb{R}^+) = ((\rho,\rho), m^+(\dot{\gamma}_l)-m^-(\dot{\gamma}_r) mod.\mathbb{R}^+, m^+(\dot{\gamma}_l)-m^-(\dot{\gamma}_r) mod.\mathbb{R}^+)$$

is a diffeomorphism from a neighborhood of $N^1(F(S))$ in $Bl_{F(S)}(\mathcal{T}(S)\times\mathcal{T}(S))$ to a neighborhood of $\Delta(\mathcal{T}(S))$ in $Bl_{\Delta(\mathcal{T}(S))}(\mathcal{T}(S) \times \mathcal{T}(S))$.

One can show, considering the Taylor expansion of $m^+(\gamma_l(s)) - m^-(\gamma_r(s))$ near $s = 0$, that modulo some positive multiplicative constant, $m^+(\dot{\gamma}_l) - m^-(\dot{\gamma}_r)$ equals the difference

$$\nabla_{e_l} e_l(\rho) - \nabla_{e_r} e_r(\rho)$$

of curvature vectors of flow lines (i.e. integral curves) of the respective left and right earthquake vector fields e_l and e_r such that

$$e_l(\rho) = e_r(\rho) = u.$$

(∇ is the Levi-Civita connection of the chosen Riemannian metric.) Recall [21] that for fixed ρ and u, such vector fields exist and are unique.

It can be furthermore shown that

$$nabla_{e_l} e_l(\rho) - nabla_{e_r} e_r(\rho)$$

equals the hessian

$$Hess_\rho(L_l + L_r)(u)$$

at (ρ, u) of the sum of the two length functions L_l and L_r associated to the earthquake paths defined respectively by e_l and e_r.(Recall that by definition of e_l and e_r, ρ is a critical point of $L_l + L_r$, which allows to define the Hessian independently of the connection ∇.)

Thus the previous conjecture amounts to the following one, purely in terms of Teichmüller theory.

Conjecture 6.1.4. *The map from the unit tangent bundle* $UT\mathcal{T}(S)$ *of* $\mathcal{T}(S)$ *to itself which sends* (ρ, u) *to* $(\rho, Hess_\rho(L_l + L_r)(u) mod \mathbb{R}^+)$ *is a homeomorphism.*

Note that the linear map $Hess_\rho(L_l + L_r)$ depends on both ρ and u.

Chapter 7

Conclusion and related topics

We have shown that any pair of conformal structures on S can be obtained as the boundary metrics of (at least) one GHMC AdS manifold.

It could be interesting to extend this result to manifolds with conical singularities along timelike lines. One would have to define a good notion of convex core and extend the results of [2] to this setting.

Another interesting question would be to prescribe the metric on a pair of convex smooth surfaces (with non necessarily constant curvature) isometrically embedded respectively in the past and the future of the convex core of a GHMC AdS manifold. Techniques such as in [23] could apply, since we do not know how to extend Schlenker's unique existence results in [29] to the AdS setting.

Bibliography

[1] Lars Andersson, Thierry Barbot, Riccardo Benedetti, Francesco Bonsante, William M. Goldman, François Labourie, Kevin P. Scannell, and Jean-Marc Schlenker. Notes on: "Lorentz spacetimes of constant curvature" [Geom. Dedicata **126** (2007), 3–45; mr2328921] by G. Mess. *Geom. Dedicata*, 126:47–70, 2007.

[2] Thierry Barbot, François Béguin, and Abdelghani Zeghib. Prescribing Gauss curvature of surfaces in 3-dimensional spacetimes: application to the Minkowski problem in the Minkowski space. *Ann. Inst. Fourier (Grenoble)*, 61(2):511–591, 2011.

[3] Lipman Bers. Simultaneous uniformization. *Bull. Amer. Math. Soc.*, 66:94–97, 1960.

[4] Arthur L. Besse. *Einstein manifolds*, volume 10 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1987.

[5] Francis Bonahon. Shearing hyperbolic surfaces, bending pleated surfaces and Thurston's symplectic form. *Ann. Fac. Sci. Toulouse Math. (6)*, 5(2):233–297, 1996.

[6] Francis Bonahon. Transverse Hölder distributions for geodesic laminations. *Topology*, 36(1):103–122, 1997.

[7] Francis Bonahon. Geodesic laminations on surfaces. In *Laminations and foliations in dynamics, geometry and topology (Stony Brook, NY, 1998)*, volume 269 of *Contemp. Math.*, pages 1–37. Amer. Math. Soc., Providence, RI, 2001.

[8] Francis Bonahon. Kleinian groups which are almost Fuchsian. *J. Reine Angew. Math.*, 587:1–15, 2005.

[9] Francis Bonahon and Jean-Pierre Otal. Laminations mesurées de plissage des variétés hyperboliques de dimension 3. *Annals of mathematics*, pages 1013–1055, 2004.

[10] F. Bonsante, G. Mondello, and J.-M. Schlenker. A cyclic extension of the earthquake flow. *ArXiv e-prints.* To appear*, Geometry and Topology*, June 2011.

[11] Francesco Bonsante and Jean-Marc Schlenker. Maximal surfaces and the universal teichmüller space. *Inventiones mathematicae*, 182(2):279–333, 2010.

[12] Francesco Bonsante and Jean-Marc Schlenker. Fixed points of compositions of earthquakes. *Duke Math. J.*, 161(6):1011–1054, 2012.

[13] R. D. Canary, D. B. A. Epstein, and P. L. Green. Notes on notes of Thurston [mr0903850]. In *Fundamentals of hyperbolic geometry: selected expositions*, volume 328 of *London Math. Soc. Lecture Note Ser.*, pages 1–115. Cambridge Univ. Press, Cambridge, 2006. With a new foreword by Canary.

[14] Yvonne Choquet-Bruhat and Robert Geroch. Global aspects of the Cauchy problem in general relativity. *Comm. Math. Phys.*, 14:329–335, 1969.

[15] D. B. A. Epstein and A. Marden. Convex hulls in hyperbolic space, a theorem of Sullivan, and measured pleated surfaces. In *Analytical and geometric aspects of hyperbolic space (Coventry/Durham, 1984)*, volume 111 of *London Math. Soc. Lecture Note Ser.*, pages 113–253. Cambridge Univ. Press, Cambridge, 1987.

[16] D. B. A. Epstein and V. Markovic. The logarithmic spiral: a counterexample to the $K = 2$ conjecture. *Ann. of Math. (2)*, 161(2):925–957, 2005.

[17] Albert Fathi, François Laudenbach, and Valentin Poénaru. *Thurston's work on surfaces*, volume 48 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 2012. Translated from the 1979 French original by Djun M. Kim and Dan Margalit.

[18] William Mark Goldman. *Discontinuous groups and the Euler class*. PhD thesis, University of California, Berkeley, 1980.

[19] Morris W Hirsch. *Differential topology*, volume 1991. Springer New York, 1976.

[20] Steven P Kerckhoff. The Nielsen realization problem. *Annals of mathematics*, pages 235–265, 1983.

[21] Steven P. Kerckhoff. Earthquakes are analytic. *Comment. Math. Helv.*, 60(1):17–30, 1985.

[22] Kirill Krasnov and Jean-Marc Schlenker. Minimal surfaces and particles in 3-manifolds. *Geom. Dedicata*, 126:187–254, 2007.

[23] François Labourie. Métriques prescrites sur le bord des variétés hyperboliques de dimension 3. *J. Differential Geom.*, 35(3):609–626, 1992.

[24] Geoffrey Mess. Lorentz spacetimes of constant curvature. *Geom. Dedicata*, 126:3–45, 2007.

[25] Y Minsky, M Sakuma, and C Series. Thurstons bending measure conjecture for once punctured torus groups. *Spaces of Kleinian groups*, 329:75, 2006.

[26] Barrett O'Neill. *Semi-Riemannian geometry*, volume 103 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983. With applications to relativity.

[27] Jean-Pierre Otal. *The hyperbolization theorem for fibered 3-manifolds*, volume 7 of *SMF/AMS Texts and Monographs*. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2001. Translated from the 1996 French original by Leslie D. Kay.

[28] John G. Ratcliffe. *Foundations of hyperbolic manifolds*, volume 149 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2006.

[29] Jean-Marc Schlenker. Hyperbolic manifolds with convex boundary. *Invent. Math.*, 163(1):109–169, 2006.

[30] W. P. Thurston. Minimal stretch maps between hyperbolic surfaces. *ArXiv Mathematics e-prints*, January 1998.

[31] William P. Thurston. Three-dimensional geometry and topology. Originally notes of lectures at Princeton University, 1979. Recent version available on http://www.msri.org/publications/books/gt3m/, 1980.

[32] William P. Thurston. Earthquakes in two-dimensional hyperbolic geometry. In *Low-dimensional topology and Kleinian groups (Coventry/Durham, 1984)*, volume 112 of *London Math. Soc. Lecture Note Ser.*, pages 91–112. Cambridge Univ. Press, Cambridge, 1986.

[33] Edward Witten. $2+1$-dimensional gravity as an exactly soluble system. *Nuclear Phys. B*, 311(1):46–78, 1988/89.