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de Monge-Ampère complexes

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Résumé

Cette thèse est consacrée à l'étude de la régularité des solutions des équations de Monge-Ampère complexes ainsi que des équations hessiennes complexes dans un domaine borné de $\mathbb{C}^n$.
Dans le premier chapitre, on donne des rappels sur la théorie du pluripotentiel.
Dans le deuxième chapitre, on étudie le module de continuité des solutions du problème de Dirichlet pour les équations de Monge-Ampère lorsque le second membre est une mesure à densité continue par rapport à la mesure de Lebesgue dans un domaine strictement hyperconvexe lipschitzien.
Dans le troisième chapitre, on prouve la continuité hölderienne des solutions de ce problème pour certaines mesures générales.
Dans le quatrième chapitre, on considère le problème de Dirichlet pour les équations hessiennes complexes plus générales où le second membre dépend de la fonction inconnue. On donne une estimation précise du module de continuité de la solution lorsque la densité est continue. De plus, si la densité est dans L^p, on démontre que la solution est Hölder-continue jusqu'au bord.

Mots-clés

Problème de Dirichlet, Opérateur de Monge-Ampère, Mesure de Hausdorff-Riesz, Fonction m-sousharmonique, Opérateur hessien, Capacité, Module de continuité, Principe de comparaison, Théorème de stabilité, Domaine strictement hyperconvexe lipschitzien, Domaine strictement m-pseudoconvexe.

Abstract

In this thesis we study the regularity of solutions to the Dirichlet problem for complex Monge-Ampère equations and also for complex Hessian equations in a bounded domain of $\mathbb{C}^n$.
In the first chapter, we give basic facts in pluripotential theory.
In the second chapter, we study the modulus of continuity of solutions to the Dirichlet problem for complex Monge-Ampère equations when the right hand side is a measure with continuous density with respect to the Lebesgue measure in a bounded strongly hyperconvex Lipschitz domain.
In the third chapter, we prove the Hölder continuity of solutions to this problem for some general measures.
In the fourth chapter, we consider the Dirichlet problem for complex Hessian equations when the right hand side depends on the unknown function. We give a sharp estimate of the modulus of continuity of the solution as the density is continuous. Moreover, for the case of L^p-density we demonstrate that the solution is Hölder continuous up to the boundary.

Keywords

Dirichlet problem, Monge-Ampère operator, Hausdorff-Riesz measure, m-subharmonic function, Hessian operator, Capacity, Modulus of continuity, Comparison principle, Stability theorem, Strongly hyperconvex Lipschitz domain, Strongly m-pseudoconvex domain.

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Contents

Chapter 0

Introduction

In this thesis we study the regularity of solutions to the Dirichlet problem for complex Monge-Ampère equations and, more generally, for complex Hessian equations in a bounded domain of $\mathbb{C}^n$.

Pluripotential theory became a branch of mathematical research in the last decades and the complex Monge-Ampère equation was studied extensively by many mathematicians.

Two influential works have been the work by Yau [Yau78] on non-degenerate equations on compact Kähler manifolds, and by Bedford-Taylor [BT76] on generalized weak solutions in the sense of pluripotential theory. They proved [BT76] that the complex Monge-Ampère operator has a sense for a non-smooth locally bounded plurisubharmonic function and there exists a continuous solution to the Dirichlet problem in a bounded strongly pseudoconvex domain with smooth boundary.

Since then, there has been considerable further progress, it was proved in [CKNS85] the smoothness of the solution to the Dirichlet problem in the case of non-degenerate smooth density and smooth boundary data.

Kołodziej demonstrated [Ko98, Ko99] that the Dirichlet problem still admits a unique weak continuous solution when the right hand side of the complex Monge-Ampère equation is a measure satisfying some sufficient condition which is close to be best possible. Furthermore, for the degenerate complex Monge-Ampère equation on compact Kähler manifolds he established [Ko98] a uniform a priori estimate which generalizes the celebrated a priori estimate of Yau [Yau78].

A viscosity approach to the complex Monge-Ampère equation has been developed by Eyssidieux, Guedj and Zeriahi in [EGZ11] on compact Kähler manifolds and they compare viscosity and potential solutions. In the local context, Wang [Wan12] studied the existence of a viscosity solution to the Dirichlet problem for the complex Monge-Ampère equation and estimated the modulus of continuity of the solution in terms of that of a given subsolution and of the right hand side.

Some results have been known about the Hölder regularity of the solution to this problem for measures absolutely continuous with respect to the Lebesgue measure. Bedford and Taylor [BT76] studied the Hölder continuity of the solution by means of Hölder continuity of the density and the boundary data. Guedj, Kołodziej and Zeriahi [GKZ08] established Hölder regularity of solutions for L^p- densities bounded near the boundary of strongly pseudoconvex domain.

In the compact case, there are many works in this area [Ko08, Ph10, DDGHKZ14] which exceed the scope of this thesis.

We are also interested in studying the complex Hessian equation in a bounded domain of $\mathbb{C}^n$. This equation corresponds to the elementary symmetric function of degree $1 \leq m \leq n$. When $m = 1$, this equation corresponds to the Poisson equation which is classical. The case $m = n$ corresponds to the complex Monge-Ampère equation.

The complex Hessian equation is a natural generalisation of the complex Monge-Ampère equation and has some geometrical applications. For examples, this equation appears in problems related to quaternionic geometry [AV10] and in the work [STW15] for solving Gauduchon's conjecture. Its real counterpart has been developed in the works of Trudinger, Wang and others (see for example [W09]). This all gives us a strong motivation to study the existence and regularity of weak solutions to complex Hessian equations.

The complex Hessian equation is a new subject and is much more difficult to handle than the complex Monge-Ampère equation (e.g. the m-subharmonic functions are not invariant under holomorphic change of variables, for $m < n$). Despite these difficulties, the pluripotential theory which was developed for the complex Monge-Ampère equation can be adapted to the complex Hessian equation [Bł05, DK14, Lu12, SA12]. Błocki [Bł05] introduced some elements of the potential theory for m-subharmonic functions and proved the existence of continuous solution for the homogeneous Dirichlet problem in the unit ball. Dinew and Kołodziej [DK14] used pluripotential techniques adapted for the complex Hessian equation to settle the question of the existence of weak solutions to the Dirichlet problem. H. C. Lu introduced in [Lu12, Lu15] finite energy classes of m-subharmonic functions and developed a variational approach to complex Hessian equations. The non-degenerate complex Hessian equation on compact Kähler manifold with smooth density has been studied in [Hou09], [HMW10], [Jb12] and the degenerate case was treated in [Lu13a] and [DK14]. H.C. Lu persisted in investigating a viscosity approach to complex Hessian equations in his paper [Lu13b].

Now we will present an overview of the main results of this thesis. First, for the sake of convenience we recall some notations. We denote by dV_{2n} the Lebesgue measure in $\mathbb{C}^n$ and $L^p(\Omega)$ stands for the usual L^p-space with respect to the Lebesgue measure in a bounded domain Ω. We use $d = \partial + \bar{\partial}$ and $d^c = (i/4)(\bar{\partial} - \partial)$, where ∂ and $\bar{\partial}$ are the usual differential operators. Here and subsequently, we use the notation :

$$\mathcal{C}^{0,\beta}(\bar{\Omega}) = \{v \in \mathcal{C}(\bar{\Omega});\ \|v\|_\beta < +\infty\},$$

for $0 < \beta \leq 1$, and the β-Hölder norm is given by

$$\|v\|_\beta = \sup\left\{|v(z)| : z \in \bar{\Omega}\right\} + \sup\left\{\frac{|v(z) - v(y)|}{|z - y|^\beta} : z, y \in \bar{\Omega}, z \neq y\right\}.$$

We mean by $\mathcal{C}^{k,\beta}(\bar{\Omega})$, with $k \geq 1$ and $0 < \beta \leq 1$, the class of functions which have continuous partial derivatives of order less than k, and whose k-th order partial derivatives satisfy a Hölder condition of order β.

The Dirichlet problem for complex Monge-Ampère equations. It asks for a function, u, plurisubharmonic on Ω and continuous on $\bar{\Omega}$ such that

$$(dd^c u)^n = f d\mu, \text{ and } u = \varphi \text{ on } \partial\Omega, \tag{0.0.1}$$

where $\varphi \in \mathcal{C}(\partial\Omega)$, μ is a nonnegative finite Borel measure on Ω and $0 \leq f \in L^1(\Omega, \mu)$.

In Chapter 2, we consider this problem in a bounded strongly hyperconvex Lipschitz domain of $\mathbb{C}^n$ with continuous densities with respect to the Lebesgue measure. Then we prove in Section 2.5 a sharp estimate for the modulus of continuity of the solution.

Theorem 0.0.1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded strongly hyperconvex Lipschitz domain, $\varphi \in \mathcal{C}(\partial\Omega)$ and $0 \leq f \in \mathcal{C}(\bar{\Omega})$. Assume that ω_φ is the modulus of continuity of φ and $\omega_{f^{1/n}}$ is the modulus of continuity of $f^{1/n}$. Then the modulus of continuity of the unique solution* $\mathtt{U}$ *to (0.0.1) has the following estimate*

$$\omega_{\mathtt{U}}(t) \leq \eta(1 + \|f\|^{1/n}_{L^\infty(\bar{\Omega})}) \max\{\omega_\varphi(t^{1/2}), \omega_{f^{1/n}}(t), t^{1/2}\},$$

where η is a positive constant depending on Ω.

In [GKZ08], Guedj, Kołodziej and Zeriahi proved the Hölder continuity of the solution to (0.0.1) when $\varphi \in \mathcal{C}^{1,1}(\partial\Omega)$ and $f \in L^p(\Omega)$, for $p > 1$, is bounded near the boundary $\partial\Omega$. Recently N.C. Nguyen [N14] proved that the solution is Hölder continuous when the density f satisfies a growth condition near $\partial\Omega$. Our next result in Chapter 3 concerns the Hölder regularity of the solution when the density is merely in $L^p(\Omega)$, $p > 1$. Moreover, we improve the Hölder exponent while $p \geq 2$ by using the relation between real and complex Monge-Ampère operators.

Theorem 0.0.2. *Let $\Omega \subset \mathbb{C}^n$ be a bounded strongly hyperconvex Lipschitz domain. Assume that $\varphi \in \mathcal{C}^{1,1}(\partial\Omega)$ and $f \in L^p(\Omega)$ for some $p > 1$. Then the unique solution* $\mathtt{U}$ *to (0.0.1) is γ-Hölder continuous on $\bar{\Omega}$ for any $0 < \gamma < 1/(nq+1)$ where $1/p+1/q = 1$. Moreover, if $p \geq 2$, then the solution* $\mathtt{U}$ *is Hölder continuous on $\bar{\Omega}$ of exponent less than* $\min\{1/2, 2/(nq+1)\}$.

In the same chapter, we study the Hölder regularity of the solution to the Dirichlet problem for a Hausdorff-Riesz measure of order $2n - 2 + \epsilon$, with $0 < \epsilon \leq 2$, that is a non-negative Borel measure satisfies the condition

$$\mu(B(z, r) \cap \Omega) \leq Cr^{2n-2+\epsilon}, \ \ \forall z \in \bar{\Omega}, \ \forall 0 < r < 1,$$

for some positive constant C. These measures are singular with respect to the Lebesgue measure, for $0 < \epsilon < 2$, and there are many nice examples (see Example 3.5.6).

More precisely, we prove in Section 3.5 the following theorems.

Theorem 0.0.3. *Let Ω be a bounded strongly hyperconvex Lipschitz domain in $\mathbb{C}^n$ and μ be a Hausdorff-Riesz measure of order $2n-2+\epsilon$, for $0 < \epsilon \leq 2$. Suppose that $\varphi \in \mathcal{C}^{1,1}(\partial\Omega)$ and $0 \leq f \in L^p(\Omega, \mu)$ for some $p > 1$, then the unique solution to the Dirichlet problem (0.0.1) is Hölder continuous on $\bar{\Omega}$ of exponent $\epsilon\gamma/2$, for any $0 < \gamma < 1/(nq + 1)$ where $1/p + 1/q = 1$.*

When the boundary data is merely Hölder continuous, we can still prove the Hölder regularity of the solution using the last theorem.

Theorem 0.0.4. *Let Ω be a bounded strongly hyperconvex Lipschitz domain in $\mathbb{C}^n$ and μ be a Hausdorff-Riesz measure of order $2n - 2 + \epsilon$, for $0 < \epsilon \leq 2$. Suppose that $\varphi \in \mathcal{C}^{0,\alpha}(\partial\Omega)$, $0 < \alpha \leq 1$ and $0 \leq f \in L^p(\Omega, \mu)$, for some $p > 1$, then the unique solution to the Dirichlet problem (0.0.1) is Hölder continuous on $\bar{\Omega}$ of exponent $\frac{\epsilon}{\epsilon+6} \min\{\alpha, \epsilon\gamma\}$, for any $0 < \gamma < 1/(nq + 1)$ where $1/p + 1/q = 1$.*

Moreover, when Ω is a smooth strongly pseudoconvex domain the Hölder exponent will be $\frac{\epsilon}{\epsilon+2} \min\{\alpha, \epsilon\gamma\}$, for any $0 < \gamma < 1/(nq + 1)$.

A natural question is that if we have a Hölder continuous subsolution to the Dirichlet problem, can we get a Hölder continuous solution in the whole domain?
This question is still open in the local case (see [DDGHKZ14] for a positive answer in the compact setting). However, we prove some particular case.

Theorem 0.0.5. *Let μ be a nonnegative finite Borel measure on a bounded strongly hyperconvex Lipschitz domain Ω. Let also $\varphi \in \mathcal{C}^{0,\alpha}(\partial\Omega)$, $0 < \alpha \leq 1$ and $0 \leq f \in L^p(\Omega, \mu)$, $p > 1$. Assume that there exists a Hölder continuous plurisubharmonic function w in Ω such that $(dd^c w)^n \geq \mu$. If, near the boundary, μ is Hausdorff-Riesz of order $2n - 2 + \epsilon$ for some $0 < \epsilon \leq 2$, then the solution $\mathtt{U}$ to (0.0.1) is Hölder continuous on $\bar{\Omega}$.*

The Dirichlet problem for complex Hessian equations. It consists in finding a function u which is m-subharmonic in Ω and continuous on $\bar{\Omega}$ such that

$$(dd^c u)^m \wedge \beta^{n-m} = f dV_{2n} \text{ and } u = \varphi \text{ on } \partial\Omega, \tag{0.0.2}$$

where $\varphi \in \mathcal{C}(\partial\Omega)$ and $0 \leq f \in L^1(\Omega)$.

We first prove in Chapter 4 a sharp estimate for the modulus of continuity of the solution when the density is continuous and depends on the unknown function.

Theorem 0.0.6. *Let Ω be a smoothly bounded strongly m-pseudoconvex domain in $\mathbb{C}^n$, $\varphi \in \mathcal{C}(\partial\Omega)$ and $0 \leq F \in \mathcal{C}(\bar{\Omega} \times \mathbb{R})$ be a nondecreasing function in the second variable. Then the modulus of continuity $\omega_{\mathtt{U}}$ of the solution $\mathtt{U}$ to*

$$\begin{cases} u \in SH_m(\Omega) \cap \mathcal{C}(\bar{\Omega}), \\ (dd^c u)^m \wedge \beta^{n-m} = F(z,u) dV_{2n} & in \quad \Omega, \\ u = \varphi & on \quad \partial\Omega, \end{cases}$$

satisfies the following estimate

$$\omega_{\mathtt{U}}(t) \leq \gamma(1 + \|F\|^{1/m}_{L^\infty(K)}) \ \max\{\omega_\varphi(t^{1/2}), \omega_{F^{1/m}}(t), t^{1/2}\},$$

where γ is a positive constant depending only on Ω, $K = \bar{\Omega} \times \{a\}$, $a = \sup_{\partial\Omega} |\varphi|$ and $\omega_{F^{1/m}}(t)$ is given by

$$\omega_{F^{1/m}}(t) := \sup_{y \in [-M,M]} \sup_{|z_1 - z_2| \leq t} |F^{1/m}(z_1, y) - F^{1/m}(z_2, y)|,$$

with $M = a + 2\,\mathrm{diam}(\Omega)^2 \sup_{\bar{\Omega}} F^{1/m}(., -a)$.

For densities in $L^p(\Omega)$, $p > n/m$, N. C. Nguyen [N14] proved that the solution to (0.0.2) is Hölder continuous when the density f satisfies some condition near the boundary. Here, we prove the general case.

Theorem 0.0.7. *Let $\Omega \subset \mathbb{C}^n$ be a bounded strongly m-pseudoconvex domain with smooth boundary, $\varphi \in \mathcal{C}^{1,1}(\partial\Omega)$ and $0 \leq f \in L^p(\Omega)$, for some $p > n/m$. Then the solution to (0.0.2), $\mathtt{U} \in \mathcal{C}^{0,\alpha}(\bar{\Omega})$ for any $0 < \alpha < \gamma_1$, where γ_1 is a constant depending on m, n, p defined by (4.5.1).*

Moreover, if $p \geq 2n/m$ then the solution to the Dirichlet problem $\mathtt{U} \in \mathcal{C}^{0,\alpha}(\bar{\Omega})$, for any $0 < \alpha < \min\{\frac{1}{2}, 2\gamma_1\}$.

In the particular case of radially symmetric solution in the unit ball, we are able to find a better Hölder exponent which turns out to be optimal.

Theorem 0.0.8. *Let $\mathbb{B}$ be the unit ball and $0 \le f \in L^p(\mathbb{B})$ be a radial function, where $p > n/m$. Then the unique solution $\mathtt{U}$ for (0.0.2) with zero boundary values is given by the explicit formula*

$$\mathtt{U}(r) = -B \int_r^1 \frac{1}{t^{2n/m-1}} \left(\int_0^t \rho^{2n-1} f(\rho) d\rho \right)^{1/m} dt,$$

where $B = \left(\frac{C_n^m}{2^{m+1}n}\right)^{-1/m}$. Moreover, $\mathtt{U} \in \mathcal{C}^{0,2-\frac{2n}{mp}}(\bar{\mathbb{B}})$ for $n/m < p < 2n/m$ and $\mathtt{U} \in Lip(\bar{\mathbb{B}})$ for $p \ge 2n/m$.

Chapter 1

Preliminaries

1.1 Basic facts in pluripotential theory

In this section, some useful facts from pluripotential theory will be stated and then used throughout this thesis. For further information about pluripotential theory, see for example [Kl91], [De89], [Ko05] and [GZ15].
Note that, with a domain we mean a nonempty, open and connected set.

Definition 1.1.1. Let $\Omega \subset \mathbb{R}^n$ be a domain. An upper semicontinuous function $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ is said to be *subharmonic* if, for every relatively compact open subset U of Ω and every continuous function $h : \bar{U} \to \mathbb{R}$ that is harmonic on U, we have the implication

$$u \leq h \text{ on } \partial U \Rightarrow u \leq h \text{ on } U.$$

It is well known in several complex variables that the class of subharmonic functions is very large and the fact that the property of being subharmonic is then not invariant under biholomorphic mappings. This fact motivates the theory of plurisubharmonic functions and pluripotential theory.

In pluripotential theory one therefore studies a smaller class of subharmonic functions whose composition with biholomorphic mappings are subharmonic. This class is precisely the class of plurisubharmonic functions that will be defined below.

Definition 1.1.2. A function $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ is called *plurisubharmonic* (briefly psh) if it is upper semicontinuous in Ω and subharmonic on the intersection of Ω with any complex line $\{a + b\xi; \xi \in \mathbb{C}\}$ where $a, b \in \mathbb{C}^n$.

We denote by $PSH(\Omega)$ the set of all plurisubharmonic functions in Ω. We state here some basic properties of psh functions.

Proposition 1.1.3.

1. *If $u, v \in PSH(\Omega)$ then $\lambda u + \eta v \in PSH(\Omega)$, $\forall \lambda, \eta \geq 0$.*
2. *If $u \in PSH(\Omega)$ and $\chi : \mathbb{R} \to \mathbb{R}$ is convex increasing function then $\chi \circ u \in PSH(\Omega)$.*
3. *Let $\{u_j\}_{j\in\mathbb{N}}$ be a decreasing sequence of psh functions in Ω. Then $u := \lim_{j\to+\infty} u_j$ is psh function in Ω.*
4. *If $u \in PSH(\Omega)$ then the standard regularizations $u_\epsilon = u * \rho_\epsilon$ are psh in $\Omega_\epsilon := \{z \in \Omega |\operatorname{dist}(z, \partial\Omega) > \epsilon\}$, for $0 < \epsilon \ll 1$.*

5. *Let U be a non-empty proper open subset of Ω, if $u \in PSH(\Omega)$, $v \in PSH(U)$ and $\limsup_{\substack{z\to y\\ z\in U}} v(z) \le u(y)$ for every $y \in \partial U \cap \Omega$, then the function*

$$w = \begin{cases} \max\{u, v\} & \text{in } U, \\ u & \text{in } \Omega \setminus U, \end{cases}$$

is psh in Ω.

6. *Let $\{u_\alpha\} \subset PSH(\Omega)$ be locally uniformly bounded from above and $u = \sup u_\alpha$. Then the upper semi-continuous regularization u^* is psh and equal to u almost everywhere.*

One of the important reasons to study plurisubharmonic functions is that we can use them to define pseudoconvex domains.

Definition 1.1.4. A domain $\Omega \subset \mathbb{C}^n$ is called *pseudoconvex* if there exists a continuous plurisubharmonic function φ in Ω such that $\{z \in \Omega; \varphi(z) < c\} \Subset \Omega$, for all $c \in \mathbb{R}$.

An important class of pseudoconvex domains is the class of hyperconvex domains.

Definition 1.1.5. A domain $\Omega \subset \mathbb{C}^n$ is called *hyperconvex* if there exists a negative continuous plurisubharmonic function ψ in Ω such that $\{z \in \Omega; \psi(z) < c\} \Subset \Omega$, for all real $c < 0$.

It is known that the Hartogs triangle is a pseudoconvex domain but not hyperconvex. However, Demailly [De87] proved that any pseudoconvex domain with Lipschitz boundary is a hyperconvex domain.

1.2 The complex Monge-Ampère operator

Let $\partial, \bar\partial$ be the usual differential operators, $d = \partial + \bar\partial$ and $d^c = (i/4)(\bar\partial - \partial)$. Then $dd^c = (i/2)\partial\bar\partial$.

If $u \in \mathcal{C}^2(\Omega)$ is a plurisubharmonic function, then the complex Monge-Ampère operator is defined by

$$(dd^c u)^n = (dd^c u) \wedge ... \wedge (dd^c u) = \det\left(\frac{\partial^2 u}{\partial z_j \partial \bar z_k}\right) \beta^n,$$

where $\beta := dd^c|z|^2 = (i/2)\sum_{j=1}^n dz_j \wedge d\bar{z_j}$ is the standard Kähler form in $\mathbb{C}^n$.
Note that $\beta^n = n!\, dV_{2n}$ where

$$dV_{2n} = (i/2)^n dz_1 \wedge d\bar z_1 \wedge ... \wedge dz_n \wedge d\bar z_n$$

is the usual volume form on $\mathbb{R}^{2n}$ or $\mathbb{C}^n$.

For $n = 1$, we have $dd^c u = (1/4)\Delta u dV_2$ and we know that the Laplace operator is well defined on all subharmonic functions. In the case $n \ge 2$ the complex Monge-Ampère operator can not be extended in a meaningful way to the whole class of plurisubharmonic functions and still have the range contained in the class of nonnegative Borel measures (see Example 3.1 in [Ki83]).

In 1976, Bedford and Taylor in their seminal work proved that the complex Monge-Ampère operator is well-defined on locally bounded plurisubharmonic functions. They defined inductively the following closed nonnegative current

$$dd^c u_1 \wedge dd^c u_2 \wedge ... \wedge dd^c u_n := dd^c(u_1 dd^c u_2 \wedge ... \wedge dd^c u_n),$$

where $u_1, u_2, ..., u_n \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$.

Furthermore, Cegrell [Ce04] introduced and investigated the largest class of plurisubharmonic functions on which the operator $(dd^c.)^n$ is well-defined.

The following inequality, named Chern-Levine-Nirenberg inequality, gives a bound on the local mass of the non-negative measure $dd^c u_1 \wedge ... \wedge dd^c u_n$ in terms of L^∞-norms of u_j's and hence ensures that these measures $dd^c u_1 \wedge ... \wedge dd^c u_n$, where $u_j \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$, $j = 1, ..., n$, are Radon measures.

Proposition 1.2.1. *Let $K \Subset U \Subset \Omega$, where K is compact and U is open. Let $u_j \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$, $j = 1, 2, ..., n$. Then there exists a constant C depending on K, U, Ω such that*

$$\|dd^c u_1 \wedge ... \wedge dd^c u_n\|_K \leq C \|u_1\|_{L^\infty(U)} ... \|u_n\|_{L^\infty(U)}.$$

In [BT82] Bedford and Taylor showed that the complex Monge-Ampère operator is continuous with respect to monotone sequences of locally bounded plurisubharmonic functions. Later, Xing [Xi96] found out that the convergence in capacity (defined below) entails the convergence of corresponding Monge-Ampère measures and he showed that this condition is quite sharp in some case.

Let Ω be a bounded domain in $\mathbb{C}^n$. For a Borel subset K of Ω, we introduce the Bedford-Taylor capacity

$$\text{Cap}(K, \Omega) = \sup \Big\{ \int_K (dd^c u)^n; u \in PSH(\Omega), -1 \leq u \leq 0 \Big\}.$$

By proposition 1.2.1, it is clear that the capacity is finite when K is relatively compact in Ω.

Definition 1.2.2. A sequence u_j of functions defined in Ω is said to converge in capacity to u if for any $t > 0$ and $K \Subset \Omega$

$$\lim_{j \to \infty} \text{Cap}(K \cap \{|u - u_j| > t\}, \Omega) = 0.$$

The complex Monge-Ampère operator is continuous with respect to sequences of locally uniformly bounded psh functions converging in capacity.

Theorem 1.2.3. *Let $(u_k^j)_{j=1}^\infty$, $k = 1, ..., n$ be a locally uniformly bounded sequence of psh functions in Ω and $u_k^j \to u_k \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$ in capacity as $j \to +\infty$ for $k = 1, ..., n$. Then*

$$\lim_{j \to \infty} dd^c u_1^j \wedge ... \wedge dd^c u_n^j = dd^c u_1 \wedge ... \wedge dd^c u_n$$

in the weak sense of currents in Ω.

We mention some useful theorems about the quasi-continuity of psh functions and the maximum principle.

Theorem 1.2.4. *Let u be a psh function in Ω. Then for all $\epsilon > 0$, there exists an open set $G \subset \Omega$ such that $Cap(G, \Omega) < \epsilon$ and $u|_{(\Omega \setminus G)}$ is continuous.*

Theorem 1.2.5. *Let $u, v \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$. Then we have the following inequality in the sense of Borel measures in Ω*

$$(dd^c \max\{u, v\})^n \geq \mathbf{1}_{\{u \geq v\}}(dd^c u)^n + \mathbf{1}_{\{u<v\}}(dd^c v)^n.$$

One of the most effective tools in pluripotential theory is the following comparison principle

Theorem 1.2.6. *Assume that $u, v \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$ are such that $\liminf_{z \to \partial\Omega}(u(z) - v(z)) \geq 0$, then*

$$\int_{\{u<v\}} (dd^c v)^n \leq \int_{\{u<v\}} (dd^c u)^n.$$

Corollary 1.2.7. *Assume that $u, v \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$ are such that $\liminf_{z \to \partial\Omega}(u(z) - v(z)) \geq 0$. If $(dd^c u)^n \leq (dd^c v)^n$ as Radon measures on Ω, then $v \leq u$ in Ω.*

Finally, we introduce Dinew's inequality for mixed Monge-Ampère measures [Di09].

Theorem 1.2.8. *Let $u, v \in PSH(\Omega) \cap L^\infty(\Omega)$. Let also $f, g \in L^1(\Omega)$ be nonnegative functions such that the following inequalities hold,*

$$(dd^c u)^n \geq f dV_{2n}, \ \ (dd^c v)^n \geq g dV_{2n}.$$

Then

$$(dd^c u)^k \wedge (dd^c v)^{n-k} \geq f^{\frac{k}{n}} g^{\frac{n-k}{n}} dV_{2n}, \ \ k = 1, ..., n.$$

1.3 Basic facts about m-subharmonic functions

In this section, we briefly recall some facts from linear algebra and basic results from potential theory for m-subharmonic functions. We refer the reader to [Bł05, SA12, Lu12, DK12, Lu13a, N13, DK14, Lu15] for more details and recent results.
We set

$$H_m(\lambda) = \sum_{1 \leq j_1 < ... < j_m \leq n} \lambda_{j_1} ... \lambda_{j_m},$$

where $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n$.
Thus $(t + \lambda_1)...(t + \lambda_n) = \sum\limits_{m=0}^{n} H_m(\lambda) t^{n-m}$ for $t \in \mathbb{R}$, where $H_0(\lambda) = 1$.
We denote by Γ_m the closure of the connected component of $\{H_m > 0\}$ containing $(1, 1, ..., 1)$. One can show that

$$\Gamma_m = \{\lambda \in \mathbb{R}^n : H_m(\lambda_1 + t, ..., \lambda_n + t) \geq 0, \ \forall t \geq 0\}.$$

It follows from the identity

$$H_m(\lambda_1 + t, ..., \lambda_n + t) = \sum_{p=0}^{m} \binom{n-p}{m-p} H_p(\lambda) t^{m-p},$$

that

$$\Gamma_m = \{\lambda \in \mathbb{R}^n : H_j(\lambda) \geq 0, \ \forall 1 \leq j \leq m\}.$$

It is clear that $\Gamma_n \subset \Gamma_{n-1} \subset ... \subset \Gamma_1$, where $\Gamma_n = \{\lambda \in \mathbb{R}^n : \lambda_i \geq 0, \ \forall i\}$.

By the paper of Gårding [G59], the set Γ_m is a convex cone in $\mathbb{R}^n$ and $H_m^{1/m}$ is concave on Γ_m. By Maclaurin's inequality, we get

$$\binom{n}{m}^{-1/m} H_m^{1/m} \leq \binom{n}{p}^{-1/p} H_p^{1/p}; \ \ 1 \leq p \leq m \leq n.$$

Let $\mathcal{H}$ be the vector space over $\mathbb{R}$ of complex Hermitian $n \times n$ matrices. For any $A \in \mathcal{H}$, let $\lambda(A) = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n$ be the eigenvalues of A. We set

$$\tilde{H}_m(A) = H_m(\lambda(A)).$$

Now, we define the cone

$$\tilde{\Gamma}_m := \{A \in \mathcal{H} : \lambda(A) \in \Gamma_m\} = \{A \in \mathcal{H} : \tilde{H}_j(A) \geq 0, \ \forall 1 \leq j \leq m\}.$$

Let α be a real (1,1)-form determined by

$$\alpha = \frac{i}{2} \sum_{i,j} a_{i\bar{j}} dz_i \wedge d\bar{z}_j,$$

where $A = (a_{i\bar{j}})$ is a Hermitian matrix. After diagonalizing the matrix $A = (a_{i\bar{j}})$, we see that

$$\alpha^m \wedge \beta^{n-m} = \tilde{S}_m(\alpha)\beta^n,$$

where β is the standard Kähler form in $\mathbb{C}^n$ and $\tilde{S}_m(\alpha) = \frac{m!(n-m)!}{n!}\tilde{H}_m(A)$.
The last equality allows us to define

$$\hat{\Gamma}_m := \{\alpha \in \mathbb{C}_{(1,1)} : \alpha \wedge \beta^{n-1} \geq 0, \alpha^2 \wedge \beta^{n-2} \geq 0, ..., \alpha^m \wedge \beta^{n-m} \geq 0\},$$

where $\mathbb{C}_{(1,1)}$ is the space of real (1,1)-forms with constant coefficients.

Let $M : \mathbb{C}^m_{(1,1)} \to \mathbb{R}$ be the polarized form of $\tilde{S}_m$, i.e. M is linear in every variable, symmetric and $M(\alpha, ..., \alpha) = \tilde{S}_m(\alpha)$, for any $\alpha \in \mathbb{C}_{(1,1)}$.
The Gårding inequality (see [G59]) asserts that

$$M(\alpha_1, \alpha_2, ..., \alpha_m) \geq \tilde{S}_m(\alpha_1)^{1/m} ... \tilde{S}_m(\alpha_m)^{1/m}, \ \ \alpha_1, \alpha_2, ..., \alpha_m \in \hat{\Gamma}_m. \tag{1.3.1}$$

Proposition 1.3.1. *([Bł05]). If $\alpha_1, ..., \alpha_p \in \hat{\Gamma}_m$, $1 \leq p \leq m$, then we have*

$$\alpha_1 \wedge \alpha_2 \wedge ... \wedge \alpha_p \wedge \beta^{n-m} \geq 0.$$

Let us set

$$\Sigma_m := \{\alpha \in \hat{\Gamma}_m \text{ of constant coefficients such that } \tilde{S}_m(\alpha) = 1\}.$$

Recall the following elementary lemma whose proof is included for the convenience of the reader.

Lemma 1.3.2. *Let $\alpha \in \hat{\Gamma}_m$. Then the following identity holds*

$$\tilde{S}_m(\alpha)^{1/m} = \inf \left\{ \frac{\alpha \wedge \alpha_1 \wedge ... \wedge \alpha_{m-1} \wedge \beta^{n-m}}{\beta^n}; \alpha_i \in \Sigma_m, \forall i \right\}.$$

Proof. Let M be a polarized form of $\tilde{S}_m$ defined by

$$M(\alpha, \alpha_1, ..., \alpha_{m-1}) = \frac{\alpha \wedge \alpha_1 \wedge ... \wedge \alpha_{m-1} \wedge \beta^{n-m}}{\beta^n},$$

for $\alpha_1, ..., \alpha_{m-1} \in \Sigma_m, \alpha \in \hat{\Gamma}_m$. By Gårding's inequality (1.3.1), we have

$$M(\alpha, \alpha_1, ..., \alpha_{m-1}) \geq \tilde{S}_m(\alpha)^{1/m}.$$

Then we obtain that

$$\tilde{S}_m(\alpha)^{1/m} \leq \inf \left\{ \frac{\alpha \wedge \alpha_1 \wedge ... \wedge \alpha_{m-1} \wedge \beta^{n-m}}{\beta^n}; \alpha_i \in \Sigma_m, \forall i \right\}.$$

Now, setting $\alpha_1 = ... = \alpha_{m-1} = \frac{\alpha}{\tilde{S}_m(\alpha)^{1/m}}$, we can ensure that

$$M(\alpha, \alpha_1, ..., \alpha_{m-1}) = \tilde{S}_m(\alpha)^{1/m}.$$

This completes the proof of lemma. □

Aspects about m-subharmonic functions. Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. Let also $\beta := dd^c|z|^2$ be the standard Kähler form in $\mathbb{C}^n$.

Definition 1.3.3. ([Bł05]). Let u be a subharmonic function in Ω.
1) For smooth case, $u \in \mathcal{C}^2(\Omega)$ is said to be m-subharmonic (briefly m-sh) if the form $dd^c u$ belongs pointwise to $\hat{\Gamma}_m$.
2) For non-smooth case, u is called m-sh if for any collection $\alpha_1, \alpha_2, ..., \alpha_{m-1} \in \hat{\Gamma}_m$, the inequality

$$dd^c u \wedge \alpha_1 \wedge ... \wedge \alpha_{m-1} \wedge \beta^{n-m} \geq 0$$

holds in the weak sense of currents in Ω.

We denote by $SH_m(\Omega)$ the set of all m-sh functions in Ω. Błocki observed that up to a point pluripotential theory can by adapted to m-subharmonic functions. We recall some properties of m-sh functions.

Proposition 1.3.4 ([Bł05]). *1. $PSH = SH_n \subset SH_{n-1} \subset ... \subset SH_1 = SH$.*

2. If $u, v \in SH_m(\Omega)$ then $\lambda u + \eta v \in SH_m(\Omega)$, $\forall \lambda, \eta \geq 0$.

3. If $u \in SH_m(\Omega)$ and $\gamma : \mathbb{R} \to \mathbb{R}$ is convex increasing function then $\gamma \circ u \in SH_m(\Omega)$.

4. Let $\{u_j\}_{j \in \mathbb{N}}$ be a decreasing sequence of m-subharmonic functions in Ω. Then $u := \lim_{j \to +\infty} u_j$ is m-subharmonic function in Ω.

5. If $u \in SH_m(\Omega)$ then the standard regularizations $u_\epsilon = u * \rho_\epsilon$ are m-subharmonic in $\Omega_\epsilon := \{z \in \Omega | \operatorname{dist}(z, \partial\Omega) > \epsilon\}$, for $0 < \epsilon \ll 1$.

6. *Let U be a nonempty proper open subset of Ω. If $u \in SH_m(\Omega)$, $v \in SH_m(U)$, and $\overline{\lim\limits_{\substack{z\to y\\ z\in U}}} v(z) \le u(y)$ for every $y \in \partial U \cap \Omega$, then the function*

$$w = \begin{cases} \max\{u, v\} & \text{in } U, \\ u & \text{in } \Omega \setminus U, \end{cases}$$

is m-sh in Ω.

7. *Let $\{u_\alpha\} \subset SH_m(\Omega)$ be locally uniformly bounded from above and $u = \sup u_\alpha$. Then the upper semi-continuous regularization u^* is m-sh and equal to u almost everywhere.*

The following example was presented by S. Dinew in the international conference in complex analysis and geometry AGC-2013 in Monastir (Tunisia).

Example 1.3.5. Let A be a nonnegative constant and define in $\mathbb{C}^n$ the function

$$u(z) = \frac{-1}{(Im(z_1)^2 + Im(z_2)^2 + ... + Im(z_n)^2)^A}.$$

We claim that u is m-sh in $\mathbb{C}^n$ when $A \le \frac{n-2m}{2m}$ and $m \le \lfloor \frac{n}{2} \rfloor$. In fact, set

$$v_\epsilon(z) = Im(z_1)^2 + Im(z_2)^2 + ... + Im(z_n)^2 + \epsilon,$$

and $\chi : \mathbb{R}^+ \to \mathbb{R}^-$ such that $\chi(t) = -t^{-A}$. An easy computation shows that

$$(dd^c(\chi \circ v_\epsilon))^k \wedge \beta^{n-k} = \frac{k}{2^{k-1}} \chi''(v_\epsilon)(\chi'(v_\epsilon))^{k-1} dv_\epsilon \wedge d^c v_\epsilon \wedge \beta^{n-1} + \frac{(\chi'(v_\epsilon))^k}{2^k} \beta^n.$$

Hence we get

$$(dd^c(\chi \circ v_\epsilon))^k \wedge \beta^{n-k} = \frac{A^k}{2^k}(n-1)! v_\epsilon^{-k(A+1)}(n - 2k(A+1)) dV_{2n}.$$

Then we can conclude that for any $\epsilon > 0$ the function $\chi \circ v_\epsilon$ is m-sh in $\mathbb{C}^n$ if we have $A \le (n-2m)/(2m)$ and $m \le \lfloor n/2 \rfloor$.

Since χ is increasing and v_ϵ decreases as ϵ tends to zero, we get $\chi \circ v_\epsilon \searrow u$ in $\mathbb{C}^n$, thus this yields $u \in SH_m(\mathbb{C}^n)$ when $A \le (n-2m)/(2m)$ and $m \le \lfloor n/2 \rfloor$.

The following example shows that $SH_m(\Omega)$ is not invariant under a holomorphic mapping.

Example 1.3.6. We define the function

$$u(z) = |z_1|^2 + |z_2|^2 - \frac{1}{2}|z_3|^2, \ z \in \mathbb{C}^3.$$

A simple computation shows that $u \in SH_2(\mathbb{C}^3)$ and $u \notin PSH(\mathbb{C}^3)$.

Let f be a holomorphic mapping from $\mathbb{C}^3$ to $\mathbb{C}^3$ such that $f(z) = (z_1, z_2, \sqrt{2} z_3)$. Then it is easy to see that $u \circ f$ is subharmonic but not 2-subharmonic.

For locally bounded m-subharmonic functions, we can inductively define a closed nonnegative current (following Bedford and Taylor for plurisubharmonic functions).

$$dd^c u_1 \wedge ... \wedge dd^c u_p \wedge \beta^{n-m} := dd^c(u_1 dd^c u_2 \wedge ... \wedge dd^c u_p \wedge \beta^{n-m}),$$

where $u_1, u_2, ..., u_p \in SH_m(\Omega) \cap L^\infty_{loc}(\Omega)$, $p \leq m$.
In particular, we define the nonnegative Hessian measure of $u \in SH_m(\Omega) \cap L^\infty_{loc}(\Omega)$ to be

$$H_m(u) := (dd^c u)^m \wedge \beta^{n-m}.$$

We can also use the following identity

$$du \wedge d^c u := (1/2) dd^c (u + C)^2 - (u + C) dd^c u, \text{ where } C \text{ is big enough},$$

to define the nonnegative current

$$du_1 \wedge d^c u_1 \wedge dd^c u_2 \wedge ... \wedge dd^c u_p \wedge \beta^{n-m},$$

where $u_1, ..., u_p \in SH_m(\Omega) \cap L^\infty_{loc}(\Omega)$, $p \leq m$.

One of the most important properties of m-subharmonic functions is the quasicontinuity. Every m-subharmonic function is continuous outside an arbitrarily small open subset. The m-Capacity is used to measure the smallness of these sets.

Definition 1.3.7. Let $E \subset \Omega$ be a Borel subset. The m-capacity of E with respect to Ω is defined to be

$$\mathrm{Cap}_m(E, \Omega) := \sup\Big\{ \int_E (dd^c u)^m \wedge \beta^{n-m}; u \in SH_m(\Omega), -1 \leq u \leq 0 \Big\}.$$

The m-capacity shares the same elementary properties as the capacity introduced by Bedford and Taylor (see [SA12, DK14, Lu15]).

Proposition 1.3.8.

1. *$Cap_m(E_1, \Omega) \leq Cap_m(E_2, \Omega)$, if $E_1 \subset E_2$.*
2. *$Cap_m(E, \Omega) = \lim_{j\to\infty} Cap_m(E_j, \Omega)$, if $E_j \uparrow E$.*
3. *$Cap_m(E, \Omega) \leq \sum Cap_m(E_j, \Omega)$, for $E = \cup E_j$.*

Definition 1.3.9. A sequence u_j of functions defined in Ω is said to converge with respect to Cap_m to a function u if for any $t > 0$ and $K \Subset \Omega$,

$$\lim_{j\to+\infty} \mathrm{Cap}_m(K \cap \{|u - u_j| > t\}, \Omega) = 0.$$

The following results can be proved by repeating the arguments in [Ko05].

Theorem 1.3.10. *Let $(u^j_k)^\infty_{j=1}$, $k = 1, ..., m$ be a locally uniformly bounded sequence of m-sh functions in Ω and $u^j_k \to u_k \in SH_m(\Omega) \cap L^\infty_{loc}(\Omega)$ in Cap_m as $j \to +\infty$ for $k = 1, ..., m$. Then*

$$dd^c u^j_1 \wedge ... \wedge dd^c u^j_m \wedge \beta^{n-m} \rightharpoonup dd^c u_1 \wedge ... \wedge dd^c u_m \wedge \beta^{n-m}.$$

Importantly, the complex Hessian operator is continuous with respect to the decreasing convergence.

Theorem 1.3.11. *If $u_j \in SH_m(\Omega)\cap L^\infty(\Omega)$ is a sequence decreasing to a bounded function u in Ω, then $(dd^c u_j)^m \wedge \beta^{n-m}$ converges to $(dd^c u)^m \wedge \beta^{n-m}$ in the weak sense of currents in Ω.*

Theorem 1.3.12. *Every m-subharmonic function u defined in Ω is quasi-continuous. This means that for any positive number ϵ one can find an open set $U \subset \Omega$ with $Cap_m(U,\Omega) < \epsilon$ and such that $u|_{\Omega\setminus U}$ is continuous.*

Theorem 1.3.13. *Let $\{u_k^j\}_{j=1}^\infty$ be a locally uniformly bounded sequence of m-subharmonic functions in Ω for $k = 1,2,...,m$ and let $u_k^j \uparrow u_k \in SH_m(\Omega)\cap L^\infty_{\mathrm{loc}}$ almost everywhere as $j \to \infty$ for $k = 1,2,...,m$. Then*

$$dd^c u_1^j \wedge ... \wedge dd^c u_m^j \wedge \beta^{n-m} \rightharpoonup dd^c u_1 \wedge ... \wedge dd^c u_m \wedge \beta^{n-m}.$$

Definition 1.3.14. Let Ω be a bounded domain in $\mathbb{C}^n$ and $u \in SH_m(\Omega)$. We say that u is m-maximal if for every open set $G \Subset \Omega$ and for each upper semicontinuous function v on $\bar{G}$ such that $v \in SH_m(G)$ and $v \le u$ on ∂G, we have $v \le u$ in G.

Theorem 1.3.15 ([Bł05])**.** *Let $u \in SH_m(\Omega)\cap L^\infty_{loc}(\Omega)$. Then $H_m(u) = 0$ in Ω if and only if u is m-maximal.*

Theorem 1.3.16 (Integration by parts)**.** *Let $u, v \in SH_m(\Omega)\cap L^\infty_{loc}(\Omega)$ such that $\lim_{z\to\partial\Omega} u = \lim_{z\to\partial\Omega} v = 0$. Then*

$$\int_\Omega u dd^c v \wedge T = \int_\Omega v dd^c u \wedge T,$$

where $T = dd^c u_1 \wedge ... \wedge dd^c u_{m-1} \wedge \beta^{n-m}$ and $u_1,...,u_{m-1} \in SH_m(\Omega)\cap L^\infty_{loc}(\Omega)$.

Theorem 1.3.17. *For $u, v \in SH_m(\Omega)\cap L^\infty_{loc}(\Omega)$, we have*

$$(dd^c \max\{u,v\})^m \wedge \beta^{n-m} \ge \mathbf{1}_{\{u>v\}}(dd^c u)^m \wedge \beta^{n-m} + \mathbf{1}_{\{u\le v\}}(dd^c v)^m \wedge \beta^{n-m},$$

where $\mathbf{1}_E$ is the characteristic function of a set E.

Theorem 1.3.18. *Let Ω be a bounded domain in $\mathbb{C}^n$ and $u, v \in SH_m(\Omega)\cap L^\infty_{loc}(\Omega)$ be such that $\liminf_{\zeta\to\partial\Omega}(u-v)(\zeta) \ge 0$. Then*

$$\int_{\{u<v\}} (dd^c v)^m \wedge \beta^{n-m} \le \int_{\{u<v\}} (dd^c u)^m \wedge \beta^{n-m}.$$

Corollary 1.3.19. *Under the same assumption of Theorem 1.3.18, if $(dd^c u)^m \wedge \beta^{n-m} \le (dd^c v)^m \wedge \beta^{n-m}$ as Radon measures on Ω, then $v \le u$ in Ω.*

Corollary 1.3.20. *Let Ω be a bounded domain in $\mathbb{C}^n$ and $u, v \in SH_m(\Omega)\cap L^\infty_{loc}(\Omega)$ be such that $\lim_{z\to\partial\Omega} u(z) = \lim_{z\to\partial\Omega} v(z)$ and $u \le v$ in Ω. Then*

$$\int_\Omega (dd^c v)^m \wedge \beta^{n-m} \le \int_\Omega (dd^c u)^m \wedge \beta^{n-m}.$$

1.3.1 Cegrell's inequalities for m-subharmonic functions

Let Ω be a bounded m-hyperconvex domain, that is, there exists a bounded continuous m-sh function $\varphi : \Omega \to \mathbb{R}^-$ such that $\{\varphi < c\} \Subset \Omega$, for all $c < 0$.

We recall the definition of the class $\mathcal{E}_m^0(\Omega)$.

Definition 1.3.21. We let $\mathcal{E}_m^0(\Omega)$ denote the class of bounded functions v in $SH_m(\Omega)$ such that $\lim_{z\to\partial\Omega} v(z) = 0$ and $\int_\Omega (dd^c v)^m \wedge \beta^{n-m} < +\infty$.

This class was introduced by Cegrell in [Ce98], for $m = n$, and was considered by Lu in [Lu15].

Lemma 1.3.22. *Let $u, v, v_1, ..., v_{m-1} \in \mathcal{E}_m^0(\Omega)$ and $T = dd^c v_1 \wedge ... \wedge dd^c v_{m-1} \wedge \beta^{n-m}$. Then we have*

$$\int_\Omega (-u) dd^c v \wedge T \leq \left(\int_\Omega (-u) dd^c u \wedge T\right)^{1/2} \left(\int_\Omega (-v) dd^c v \wedge T\right)^{1/2}.$$

Proof. It is enough to note that

$$(u, v) := \int_\Omega (-u) dd^c v \wedge T$$

is symmetric semi positive bilinear form (using integration by parts). the required inequality follows from the classical Cauchy-Schwarz inequality for the form (u, v). □

The following proposition was proved by induction in [Ce04] for plurisubharmonic functions and we can do the same argument for m-sh functions.

Proposition 1.3.23. *Suppose that $h, u_1, u_2 \in \mathcal{E}_m^0(\Omega)$, $p, q \geq 1$ such that $p + q \leq m$ and $T = dd^c g_1 \wedge ... \wedge dd^c g_{m-p-q} \wedge \beta^{n-m}$, where $g_1, ..., g_{m-p-q} \in \mathcal{E}_m^0(\Omega)$. Then we get*

$$\int_\Omega -h(dd^c u_1)^p \wedge (dd^c u_2)^q \wedge T \leq \left[\int_\Omega -h(dd^c u_1)^{p+q} \wedge T\right]^{\frac{p}{p+q}} \left[\int_\Omega -h(dd^c u_2)^{p+q} \wedge T\right]^{\frac{q}{p+q}}.$$

Proof. We first prove the statement for $p = q = 1$. Thanks to the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\int_\Omega -h dd^c u_1 \wedge dd^c u_2 \wedge T &= \int_\Omega -u_1 dd^c u_2 \wedge dd^c h \wedge T \\ &\leq \left[\int_\Omega -u_1 dd^c u_1 \wedge dd^c h \wedge T\right]^{1/2} \left[\int_\Omega -u_2 dd^c u_2 \wedge dd^c h \wedge T\right]^{1/2} \\ &= \left[\int_\Omega -h(dd^c u_1)^2 \wedge T\right]^{1/2} \left[\int_\Omega -h(dd^c u_2)^2 \wedge T\right]^{1/2}.\end{aligned}$$

The general case follows by induction in the same way as in [Ce04]. □

We will need in this thesis the following particular case.

Corollary 1.3.24. *Let $u_1, u_2 \in \mathcal{E}_m^0(\Omega)$. Then we have*

$$\int_\Omega dd^c u_1 \wedge (dd^c u_2)^{m-1} \wedge \beta^{n-m} \leq \left[\int_\Omega (dd^c u_1)^m \wedge \beta^{n-m}\right]^{\frac{1}{m}} \left[\int_\Omega (dd^c u_2)^m \wedge \beta^{n-m}\right]^{\frac{m-1}{m}}.$$

For $m = n$, we have the following result proved by Cegrell [Ce04].

Corollary 1.3.25. *Let $u_1, u_2 \in \mathcal{E}_0(\Omega)$. Then we have*

$$\int_\Omega dd^c u_1 \wedge (dd^c u_2)^{n-1} \leq \left[\int_\Omega (dd^c u_1)^n\right]^{\frac{1}{n}} \left[\int_\Omega (dd^c u_2)^n\right]^{\frac{n-1}{n}}.$$

Chapter 2

Modulus of continuity of the solution to the Dirichlet problem

2.1 Introduction

Let Ω be a bounded domain in $\mathbb{C}^n$. Given $\varphi \in \mathcal{C}(\partial\Omega)$ and $0 \le f \in L^1(\Omega)$, we consider the following Dirichlet problem:

$$Dir(\Omega, \varphi, f) : \begin{cases} u \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega}), & \\ (dd^c u)^n = f\beta^n & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

This problem was studied in the last decades by many authors. When Ω is a bounded strongly pseudoconvex domain with smooth boundary and $f \in \mathcal{C}(\bar{\Omega})$, Bedford and Taylor showed that $Dir(\Omega, \varphi, f)$ has a unique continuous solution $\mathtt{U} := \mathtt{U}(\Omega, \varphi, f)$. Furthermore, it was proved in [BT76] that $\mathtt{U} \in Lip_\alpha(\bar{\Omega})$ when $\varphi \in Lip_{2\alpha}(\partial\Omega)$ and $f^{1/n} \in Lip_\alpha(\bar{\Omega})$ $(0 < \alpha \le 1)$. In the nondegenerate case, i.e. $0 < f \in \mathcal{C}^\infty(\bar{\Omega})$ and $\varphi \in \mathcal{C}^\infty(\partial\Omega)$, Caffarelli, Kohn, Nirenberg and Spruck proved in [CKNS85] that $\mathtt{U} \in \mathcal{C}^\infty(\bar{\Omega})$. However a simple example of Gamelin and Sibony shows that the solution is not, in general, better than $\mathcal{C}^{1,1}$-smooth when $f \ge 0$ and smooth (see [GS80]). Krylov proved that if $\varphi \in \mathcal{C}^{3,1}(\partial\Omega)$ and $f^{1/n} \in \mathcal{C}^{1,1}(\bar{\Omega})$, $f \ge 0$ then $\mathtt{U} \in \mathcal{C}^{1,1}(\bar{\Omega})$ (see [Kr89]).

For B-regular domains, Błocki [Bł96] proved the existence of a continuous solution to the Dirichlet problem $Dir(\Omega, \varphi, f)$ when $0 \le f \in \mathcal{C}(\bar{\Omega})$.

In this chapter which is based on my paper [Ch15a], we consider the more general case where Ω is a bounded strongly hyperconvex Lipschitz domain for which the boundary does not need to be smooth (see the definition below) and we study the existence and regularity of solutions to $Dir(\Omega, \varphi, f)$ when $0 \le f \in \mathcal{C}(\bar{\Omega})$.

The principal result in this chapter gives a sharp estimate for the modulus of continuity of the solution in terms of the modulus of continuity of the data φ, f.

Theorem 2.1.1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded strongly hyperconvex Lipschitz domain, $\varphi \in \mathcal{C}(\partial\Omega)$ and $0 \le f \in \mathcal{C}(\bar{\Omega})$. Assume that ω_φ is the modulus of continuity of φ and $\omega_{f^{1/n}}$ is the modulus of continuity of $f^{1/n}$. Then the modulus of continuity of the unique solution $\mathtt{U}$ to $Dir(\Omega, \varphi, f)$ has the following estimate*

$$\omega_{\mathtt{U}}(t) \le \eta(1 + \|f\|^{1/n}_{L^\infty(\bar{\Omega})}) \max\{\omega_\varphi(t^{1/2}), \omega_{f^{1/n}}(t), t^{1/2}\},$$

where η is a positive constant depending on Ω.

Remark 2.1.2. Here we will use an alternative description of the solution given by Theorem 2.3.2 to get an optimal control for the modulus of continuity of this solution in a strongly hyperconvex Lipschitz domain. This result was suggested by E. Bedford [Be88] and proved in the case of strictly convex domains with $f = 0$ [Be82].

We also consider the case when the density in the Dirichlet problem depends on the unknown function:

$$\begin{cases} u \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega}), & \\ (dd^c u)^n = F(z,u)\beta^n & \text{in} \quad \Omega, \\ u = \varphi & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1.1}$$

where $F : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}^+$ is a continuous function and nondecreasing in the second variable.

We can prove a sharp estimate for the modulus of continuity of the solution to (2.1.1). Since the proof is similar to the one of Theorem 4.1.1 for complex Hessian equations, we do not mention it in this chapter.

Theorem 2.1.3. *Let Ω be a bounded strongly hyperconvex Lipschitz domain in $\mathbb{C}^n$, $\varphi \in \mathcal{C}(\partial\Omega)$ and $0 \le F \in \mathcal{C}(\bar{\Omega} \times \mathbb{R})$ be a nondecreasing function in the second variable. Then there exists a unique continuous solution $\mathtt{U}$ to (2.1.1) and its modulus of continuity satisfies the following estimate*

$$\omega_{\mathtt{U}}(t) \le \gamma(1 + \|F\|_{L^\infty(K)}^{1/n}) \ \max\{\omega_\varphi(t^{1/2}), \omega_{F^{1/n}}(t), t^{1/2}\},$$

where γ is a positive constant depending only on Ω, $K = \bar{\Omega} \times \{a\}$, $a = \sup_{\partial\Omega} |\varphi|$ and $\omega_{F^{1/n}}(t)$ is given by

$$\omega_{F^{1/n}}(t) := \sup_{y \in [-M,M]} \sup_{|z_1 - z_2| \le t} |F^{1/n}(z_1, y) - F^{1/n}(z_2, y)|,$$

with $M := a + 2\,\mathrm{diam}(\Omega)^2 \sup_{\bar{\Omega}} F^{1/n}(., -a)$.

2.2 Basic facts

Definition 2.2.1. A bounded domain $\Omega \subset \mathbb{C}^n$ is called a *strongly hyperconvex Lipschitz* (briefly SHL) domain if there exist a neighborhood Ω' of $\bar{\Omega}$ and a Lipschitz plurisubharmonic defining function $\rho : \Omega' \to \mathbb{R}$ such that

1. $\Omega = \{z \in \Omega'; \rho(z) < 0\}$ and $\partial\Omega = \{\rho = 0\}$,
2. there exists a constant $c > 0$ such that $dd^c \rho \ge c\beta$ in Ω in the weak sense of currents.

Example 2.2.2.

1. Let Ω be a strictly convex domain, that is, there exists a Lipschitz defining function ρ such that $\rho - c|z|^2$ is convex for some $c > 0$. It is clear that Ω is a strongly hyperconvex Lipschitz domain.

2. A smooth strongly pseudoconvex bounded domain is a SHL domain (see [HL84]).

3. The nonempty finite intersection of strongly pseudoconvex bounded domains with smooth boundary in $\mathbb{C}^n$ is a bounded SHL domain. In fact, it is sufficient to set $\rho = \max\{\rho_i\}$. More generally a finite intersection of SHL domains is a SHL domain.

4. The domain

$$\Omega = \{z = (z_1, \cdots, z_n) \in \mathbb{C}^n; |z_1| + \cdots + |z_n| < 1\} \ \ (n \geq 2)$$

is a bounded strongly hyperconvex Lipschitz domain in $\mathbb{C}^n$ with non-smooth boundary.

5. The unit polydisc in $\mathbb{C}^n$ $(n \geq 2)$ is hyperconvex with Lipschitz boundary but it is not strongly hyperconvex Lipschitz.

We recall the definition of B-regular domain in the sense of Sibony ([Sib87], [Bł96]).

Definition 2.2.3. A bounded domain Ω in $\mathbb{C}^n$ is called *B-regular* if for any boundary point $z_0 \in \partial\Omega$ there exists $v \in PSH(\Omega)$ such that $\lim_{z\to z_0} v(z) = 0$ and $v^*|_{\bar{\Omega}\setminus\{z_0\}} < 0$.

Remark 2.2.4. Any bounded SHL domain is B-regular in the sense of Sibony. Indeed, for any boundary point $z_0 \in \partial\Omega$ it is enough to take $v(z) = A\rho - |z - z_0|^2$ where $A > 1/c$ and $c > 0$ is as in Definition 2.2.1.

Remark 2.2.5. Kerzman and Rosay [KR81] proved that in a hyperconvex domain there exists an exhaustion function which is smooth and strictly plurisubharmonic. Furthermore, they proved that any bounded pseudoconvex domain with $\mathcal{C}^1$-boundary is hyperconvex domain. Later, Demailly [De87] proved that any bounded pseudoconvex domain with Lipschitz boundary is hyperconvex. It is obvious that such a domain can contain a germ of analytic subvariety in the boundary, hence it can not be a bounded SHL domain (for example, we smooth out the boundary of a polydisc) since the condition (2) in Definition 2.2.1 fails.

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. If $u \in PSH(\Omega)$ then $dd^c u \geq 0$ in the sense of currents. We define

$$\Delta_H u := \sum_{j,k=1}^{n} h_{j\bar{k}} \frac{\partial^2 u}{\partial z_k \partial \bar{z}_j} = \mathrm{tr}\,(H.\mathrm{Hess}_{\mathbb{C}} u) \tag{2.2.1}$$

for every positive definite Hermitian matrix $H = (h_{j\bar{k}})$. We can view $\Delta_H u$ as a nonnegative Radon measure in Ω.

The following lemma is elementary and important for what follows (see [Gav77]).

Lemma 2.2.6. *Let Q be a $n \times n$ nonnegative Hermitian matrix. Then*

$$(\det Q)^{\frac{1}{n}} = \inf\{\mathrm{tr}(H.Q) : H \in H_n^+ \text{ and } \det(H) = n^{-n}\},$$

where H_n^+ denotes the set of all positive Hermitian $n \times n$ matrices.

Proof. For every matrix $H \in H_n^+$, there is $C \in H_n^+$ such that $C^2 = H$. We set $H^{1/2} := C$, hence $H^{1/2}.Q.H^{1/2} \in H_n^+$. After diagonalizing the matrix $H^{1/2}.Q.H^{1/2}$ and by the inequality of arithmetic and geometric means, we get

$$(\det Q)^{\frac{1}{n}}(\det H)^{\frac{1}{n}} = (\det(H^{1/2}.Q.H^{1/2}))^{\frac{1}{n}} \leq \frac{1}{n}\operatorname{tr}(H^{1/2}.Q.H^{1/2}).$$

Then

$$(\det Q)^{\frac{1}{n}}\,(\det H)^{\frac{1}{n}} \leq \frac{1}{n}\operatorname{tr}(Q.H).$$

Consequently, we have

$$(\det Q)^{\frac{1}{n}} \leq \inf\{\operatorname{tr}(H.Q) : H \in H_n^+ \text{ and } \det(H) = n^{-n}\}.$$

Since $Q \in H_n^+$, we diagonalize it, then we get $A = (\lambda_{ii}) \in H_n^+$ such that $Q = P.A.P^{-1}$ where P is the transformation matrix. One can find a matrix $H = (\alpha_{ii}) \in H_n^+$ such that $\det(H) = n^{-n}$ and $(\det A)^{\frac{1}{n}} = \operatorname{tr}(A.H)$. Indeed, it suffices to set

$$\alpha_{ii} = \frac{(\prod_i \lambda_{ii})^{\frac{1}{n}}}{n\lambda_{ii}}.$$

Finally,

$$(\det Q)^{\frac{1}{n}} = (\det A)^{\frac{1}{n}} = \operatorname{tr}(H.A) = \operatorname{tr}(H.P.A.P^{-1}) = \operatorname{tr}(H.Q).$$

□

Example 2.2.7. We calculate $\Delta_H(|z|^2)$ for every matrix $H \in H_n^+$ and $\det H = n^{-n}$.

$$\Delta_H(|z|^2) = \sum_{j,k=1}^{n} h_{j\bar{k}}.\delta_{k\bar{j}} = \operatorname{tr}(H).$$

Using the inequality of arithmetic and geometric means, we have :

$$1 = (\det I)^{\frac{1}{n}} \leq \operatorname{tr}(H),$$

hence $\Delta_H(|z|^2) \geq 1$ for every matrix $H \in H_n^+$ and $\det(H) = n^{-n}$.

The following result is well known (see [Bł96]), but we will give here an alternative proof using ideas from the theory of viscosity due to Eyssidieux, Guedj and Zeriahi [EGZ11].

Proposition 2.2.8. *Let $u \in PSH(\Omega) \cap L^\infty(\Omega)$ and $0 \leq f \in \mathcal{C}(\Omega)$. Then the following conditions are equivalent:*
(1) $\Delta_H u \geq f^{1/n}$ in the weak sense of distributions, for any $H \in H_n^+$ and $\det H = n^{-n}$.
(2) $(dd^c u)^n \geq f\beta^n$ in the weak sense of currents on Ω.

Proof. First, suppose that $u \in \mathcal{C}^2(\Omega)$. Then by Lemma 2.2.6 the inequality

$$\Delta_H u = \sum_{j,k=1}^{n} h^{j\bar{k}} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \geq f^{1/n}, \forall H \in H_n^+, \det(H) = n^{-n},$$

is equivalent to

$$\left(\det(\frac{\partial^2 u}{\partial z_j \partial \bar{z_k}})\right)^{1/n} \geq f^{1/n}.$$

The latter means that

$$(dd^c u)^n \geq f\beta^n.$$

(1)⇒(2). Let (ρ_ϵ) be the standard family of regularizing kernels with $\operatorname{supp} \rho_\epsilon \subset B(0,\epsilon)$ and $\int_{B(0,\epsilon)} \rho_\epsilon = 1$. Then the sequence $u_\epsilon = u * \rho_\epsilon$ decreases to u, and we see that (1) implies $\Delta_H u_\epsilon \geq (f^{1/n})_\epsilon$. Since u_ϵ is smooth, we use the first case and get $(dd^c u_\epsilon)^n \geq ((f^{1/n})_\epsilon)^n \beta^n$, hence by applying the convergence theorem of Bedford and Taylor (Theorem 7.4 in [BT82]) we obtain $(dd^c u)^n \geq f\beta^n$.

(2)⇒(1). Fix $x_0 \in \Omega$, and let q be a $\mathcal{C}^2$-function in a neighborhood B of x_0 such that $u \leq q$ in this neighborhood and $u(x_0) = q(x_0)$.
First step: We will show that $dd^c q_{x_0} \geq 0$. Indeed, for every small enough ball $B' \subset B$ centered at x_0, we have

$$u(x_0) - q(x_0) \geq \frac{1}{V(B')} \int_{B'} (u-q) dV_{2n},$$

therefore

$$\frac{1}{V(B')} \int_{B'} q dV_{2n} - q(x_0) \geq \frac{1}{V(B')} \int_{B'} u dV_{2n} - u(x_0) \geq 0.$$

Since q is $\mathcal{C}^2$-smooth and the radius of B' tends to 0, it follows from Proposition 3.2.10 in [H94] that $\Delta q_{x_0} \geq 0$. For every positive definite Hermitian matrix H with $\det H = n^{-n}$, we make a linear change of complex coordinates T such that $\operatorname{tr}(HQ) = \operatorname{tr}(\tilde{Q})$ where $\tilde{Q} = (\partial^2 \tilde{q} / \partial w_j \partial \bar{w}_k)$ and $\tilde{q} = q \circ T^{-1}$. Then

$$\Delta_H q(x_0) = \operatorname{tr}(H.Q) = \operatorname{tr}(\tilde{Q}) = \Delta \tilde{q}(y_0).$$

Indeed, we first make a unitary transformation T_1 such that $\operatorname{tr}(H.Q) = \operatorname{tr}(S.Q_1)$ where S is a diagonal matrix with positive eigenvalues $\lambda_1, ..., \lambda_n$ and $Q_1 := (\partial^2 q_1 / \partial x_j \partial \bar{x_k})$ with $q_1 = q \circ T_1^{-1}$. Then we do another linear transformation $T_2 : \mathbb{C}^n \to \mathbb{C}^n$ such that

$$T_2(x_1, ..., x_n) := \left(\frac{x_1}{\sqrt{\lambda_1}}, ..., \frac{x_n}{\sqrt{\lambda_n}} \right).$$

Let us set $\tilde{q} = q_1 \circ T_2^{-1}$. We get that

$$\operatorname{tr}(S.Q_1) = \lambda_1 \frac{\partial^2 q_1}{\partial x_1 \partial \bar{x}_1} + ... + \lambda_n \frac{\partial^2 q_1}{\partial x_n \partial \bar{x}_n} = \frac{\partial^2 \tilde{q}}{\partial w_1 \partial \bar{w}_1} + ... + \frac{\partial^2 \tilde{q}}{\partial w_n \partial \bar{w}_n} = \operatorname{tr}(\tilde{Q}).$$

Hence $\Delta_H q(x_0) \geq 0$ for every $H \in H_n^+$ and $\det H = n^{-n}$, so $dd^c q_{x_0} \geq 0$.
Second step: We claim that $(dd^c q)^n_{x_0} \geq f(x_0)\beta^n$. Suppose that there exists a point $x_0 \in \Omega$ and a $\mathcal{C}^2$-function q which satisfies $u \leq q$ in a neighborhood of x_0 and $u(x_0) = q(x_0)$ such that $(dd^c q)^n_{x_0} < f(x_0)\beta^n$. We put

$$q^\epsilon(x) = q(x) + \epsilon \left(\|x - x_0\|^2 - r^2/2 \right)$$

for $0 < \epsilon \ll 1$ small enough, we see that

$$0 < (dd^c q^\epsilon)^n_{x_0} < f(x_0)\beta^n.$$

Since f is lower semi-continuous on Ω, there exists $r > 0$ such that

$$(dd^c q^\epsilon)^n_x \leq f(x)\beta^n,\ x \in B(x_0, r).$$

Then $(dd^c q^\epsilon)^n \leq f\beta^n \leq (dd^c u)^n$ in $B(x_0, r)$ and $q^\epsilon = q + \epsilon\frac{r^2}{2} \geq q \geq u$ on $\partial B(x_0, r)$, hence $q^\epsilon \geq u$ on $B(x_0, r)$ by the comparison principle. But $q^\epsilon(x_0) = q(x_0) - \epsilon\frac{r^2}{2} = u(x_0) - \epsilon\frac{r^2}{2} < u(x_0)$, a contradiction.

Hence, from the first part of the proof, we get $\Delta_H q(x_0) \geq f^{1/n}(x_0)$ for every point $x_0 \in \Omega$ and every $\mathcal{C}^2$-function q in a neighborhood of x_0 such that $u \leq q$ in this neighborhood and $u(x_0) = q(x_0)$.

Assume that $f > 0$ and $f \in \mathcal{C}^\infty(\Omega)$, then there exists $g \in \mathcal{C}^\infty(\Omega)$ such that $\Delta_H g = f^{1/n}$. Hence $\varphi = u - g$ is Δ_H-subharmonic (by Proposition 3.2.10', [H94]), from which it follows that $\Delta_H \varphi \geq 0$ and $\Delta_H u \geq f^{1/n}$.

In case $f > 0$ is merely continuous, we observe that

$$f = \sup\{w; w \in \mathcal{C}^\infty, f \geq w > 0\},$$

so $(dd^c u)^n \geq f\beta^n \geq w\beta^n$. Since $w > 0$ is smooth, we have $\Delta_H u \geq w^{1/n}$. Therefore, we get $\Delta_H u \geq f^{1/n}$.

In the general case $0 \leq f \in \mathcal{C}(\Omega)$, we observe that $u^\epsilon(z) = u(z) + \epsilon|z|^2$ satisfies

$$(dd^c u^\epsilon)^n \geq (f + \epsilon^n)\beta^n,$$

and so

$$\Delta_H u^\epsilon \geq (f + \epsilon^n)^{1/n}.$$

Letting ϵ converge to 0, we get $\Delta_H u \geq f^{1/n}$ for all $H \in H_n^+$ and $\det H = n^{-n}$. □

As a consequence of Proposition 2.2.8, we give an alternative description of the classical Perron-Bremermann family of subsolutions to the Dirichlet problem $Dir(\Omega, \varphi, f)$.

Definition 2.2.9. We denote by $\mathcal{V}(\Omega, \varphi, f)$ the family of subsolutions of $Dir(\Omega, \varphi, f)$, that is

$$\mathcal{V}(\Omega, \varphi, f) = \{v \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega}), v|_{\partial\Omega} \leq \varphi \text{ and } \Delta_H v \geq f^{1/n}, \forall H \in H_n^+, \det H = n^{-n}\}.$$

Remark 2.2.10. We observe that $\mathcal{V}(\Omega, \varphi, f) \neq \emptyset$. Indeed, let ρ be as in Definition 2.2.1 and $A, B > 0$ big enough, then $A\rho - B \in \mathcal{V}(\Omega, \varphi, f)$.

Furthermore, the family $\mathcal{V}(\Omega, \varphi, f)$ is stable under finite maximum, that is if $u, v \in \mathcal{V}(\Omega, \varphi, f)$ then $\max\{u, v\} \in \mathcal{V}(\Omega, \varphi, f)$. It is enough to show that

$$\Delta_H(\max\{u, v\}) \geq \min\{\Delta_H u, \Delta_H v\} \tag{2.2.2}$$

We set $\mu := \min\{\Delta_H u, \Delta_H v\}$ and suppose that $\mu(\{z; u(z) = v(z)\}) = 0$. Then in the open set $\Omega_1 = \{u < v\}$, we have $\Delta_H(\max\{u, v\}) = \Delta_H v \geq \mu$, and a similar consequence in the

set $\Omega_2 = \{v < u\}$.
Since $\mu(\Omega \setminus (\Omega_1 \cup \Omega_2)) = 0$ and $\Delta_H(\max\{u, v\}) \geq 0$, we get $\Delta_H(\max\{u, v\}) \geq \min\{\Delta_H u, \Delta_H v\}$.
In the general case, we replace v by $v + \epsilon$, where $\epsilon > 0$ is a small constant, then $\max\{u, v + \epsilon\} \to \max\{u, v\}$. Thus $\Delta_H(\max\{u, v + \epsilon\})$ converges to $\Delta_H(\max\{u, v\})$ in the sense of distributions.
We set $\mu = \min\{\Delta_H u, \Delta_H(v+\epsilon)\}$, by the first case the inequality is true for $\max\{u, v+\epsilon\}$ for all $\epsilon > 0$ such that $\mu(\{z; u(z) = v(z) + \epsilon\}) = 0$. On the other hand, $\mu(\{z; u(z) = v(z) + \epsilon\}) = 0$ for all $\epsilon > 0$ except at most countably many $\epsilon > 0$, then we obtain (2.2.2) by passing to the limit when $\epsilon \to 0$ (avoiding these countably many values of $\epsilon > 0$).

2.3 The Perron-Bremermann envelope

Bedford and Taylor proved in [BT76] that the unique solution to $Dir(\Omega, \varphi, f)$ in a bounded strongly pseudoconvex domain with smooth boundary, is given as the *Perron-Bremermann envelope*

$$u = \sup\{v; v \in \mathcal{B}(\Omega, \varphi, f)\},$$

where $\mathcal{B}(\Omega, \varphi, f) = \{v \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega}) : v|_{\partial\Omega} \leq \varphi \text{ and } (dd^c v)^n \geq f\beta^n\}$.
Thanks to Proposition 2.2.8, we get the following corollary

Corollary 2.3.1. *The two families $\mathcal{V}(\Omega, \varphi, f)$ and $\mathcal{B}(\Omega, \varphi, f)$ coincide, that is*

$$\mathcal{V}(\Omega, \varphi, f) = \mathcal{B}(\Omega, \varphi, f).$$

The context of this section is classical and follows the main scheme of Bedford and Taylor's approach. A simplification of their proof was given by Demailly (for the homogeneous case ([De89])) and by Błocki for the general case (see [Bł96]). Here we will prove the following theorem using an alternative description of the Perron-Bremermann envelope in a bounded SHL domain.

Theorem 2.3.2. *Let $\Omega \subset \mathbb{C}^n$ be a bounded SHL domain, $0 \leq f \in \mathcal{C}(\bar{\Omega})$ and $\varphi \in \mathcal{C}(\partial\Omega)$. Then the Dirichlet problem $Dir(\Omega, \varphi, f)$ has a unique solution $\mathtt{U}$. Moreover the solution is given by*

$$\mathtt{U} = \sup\{v; v \in \mathcal{V}(\Omega, \varphi, f)\},$$

where $\mathcal{V}$ is defined in Definition 2.2.9 and Δ_H is the Laplacian associated to a positive definite Hermitian matrix H as in (2.2.1).

The uniqueness of the solution to $Dir(\Omega, \varphi, f)$ is a consequence of the comparison principle (Corollary 1.2.7).

The first step to prove this theorem is to ensure that $\mathtt{U} \in \mathcal{V}(\Omega, \varphi, f)$. For this purpose, we use the argument of Walsh (see [Wal69] and [Bł96]) to prove the continuity of the upper envelope.

2.3.1 Continuity of the upper envelope

Proposition 2.3.3. *Let $\Omega \subset \mathbb{C}^n$ be a bounded SHL domain, $0 \leq f \in \mathcal{C}(\bar{\Omega})$ and $\varphi \in \mathcal{C}(\partial\Omega)$. Then the upper envelope*

$$\mathtt{U} = \sup\{v;\ v \in \mathcal{V}(\Omega, \varphi, f)\}$$

belongs to $\mathcal{V}(\Omega, \varphi, f)$ and $\mathtt{U} = \varphi$ on $\partial\Omega$.

Proof. Let $g \in \mathcal{C}^2(\bar{\Omega})$ be an approximation of φ such that $|g - \varphi| \leq \epsilon$ on $\partial\Omega$, for fixed $\epsilon > 0$. Let also ρ be the defining function as in Definition 2.2.1 and $A > 0$ large enough such that $v_0 := A\rho + g - \epsilon$ belongs to $\mathcal{V}(\Omega, \varphi, f)$ and $\Delta_H v_0 \geq \max\{\sup_{\bar{\Omega}} f^{1/n}, 1\}$.

A similar construction gives that $v_1 := -B\rho + g + \epsilon$ is plurisuperharmonic in Ω when $B > 0$ is big enough. We claim that $\mathtt{U} \leq v_1$ in Ω. Suppose that $v \in \mathcal{V}(\Omega, \varphi, f)$, then $v - v_1 \leq \varphi - g - \epsilon \leq 0$ on $\partial\Omega$. Hence, by the maximum principle we get $v - v_1 \leq 0$ in Ω. This yields $\mathtt{U} \leq v_1$ in Ω. Consequently, we get $v_0 \leq \mathtt{U} \leq v_1$. Then on the boundary $\partial\Omega$ we have

$$\varphi - 2\epsilon \leq g - \epsilon \leq \mathtt{U} \leq g + \epsilon \leq \varphi + 2\epsilon.$$

Letting ϵ tend to 0, we obtain that $\mathtt{U} = \varphi$ on $\partial\Omega$ and $\lim_{z\to\xi} \mathtt{U}(z) = \varphi(\xi)$ for all $\xi \in \partial\Omega$.

We will prove that $\mathtt{U}$ is continuous on Ω. Fix $\epsilon > 0$ and z_0 in a compact set $K \Subset \Omega$. Thanks to the continuity of v_1 and v_0 on $\bar{\Omega}$, one can find $\delta > 0$ such that for any $z_1, z_2 \in \bar{\Omega}$ we have

$$|v_1(z_1) - v_1(z_2)| \leq \epsilon,\ |v_0(z_1) - v_0(z_2)| \leq \epsilon, \text{ if } |z_1 - z_2| \leq \delta.$$

Let $a \in \mathbb{C}^n$ such that $|a| < \min\{\delta, \operatorname{dist}(K, \partial\Omega)\}$. Since $\mathtt{U}$ is the upper envelope, we can find $\tilde{v} \in \mathcal{V}(\Omega, \varphi, f)$ such that $\tilde{v}(z_0 + a) \geq \mathtt{U}(z_0 + a) - \epsilon$. Let us set $v = \max\{\tilde{v}, v_0\}$.
Hence, for all $z \in \bar{\Omega}$ and $w \in \partial\Omega$ such that $|z - w| \leq \delta$ we get

$$-3\epsilon \leq v_0(z) - \varphi(w) \leq v(z) - \varphi(w) \leq v_1(z) - \varphi(w) \leq 3\epsilon.$$

This implies that

$$|v(z) - \varphi(w)| \leq 3\epsilon, \text{ if } |z - w| \leq \delta. \tag{2.3.1}$$

Then for $z \in \Omega$ and $z + a \in \partial\Omega$, we have

$$v(z + a) \leq \varphi(z + a) \leq v(z) + 3\epsilon.$$

We define the following function

$$v_1(z) = \begin{cases} v(z) & ; z + a \notin \bar{\Omega}, \\ \max\{v(z), v(z + a) - 3\epsilon\} & ; z + a \in \bar{\Omega}, \end{cases}$$

which is well defined, plurisubharmonic on Ω, continuous on $\bar{\Omega}$ and $v_1 \leq \varphi$ on $\partial\Omega$. Indeed, if $z \in \partial\Omega$, $z+a \notin \bar{\Omega}$ then $v_1(z) = v(z) \leq \varphi(z)$. On the other hand, if $z \in \partial\Omega$ and $z+a \in \bar{\Omega}$ then we have, from (2.3.1), that $v(z+a) - 3\epsilon \leq \varphi(z)$, so $v_1(z) = \max\{v(z), v(z+a) - 3\epsilon\} \leq \varphi(z)$. Moreover, we note by (2.2.2) that

$$\Delta_H v_1(z) \geq \min(f^{1/n}(z), f^{1/n}(z + a)) \text{ if } z, z + a \in \Omega.$$

Let ω be the modulus of continuity of $f^{1/n}$. Then we conclude that

$$\Delta_H v_1(z) \geq f^{1/n}(z) - \omega(|a|) \text{ in } \Omega. \tag{2.3.2}$$

Now, let us define

$$v_2 = v_1 + \omega(|a|)(v_0 - \|v_0\|_{L^\infty(\bar{\Omega})}).$$

It is clear that $v_2 \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega})$ and $v_2 \leq \varphi$ on $\partial\Omega$. Furthermore, using (2.3.2) we see that

$$\Delta_H v_2 = \Delta_H v_1 + \omega(|a|)\Delta_H v_0 \geq f^{1/n}.$$

This yields that $v_2 \in \mathcal{V}(\Omega, \varphi, f)$.
For small enough $|a|$ we can assume $\omega(|a|) \leq \epsilon/\|v_0\|$ and infer that

$$\begin{aligned} \mathtt{U}(z_0) &\geq v_1(z_0) + \omega(|a|)v_0(z_0) - \omega(|a|)\|v_0\| \\ &\geq v(z_0 + a) - 5\epsilon \\ &\geq \mathtt{U}(z_0 + a) - 6\epsilon. \end{aligned}$$

The last inequality is true for every $z_0 \in K$, then $\mathtt{U}$ is continuous on Ω.

It follows from Choquet's lemma that there exists a sequence (u_j) in $\mathcal{V}(\Omega, \varphi, f)$ such that

$$\mathtt{U} = (\sup_j u_j)^*.$$

As the family $\mathcal{V}(\Omega, \varphi, f)$ is stable under the operation maximum, we can assume that the sequence (u_j) is increasing almost everywhere to $\mathtt{U}$, then $u_j \to \mathtt{U}$ in $L^1(\Omega)$. Hence $\Delta_H \mathtt{U} = \lim \Delta_H u_j \geq f^{1/n}$ for all $H \in H_n^+$, $detH = n^{-n}$, this implies $\mathtt{U} \in \mathcal{V}(\Omega, \varphi, f)$. □

In order to verify that $(dd^c\mathtt{U})^n = f\beta^n$ in Ω, we first ensure this statement when $\Omega = \mathbb{B}$ the unit ball in $\mathbb{C}^n$. For this end, we introduce the following theorem, which is due to Bedford and Taylor [BT76], to prove that the second order derivatives of $\mathtt{U}$ are locally bounded under extra assumptions. Here the presentation is derived from Demailly [De89].

2.3.2 Regularity in the case of the unit ball

Theorem 2.3.4. *Suppose that $\Omega = \mathbb{B}$ is the unit ball in $\mathbb{C}^n$, $f^{1/n} \in \mathcal{C}^{1,1}(\bar{\mathbb{B}})$ and $\varphi \in \mathcal{C}^{1,1}(\partial\mathbb{B})$. Then the second order partial derivatives of $\mathtt{U}$ are locally bounded, in particular $\mathtt{U} \in \mathcal{C}^{1,1}_{loc}(\mathbb{B})$.*

Proof. First, we assert that $\mathtt{U} \in \mathcal{C}^{0,1}(\bar{\mathbb{B}})$. Actually, let $\tilde{\varphi}$ be a $\mathcal{C}^{1,1}$-extension of φ to $\bar{\mathbb{B}}_2 := \bar{B}(0,2)$ such that

$$\|\tilde{\varphi}\|_{\mathcal{C}^{1,1}(\bar{\mathbb{B}}_2)} \leq C\|\varphi\|_{\mathcal{C}^{1,1}(\partial\mathbb{B})}$$

for some positive constant C (see [GT01]).
Let us set $A \gg 1$ such that $u_1 = A(|z|^2 - 1) + \tilde{\varphi}$ is plurisubharmonic on $\mathbb{B}_2$ and $u_2 = A(1 - |z|^2) + \tilde{\varphi}$ is plurisuperharmonic on $\mathbb{B}_2$. For A big enough, one can note that $u_1 \leq \mathtt{U} \leq u_2$ on $\bar{\mathbb{B}}$ by the comparison principle. We set

$$\tilde{u}(z) = \begin{cases} \mathtt{U}(z) & ; z \in \mathbb{B}, \\ u_1(z) & ; z \in \bar{\mathbb{B}}_2 \setminus \mathbb{B}. \end{cases}$$

Since $u_1 = \mathtt{U} = \varphi$ on $\partial\mathbb{B}$, we get a well defined plurisubharmonic function $\tilde{u}$ on $\mathbb{B}_2$ and $\tilde{u} \leq \max\{u_1, u_2\}$ on $\bar{\mathbb{B}}_2$. Then for all $z \in \partial\mathbb{B}$ and $|h|$ small we get

$$\begin{aligned}\tilde{u}(z+h) &\leq \varphi(z) + C_1 \max\{\|u_1\|_{\mathcal{C}^1(\bar{\mathbb{B}}_2)}, \|u_2\|_{\mathcal{C}^1(\bar{\mathbb{B}}_2)}\}|h| \\ &\leq \varphi(z) + C_2|h|,\end{aligned}$$

where $C_2 = C_1(A + C\|\varphi\|_{\mathcal{C}^{1,1}(\partial\mathbb{B})})$.
Since $f^{1/n} \in \mathcal{C}^{1,1}(\bar{\mathbb{B}})$, there exists a constant B such that

$$|f^{1/n}(z) - f^{1/n}(y)| \leq B|z-y|.$$

Now, let us define the function

$$\hat{u}(z) = \tilde{u}(z+h) - C_2|h| + B|h|(|z|^2 - 1).$$

It is clear that $\hat{u} \in PSH(\mathbb{B}) \cap \mathcal{C}(\bar{\mathbb{B}})$, $\hat{u}|_{\partial\mathbb{B}} \leq \varphi$ and $\Delta_H \hat{u} \geq f^{1/n}$ for all $H \in H_n^+$ and $\det H = n^{-n}$. Thus we have $\hat{u} \in \mathcal{V}(\mathbb{B}, \varphi, f)$ and $\hat{u} \leq \mathtt{U}$ on $\bar{\mathbb{B}}$.
This implies that

$$\tilde{u}(z+h) - \mathtt{U}(z) \leq (C_2 + B)|h| \text{ on } \bar{\mathbb{B}}.$$

By changing h into $-h$, we conclude that

$$|\mathtt{U}(z+h) - \mathtt{U}(z)| \leq (C_2 + B)|h|,$$

for $z \in \mathbb{B}$ and $|h|$ small. This yields that $\|\mathtt{U}\|_{\mathcal{C}^{0,1}(\bar{\mathbb{B}})} \leq (C_2 + B)$.

Second step, we estimate the following expression

$$\mathtt{U}(z+h) + \mathtt{U}(z-h) - 2\mathtt{U}(z).$$

But this expression is not defined in the whole ball $\mathbb{B}$, thus we use the automorphism of the unit ball. For $a \in \mathbb{B}$, we define a holomorphic automorphism T_a of the unit ball as follows;

$$T_a(z) = \frac{P_a(z) - a + \sqrt{1-|a|^2}(z - P_a(z))}{1 - \langle z, a\rangle} \quad ; \quad P_a(z) = \frac{\langle z, a\rangle a}{|a|^2},$$

where $\langle ., . \rangle$ denote the Hermitian product in $\mathbb{C}^n$.
Let $h = a - \langle z, a\rangle z$. Then we get for $|a| \ll 1$ that

$$T_a(z) = z - h + O(|a|^2),$$

where $O(|a|^2)$ is bounded and converges to 0 when $|a|$ tends to 0, i.e. $O(|a|^2) \leq c|a|^2$, for some positive constant c which is uniform for $z \in \bar{\mathbb{B}}$.
The determinant of Jacobian matrix of T_a is given by

$$\det T_a'(z) = 1 + (n+1)\langle z, a\rangle + O(|a|^2).$$

Then

$$\left(\det T_a'(z)\right)^{2/n} = 1 + \frac{2(n+1)}{n}\langle z, a\rangle + O(|a|^2).$$

Let $g \in \mathcal{C}^{0,1}(\bar{\mathbb{B}})$, so it is easy to see that

$$|g \circ T_a(z) - g(z-h)| \leq \|g\|_{\mathcal{C}^{0,1}(\bar{\mathbb{B}})}.|T_a(z) - z + h| \leq c_1 \|g\|_{\mathcal{C}^{0,1}(\bar{\mathbb{B}})}.|a|^2. \tag{2.3.3}$$

Since $f^{1/n} \in \mathcal{C}^{1,1}(\bar{\mathbb{B}})$, we get by Taylor's expansion

$$f^{1/n} \circ T_a(z) = f^{1/n}(z - h + O(|a|^2)) = f^{1/n}(z) - Df^{1/n}(z).h + O(|a|^2).$$

We set $\psi(z,a) = -Df^{1/n}(z).h$, then

$$f^{1/n} \circ T_a(z) = f^{1/n}(z) + \psi(z,a) + O(|a|^2).$$

A simple computation yields that the following expression

$$I := |\det T'_a|^{2/n}(f^{1/n} \circ T_a) + |\det T'_{-a}|^{2/n}(f^{1/n} \circ T_{-a}),$$

can be estimated as follows

$$I \geq 2f^{1/n} - \frac{4(n+1)}{n}|\langle z,a\rangle \psi(z,a)| - c_2|a|^2.$$

There exists $c_3 > 0$ depending on $\|f^{1/n}\|_{\mathcal{C}^{1,1}(\bar{\Omega})}$ such that

$$|\langle z,a\rangle \psi(z,a)| \leq c_3|z|.|a|^2 \leq c_3|a|^2.$$

Hence we get

$$|\det T'_a|^{2/n}(f^{1/n} \circ T_a) + |\det T'_{-a}|^{2/n}(f^{1/n} \circ T_{-a}) \geq 2f^{1/n} - c_4|a|^2.$$

A similar computation yields that the following inequality holds on $\partial\mathbb{B}$

$$\varphi \circ T_a + \varphi \circ T_{-a} \leq 2\varphi + c_4|a|^2, \tag{2.3.4}$$

where c_4 is large and depending also on $\|\varphi\|_{\mathcal{C}^{1,1}(\partial\mathbb{B})}$.
Let us consider

$$v_a(z) := (\mathtt{U} \circ T_a + \mathtt{U} \circ T_{-a})(z).$$

We observe that

$$\Delta_H(\mathtt{U} \circ T_a) \geq |\det T'_a|^{2/n}(f^{1/n} \circ T_a),$$

then we get

$$\Delta_H v_a \geq |\det T'_a|^{2/n}(f^{1/n} \circ T_a) + |\det T'_{-a}|^{2/n}(f^{1/n} \circ T_{-a}) \geq 2f^{1/n} - c_4|a|^2.$$

Let us put

$$v(z) := \frac{1}{2}v_a(z) - \frac{c_4}{2}|a|^2(2 - |z|^2) \in PSH(\mathbb{B}) \cap \mathcal{C}(\bar{\mathbb{B}}).$$

It follows from (2.3.4) that $v \leq \varphi$ on $\partial\mathbb{B}$. Moreover, we have

$$\Delta_H v = \frac{1}{2}\Delta_H v_a + \frac{c_4}{2}|a|^2 \Delta_H(|z|^2) \geq f^{1/n} - \frac{c_4}{2}|a|^2 + \frac{c_4}{2}|a|^2 \geq f^{1/n},$$

for every $H \in H_n^+, \det H = n^{-n}$. Hence $v \in \mathcal{V}(\mathbb{B}, \varphi, f)$, in particular $v \leq \mathtt{U}$. Consequently,

$$\frac{1}{2}v_a(z) - c_4|a|^2 \leq \frac{1}{2}v_a(z) - \frac{c_4}{2}|a|^2(2 - |z|^2) \leq \mathtt{U}.$$

Hence, we get

$$(\mathtt{U} \circ T_a + \mathtt{U} \circ T_{-a})(z) - 2\mathtt{U}(z) \leq 2c_4|a|^2.$$

Applying (2.3.3) with $g = \mathtt{U}$, we obtain

$$\begin{aligned} \mathtt{U}(z-h) + \mathtt{U}(z+h) - 2\mathtt{U}(z) &\leq (\mathtt{U} \circ T_a + \mathtt{U} \circ T_{-a})(z) - 2\mathtt{U}(z) + 2c_1\|\mathtt{U}\|_{\mathcal{C}^{0,1}(\bar{\mathbb{B}})}.|a|^2 \\ &\leq (2c_4 + 2c_1\|\mathtt{U}\|_{\mathcal{C}^{0,1}(\bar{\mathbb{B}})}).|a|^2 \\ &\leq c_5|a|^2. \end{aligned} \tag{2.3.5}$$

Since $h = a - \langle z, a\rangle z$, the inverse linear map $h \mapsto a$ has a norm less than $1/(1 - |z|^2)$. Indeed, using the Cauchy-Schwarz inequality, we have

$$|h| \geq ||a| - |\langle z, a\rangle|.|z|| \geq ||a| - |z|^2|a|| \geq |a|(1 - |z|^2).$$

Thus we conclude that

$$\mathtt{U}(z+h) + \mathtt{U}(z-h) - 2\mathtt{U}(z) \leq \frac{c_5}{(1-|z|^2)^2}|h|^2.$$

Let us fix a compact $K \subset \mathbb{B}$. For $z \in K$ and $|h|$ small enough we obtain by taking a convolution with a regularizing kernel ρ_ϵ, for small enough $\epsilon > 0$, that

$$\mathtt{U}_\epsilon(z+h) + \mathtt{U}_\epsilon(z-h) - 2\mathtt{U}_\epsilon(z) \leq \frac{c_5}{(1-(|z|+\epsilon)^2)^2}|h|^2.$$

Since $\mathtt{U}_\epsilon \in PSH \cap \mathcal{C}^\infty(\mathbb{B}_\epsilon)$ where $\mathbb{B}_\epsilon$ is the ball of radius $1 - \epsilon$ and thanks to Taylor's expansion of degree two of u_ϵ, we infer

$$D^2\mathtt{U}_\epsilon(z).h^2 \leq \frac{c_5}{(1-(|z|+\epsilon)^2)^2}|h|^2.$$

Let us set

$$A := \frac{2c_5}{\mathrm{dist}(K, \partial\mathbb{B})^2}.$$

Then for all $z \in K$ and $h \in \mathbb{C}^n$ with small enough norm we get

$$D^2\mathtt{U}_\epsilon(z).h^2 \leq A|h|^2.$$

The plurisubharmonicity of $\mathtt{U}_\epsilon$ yields

$$D^2\mathtt{U}_\epsilon(z).h^2 + D^2\mathtt{U}_\epsilon(z).(ih)^2 = 4\sum_{j,k}\frac{\partial^2\mathtt{U}_\epsilon}{\partial z_j\partial\bar{z}_k}.h_j\bar{h_k} \geq 0.$$

Hence

$$D^2\mathtt{U}_\epsilon(z).h^2 \geq -D^2\mathtt{U}_\epsilon(z).(ih)^2 \geq -A|h|^2.$$

Therefore, we have
$$|D^2\mathtt{U}_\epsilon(z)| \leq A; \forall z \in K.$$
We know that the dual space of $L^1(K)$ is $L^\infty(K)$, hence by applying the Alaoglu-Banach theorem, there exists a bounded function g such that $D^2\mathtt{U}_\epsilon$ converges weakly to g in $L^\infty(K)$. On the other hand, $D^2\mathtt{U}_\epsilon \to D^2\mathtt{U}$ in the sense of distributions, then we get $D^2\mathtt{U} = g$. Finally, the second order derivatives of $\mathtt{U}$ exist almost everywhere and are locally bounded in $\mathbb{B}$ with
$$\|D^2\mathtt{U}\|_{L^\infty(K)} \leq A,$$
where $A = 2c_5/\operatorname{dist}(K, \partial\mathbb{B})^2$ and c_5 depends on $\|\mathtt{U}\|_{\mathcal{C}^{0,1}(\bar{\mathbb{B}})}$, $\|\varphi\|_{\mathcal{C}^{1,1}(\partial\mathbb{B})}$ and $\|f^{1/n}\|_{\mathcal{C}^{1,1}(\bar{\mathbb{B}})}$. Thus we conclude that $\mathtt{U} \in \mathcal{C}^{1,1}_{loc}(\mathbb{B})$. □

Remark 2.3.5. Dufresnoy [Du89] proved that the $\mathcal{C}^{1,1}$-norm of $\mathtt{U}$ does not explode faster than $1/\operatorname{dist}(., \partial\mathbb{B})$ as we approach to the boundary. In general, $\mathtt{U}$ can not belong to $\mathcal{C}^{1,1}(\bar{\mathbb{B}})$, the next example shows that there is a necessary loss in the regularity up to the boundary.

Example 2.3.6. Let $\mathbb{B} \subset \mathbb{C}^2$ and $\varphi(z, w) = (1 + Re(w))^{1+\epsilon} \in \mathcal{C}^{2,2\epsilon}(\partial\mathbb{B})$ for small $\epsilon > 0$. We consider the following Dirichlet problem:
$$\begin{cases} u \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega}), & \\ (dd^c\mathtt{U})^2 = 0 & \text{in } \mathbb{B}, \\ \mathtt{U} = \varphi & \text{on } \partial\mathbb{B}. \end{cases}$$
Then $\mathtt{U}(z, w) = (1 + Re(w))^{1+\epsilon}$ is the solution to this problem. One can observe that $\mathtt{U}$ belongs to $\mathcal{C}^{1,\epsilon}(\bar{\mathbb{B}}) \cap \mathcal{C}^{1,1}_{loc}(\mathbb{B})$ but it is not $\mathcal{C}^{1,1}$-smooth on $\bar{\mathbb{B}}$. This can be seen by a radial approach to the boundary point $(z_0, w_0) = (0, -1)$.

We will prove in the following proposition that the Perron-Bremermann envelope is the solution to the Dirichlet problem in the unit ball $\mathbb{B}$.

Proposition 2.3.7. *Suppose* $0 \leq f^{1/n} \in \mathcal{C}^{1,1}(\bar{\mathbb{B}})$ *and* $\varphi \in \mathcal{C}^{1,1}(\partial\mathbb{B})$. *Then the envelope* $\mathtt{U}$ *is the solution to the Dirichlet problem* $Dir(\mathbb{B}, \varphi, f)$.

Proof. We have proved that $\mathtt{U} \in \mathcal{C}^{1,1}_{loc}(\mathbb{B})$ and $\mathtt{U} \in \mathcal{V}(\mathbb{B}, \varphi, f)$. It remains to show that $(dd^c\mathtt{U})^n = f\beta^n$. Proof by contradiction, suppose that there exists a point $z_0 \in \mathbb{B}$ at which $\mathtt{U}$ has second order partial derivatives and satisfies
$$(dd^c\mathtt{U})^n(z_0) > (f(z_0) + \epsilon)\beta^n,$$
for some $\epsilon > 0$. Then by Proposition 2.2.8 we have
$$\Delta_H\mathtt{U}(z_0) > (f(z_0) + \epsilon)^{1/n},$$
for all $H \in H_n^+$ and $\det(H) = n^{-n}$.
Using the Taylor expansion at z_0, we get
$$\begin{aligned} \mathtt{U}(z_0 + \xi) = \mathtt{U}(z_0) + D\mathtt{U}(z_0).\xi + \frac{1}{2}\sum_{j,k} \frac{\partial^2\mathtt{U}}{\partial z_j \partial z_k}(z_0)\xi_j\xi_k + \\ + \frac{1}{2}\sum_{j,k} \frac{\partial^2\mathtt{U}}{\partial \bar{z}_j \partial \bar{z}_k}(z_0)\bar{\xi}_j\bar{\xi}_k + \sum_{j,k} \frac{\partial^2\mathtt{U}}{\partial z_j \partial \bar{z}_k}(z_0)\xi_j\bar{\xi}_k + o(|\xi|^2). \\ = \mathtt{U}(z_0) + ReP(\xi) + L(\xi) + o(|\xi|^2), \end{aligned}$$

where P is a complex polynomial of degree 2, then ReP is pluriharmonic and

$$L(\xi) = \sum_{j,k} \frac{\partial^2 \mathtt{U}}{\partial z_j \partial \bar{z_k}}(z_0)\xi_j \bar{\xi_k} > 0.$$

Let us fix

$$s := \left(\frac{f(z_0) + \epsilon/2}{f(z_0) + \epsilon} \right)^{1/n} < 1.$$

One can find $\delta, r > 0$ small enough such that $B(z_0, r) \Subset \mathbb{B}$ and for $|\xi| = r$, we have

$$\mathtt{U}(z_0) + ReP(\xi) + sL(\xi) + \delta \leq \mathtt{U}(z_0 + \xi).$$

We define the function

$$v(z) = \begin{cases} \mathtt{U}(z) & ; z \notin B(z_0, r), \\ \max\{\mathtt{U}(z), v_1(z)\} & ; z \in B(z_0, r), \end{cases}$$

where $v_1(z) := \mathtt{U}(z_0) + ReP(z - z_0) + sL(z - z_0) + \delta$ is a psh function in $B(z_0, r)$. It is clear that v is well defined psh in $\mathbb{B}$ and satisfies $v = \varphi$ on $\partial\mathbb{B}$. We claim that $\Delta_H v \geq f^{1/n}$ for all $H \in H_n^+$ and $\det(H) = n^{-n}$. Indeed, in the ball $B(z_0, r)$ we note

$$\Delta_H v_1 \geq s \Delta_H L(z - z_0) = s \sum_{j,k} \frac{\partial^2 \mathtt{U}}{\partial z_j \partial \bar{z_k}}(z_0) h_{k\bar{j}} > s(f(z_0) + \epsilon)^{1/n} = (f(z_0) + \epsilon/2)^{1/n}.$$

Since f is uniformly continuous in $\bar{\mathbb{B}}$, shrinking r if necessary, we can get that $f(z_0) + \epsilon/2 \geq f(z)$ for $z \in B(z_0, r)$, hence $\Delta_H v_1(z) \geq f^{1/n}(z)$ in $B(z_0, r)$. Consequently, it follows from (2.2.2) that $\Delta_H v \geq f^{1/n}$. Thus we infer $v \in \mathcal{V}(\mathbb{B}, \varphi, f)$ and $v \leq \mathtt{U}$ in $\mathbb{B}$. But we observe that $v(z_0) = \mathtt{U}(z_0) + \delta > \mathtt{U}(z_0)$, this is a contradiction. □

Corollary 2.3.8. *Let $\mathbb{B}$ be the unit ball in $\mathbb{C}^n$, $0 \leq f \in \mathcal{C}(\bar{\mathbb{B}})$ and $\varphi \in \mathcal{C}(\partial\mathbb{B})$. Then the upper envelope $\mathtt{U}$ is the solution to Dirichlet problem $Dir(\mathbb{B}, \varphi, f)$.*

Proof. We choose a sequence of functions (f_j) such that $0 < f_j \in \mathcal{C}^\infty(\bar{\mathbb{B}})$ and f_j decreases to f uniformly on $\bar{\mathbb{B}}$. We also find a sequence $\mathcal{C}^\infty$-smooth functions φ_j such that φ_j increases to φ uniformly on $\partial\mathbb{B}$. Thanks to the last proposition, there exists a continuous solution $\mathtt{U}_j$ to the Dirichlet problem $Dir(\mathbb{B}, \varphi_j, f_j)$. Hence, by the comparison principle, we can conclude that the sequence $\mathtt{U}_j$ is increasing.
Fix $\epsilon > 0$ and since f_k converges uniformly to f, we find $j_0 > 0$ such that $f_j \leq f_k + \epsilon^n$ in $\bar{\mathbb{B}}$ for all $k \geq j \geq j_0$. Then we note for all $k \geq j \geq j_0$ that

$$(dd^c(\mathtt{U}_k + \epsilon(|z|^2 - 1)))^n \geq (f_k + \epsilon^n)\beta^n \geq f_j \beta^n = (dd^c \mathtt{U}_j)^n.$$

Moreover, we can find j_1 large enough such that $\varphi_j + \epsilon \geq \varphi_k$ on $\partial\mathbb{B}$ for all $k \geq j \geq j_1$. Then for $k \geq j \geq \max\{j_0, j_1\}$ we have

$$(dd^c(\mathtt{U}_k + \epsilon(|z|^2 - 1)))^n \geq (dd^c \mathtt{U}_j)^n \text{ in } \mathbb{B},$$

and

$$\mathtt{U}_k + \epsilon(|z|^2 - 1) \leq \mathtt{U}_j + \epsilon \text{ on } \partial\mathbb{B}.$$

Hence by the comparison principle we get that for all $k \geq j \geq \max\{j_0, j_1\}$

$$\mathtt{U}_k - \mathtt{U}_j \leq 2\epsilon - \epsilon|z|^2 \leq 2\epsilon \text{ in } \bar{\mathbb{B}}.$$

On the other hand, $\mathtt{U}_j \leq \mathtt{U}_k$, so we infer

$$\|\mathtt{U}_k - \mathtt{U}_j\|_{L^\infty(\bar{\mathbb{B}})} \leq 2\epsilon.$$

This implies that the sequence $(\mathtt{U}_j)$ converges uniformly in $\bar{\mathbb{B}}$.

Let us put $u = \lim_{j\to\infty} \mathtt{U}_j$ which is continuous on $\bar{\mathbb{B}}$, plurisubharmonic on $\mathbb{B}$ and $u = \varphi$ on $\partial\mathbb{B}$. Moreover, $(dd^c\mathtt{U}_j)^n$ converges to $(dd^c u)^n$ in the weak sense of currents, then $(dd^c u)^n = f\beta^n$. Consequently, u is a candidate in the Perron-Bremermann envelope, i.e. $u \in \mathcal{V}(\mathbb{B}, \varphi, f)$ and $u \leq U$ in $\mathbb{B}$. Once again the comparison principle yields $u \geq \mathtt{U}$ in $\mathbb{B}$. Finally, we conclude $u = \mathtt{U}$ in $\bar{\mathbb{B}}$ and $(dd^c\mathtt{U})^n = f\beta^n$ in $\mathbb{B}$. □

Proof of Theorem 2.3.2 . We already know as in Proposition 2.3.3 that $\mathtt{U} \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega})$, $\mathtt{U} = \varphi$ on $\partial\Omega$ and $(dd^c\mathtt{U})^n \geq f\beta^n$ in Ω. It remains to prove that $(dd^c\mathtt{U})^n = f\beta^n$ in Ω. We use the balayage procedure as follows; Fix a ball $B_0 \subset \Omega$. Thanks to Corollary 2.3.8, there exists a unique solution ψ to $Dir(B_0, \mathtt{U}, f)$, that is

$$(dd^c\psi)^n = f\beta^n \text{ in } B_0 \text{ and } \psi = \mathtt{U} \text{ on } \partial B_0.$$

By the comparison principle $\mathtt{U} \leq \psi$ on B_0. Let us define the function

$$v(z) = \begin{cases} \psi(z) & ; z \in B_0, \\ \mathtt{U}(z) & ; z \in \bar{\Omega} \setminus B_0, \end{cases}$$

which belongs to $\mathcal{V}(\Omega, \varphi, f)$ and $v = \mathtt{U} = \varphi$ on $\partial\Omega$.
In particular $v \leq \mathtt{U}$, hence $\psi \leq \mathtt{U}$ in B_0. Consequently, $\psi = \mathtt{U}$ in $\mathbb{B}_0$. Then $(dd^c\mathtt{U})^n = (dd^c\psi)^n = f\beta^n$ in B_0. Since B_0 is an arbitrary ball in Ω, we infer that $(dd^c\mathtt{U})^n = f\beta^n$ in Ω. □

2.3.3 Stability estimates

Proposition 2.3.9. *Let $\varphi_1, \varphi_2 \in \mathcal{C}(\partial\Omega)$ and $f_1, f_2 \in \mathcal{C}(\bar{\Omega})$. Then the solutions $\mathtt{U}_1 = \mathtt{U}(\Omega, \varphi_1, f_1)$, $\mathtt{U}_2 = \mathtt{U}(\Omega, \varphi_2, f_2)$ satisfy the following stability estimate*

$$\|\mathtt{U}_1 - \mathtt{U}_2\|_{L^\infty(\bar{\Omega})} \leq d^2 \|f_1 - f_2\|^{1/n}_{L^\infty(\bar{\Omega})} + \|\varphi_1 - \varphi_2\|_{L^\infty(\partial\Omega)}, \tag{2.3.6}$$

where $d := \mathrm{diam}(\Omega)$.

Proof. Let us fix $z_0 \in \Omega$ and define

$$v_1(z) = \|f_1 - f_2\|^{1/n}_{L^\infty(\bar{\Omega})}(|z - z_0|^2 - d^2) + \mathtt{U}_2(z),$$

and

$$v_2(z) = \mathtt{U}_1(z) + \|\varphi_1 - \varphi_2\|_{L^\infty(\partial\Omega)}.$$

It is clear that $v_1, v_2 \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega})$, $v_1 \leq v_2$ on $\partial\Omega$ and $(dd^c v_1)^n \geq (dd^c v_2)^n$ in Ω. Hence, by the comparison principle, we get $v_1 \leq v_2$ in Ω. Then we conclude that

$$\mathtt{U}_2 - \mathtt{U}_1 \leq d^2 \|f_1 - f_2\|^{1/n}_{L^\infty(\bar{\Omega})} + \|\varphi_1 - \varphi_2\|_{L^\infty(\partial\Omega)}.$$

By reversing the roles of $\mathtt{U}_1$ and $\mathtt{U}_2$, we get the inequality (2.3.6). □

Remark 2.3.10. We will need in the sequel an estimate, proved by Błocki in [Bł93], for the $L^n - L^1$ stability of solutions to the Dirichlet problem $Dir(\Omega, \varphi, f)$

$$\|\mathtt{U}_1 - \mathtt{U}_2\|_{L^n(\Omega)} \leq \lambda(\Omega)\|\varphi_1 - \varphi_2\|_{L^\infty(\partial\Omega)} + \frac{r^2}{4}\|f_1 - f_2\|^{1/n}_{L^1(\Omega)}, \tag{2.3.7}$$

where $r = \min\{r' > 0 : \Omega \subset B(z_0, r')$ for some $z_0 \in \mathbb{C}^n\}$.

2.4 The modulus of continuity of Perron-Bremermann envelope

Recall that a real function ω on $[0, l]$, $0 < l < \infty$, is called a modulus of continuity if ω is continuous, subadditive, nondecreasing and $\omega(0) = 0$.
In general, ω fails to be concave, we denote by $\bar{\omega}$ the minimal concave majorant of ω. We denote by ω_ψ the optimal modulus of continuity of the continuous function ψ which is defined by

$$\omega_\psi(t) = \sup_{|x-y|\leq t} |\psi(x) - \psi(y)|.$$

The following property of the minimal concave majorant $\bar{\omega}$ is well known (see [Kor82] and [Ch14]).

Lemma 2.4.1. *Let ω be a modulus of continuity on $[0, l]$ and $\bar{\omega}$ be the minimal concave majorant of ω. Then $\omega(\eta t) \leq \bar{\omega}(\eta t) \leq (1 + \eta)\omega(t)$ for any $t > 0$ and $\eta > 0$.*

Proof. Fix $t_0 > 0$ such that $\omega(t_0) > 0$. We claim that

$$\frac{\omega(t)}{\omega(t_0)} \leq 1 + \frac{t}{t_0}, \quad \forall t \geq 0.$$

For $0 < t \leq t_0$, since ω is nondecreasing, we have

$$\frac{\omega(t)}{\omega(t_0)} \leq \frac{\omega(t_0)}{\omega(t_0)} \leq 1 + \frac{t}{t_0}.$$

Otherwise , if $t_0 \leq t \leq l$, by Euclid's Algorithm, we write $t = kt_0 + \alpha$, $0 \leq \alpha < t_0$ and k is natural number with $1 \leq k \leq t/t_0$. Using the subadditivity of ω, we observe that

$$\frac{\omega(t)}{\omega(t_0)} \leq \frac{k\omega(t_0) + \omega(\alpha)}{\omega(t_0)} \leq k + 1 \leq 1 + \frac{t}{t_0}.$$

Let $l(t) := \omega(t_0) + \frac{t}{t_0}\omega(t_0)$ be a straight line, then $\omega(t) \leq l(t)$ for all $0 < t \leq l$.
Therefore,

$$\bar{\omega}(t) \leq l(t) = \omega(t_0) + \frac{t}{t_0}\omega(t_0),$$

for all $0 < t \leq l$. Hence, for any $\eta > 0$ we have

$$\omega(\eta t) \leq \bar{\omega}(\eta t) \leq (1 + \eta)\omega(t).$$

□

2.4.1 Modulus of continuity of the solution

Now, we will start the first step to establish an estimate for the modulus of continuity of the solution to $Dir(\Omega, \varphi, f)$. For this purpose, it is natural to investigate the relation between the modulus of continuity of $\mathtt{U}$ and the modulus of continuity of a subbarrier and a superbarrier. We prove the following:

Proposition 2.4.2. *Let $\Omega \subset \mathbb{C}^n$ be a bounded SHL domain, $\varphi \in \mathcal{C}(\partial\Omega)$ and $0 \le f \in \mathcal{C}(\bar{\Omega})$. Suppose that there exist $v \in \mathcal{V}(\Omega, \varphi, f)$ and $w \in SH(\Omega) \cap \mathcal{C}(\bar{\Omega})$ such that $v = \varphi = -w$ on $\partial\Omega$, then the modulus of continuity of $\mathtt{U}$ satisfies*

$$\omega_{\mathtt{U}}(t) \le (d^2+1)\max\{\omega_v(t), \omega_w(t), \omega_{f^{1/n}}(t)\},$$

where $d := \operatorname{diam}(\Omega)$.

Proof. Let us set $g(t) := \max\{\omega_v(t), \omega_w(t), \omega_{f^{1/n}}(t)\}$. As $v = \varphi = -w$ on $\partial\Omega$, we have for all $z \in \bar{\Omega}$ and $\xi \in \partial\Omega$ that

$$-g(|z-\xi|) \le v(z) - \varphi(\xi) \le \mathtt{U}(z) - \varphi(\xi) \le -w(z) - \varphi(\xi) \le g(|z-\xi|).$$

Hence

$$|\mathtt{U}(z) - \mathtt{U}(\xi)| \le g(|z-\xi|), \quad \forall z \in \bar{\Omega}, \forall \xi \in \partial\Omega. \tag{2.4.1}$$

Fix a point $z_0 \in \Omega$. For any vector $\tau \in \mathbb{C}^n$ with small enough norm, we set $\Omega_{-\tau} := \{z - \tau; z \in \Omega\}$ and define in $\Omega \cap \Omega_{-\tau}$ the function

$$v_1(z) = \mathtt{U}(z+\tau) + g(|\tau|)|z-z_0|^2 - d^2 g(|\tau|) - g(|\tau|),$$

which is a well defined psh function in $\Omega \cap \Omega_{-\tau}$ and continuous on $\bar{\Omega} \cap \bar{\Omega}_{-\tau}$. By (2.4.1), if $z \in \bar{\Omega} \cap \partial\Omega_{-\tau}$ we can see that

$$v_1(z) - \mathtt{U}(z) \le g(|\tau|) + g(|\tau|)|z-z_0|^2 - d^2 g(|\tau|) - g(|\tau|) \le 0. \tag{2.4.2}$$

Moreover, we assert that $\Delta_H v_1 \ge f^{1/n}$ in $\Omega \cap \Omega_{-\tau}$ for all $H \in H_n^+, \det H = n^{-n}$. Indeed, we have

$$\begin{aligned} \Delta_H v_1(z) &\ge f^{1/n}(z+\tau) + g(|\tau|)\Delta_H(|z-z_0|^2) \\ &\ge f^{1/n}(z+\tau) + g(|\tau|) \\ &\ge f^{1/n}(z+\tau) + |f^{1/n}(z+\tau) - f^{1/n}(z)| \\ &\ge f^{1/n}(z) \end{aligned}$$

for all $H \in H_n^+$ and $\det H = n^{-n}$. Hence, by the above properties of v_1, we find that

$$V_\tau(z) = \begin{cases} \mathtt{U}(z) & ; z \in \bar{\Omega} \setminus \Omega_{-\tau}, \\ \max\{\mathtt{U}(z), v_1(z)\} & ; z \in \bar{\Omega} \cap \Omega_{-\tau}, \end{cases}$$

is a well defined function and belongs to $PSH(\Omega) \cap \mathcal{C}(\bar{\Omega})$. It is clear that $\Delta_H V_\tau \ge f^{1/n}$ for all $H \in H_n^+, \det H = n^{-n}$. We claim that $V_\tau = \varphi$ on $\partial\Omega$. If $z \in \partial\Omega \setminus \Omega_{-\tau}$ then

$V_\tau(z) = \mathtt{U}(z) = \varphi(z)$. On the other hand, if $z \in \partial\Omega \cap \Omega_{-\tau}$, we get by (2.4.2) that $V_\tau(z) = \max\{\mathtt{U}(z), v_1(z)\} = \mathtt{U}(z) = \varphi(z)$. Consequently, $V_\tau \in \mathcal{V}(\Omega, \varphi, f)$ and this implies that

$$V_\tau(z) \le \mathtt{U}(z); \forall z \in \bar{\Omega}.$$

Then for all $z \in \bar{\Omega} \cap \Omega_{-\tau}$ we have

$$\mathtt{U}(z+\tau) + g(|\tau|)|z - z_0|^2 - d^2 g(|\tau|) - g(|\tau|) \le \mathtt{U}(z).$$

Hence,

$$\mathtt{U}(z+\tau) - \mathtt{U}(z) \le (d^2+1)g(|\tau|) - g(|\tau|)|z-z_0|^2 \le (d^2+1)g(|\tau|).$$

Reversing the roles of $z+\tau$ and z, we get

$$|\mathtt{U}(z+\tau) - \mathtt{U}(z)| \le (d^2+1)g(|\tau|), \ \forall z, z+\tau \in \bar{\Omega}.$$

Thus, finally,

$$\omega_{\mathtt{U}}(|\tau|) \le (d^2+1)\max\{\omega_v(|\tau|), \omega_w(|\tau|), \omega_{f^{1/n}}(|\tau|)\}.$$

□

Remark 2.4.3. Let H_φ be the harmonic extension of φ in a bounded SHL domain Ω. We can replace w in the last proposition by H_φ. It is known in the classical harmonic analysis (see [Ai10]) that the harmonic extension H_φ does not have, in general, the same modulus of continuity of φ.
Let us define, for small positive t, the modulus of continuity

$$\psi_{\alpha,\beta}(t) = (-\log(t))^{-\alpha} t^\beta$$

with $\alpha \ge 0$ and $0 \le \beta < 1$. It is clear that $\psi_{\alpha,0}$ is weaker than the Hölder continuity and $\psi_{0,\beta}$ is the Hölder continuity. It was shown in [Ai02] that $\omega_{H_\varphi}(t) \le c\psi_{0,\beta}(t)$ for some $c > 0$ if $\omega_\varphi(t) \le c_1\psi_{0,\beta}(t)$ for $\beta < \beta_0$ where $\beta_0 < 1$ depends only on n and the Lipschitz constant of the defining function ρ. Moreover, a similar result was proved in [Ai10] for the modulus of continuity $\psi_{\alpha,0}(t)$. However, the same argument of Aikawa gives that $\omega_{H_\varphi}(t) \le c\psi_{\alpha,\beta}(t)$ for some $c > 0$ if $\omega_\varphi(t) \le c_1\psi_{\alpha,\beta}(t)$ for $\alpha \ge 0$ and $0 \le \beta < \beta_0 < 1$.

This leads us to the conclusion that if there exists a barrier v to the Dirichlet problem such that $v = \varphi$ on $\partial\Omega$ and $\omega_v(t) \le \lambda\psi_{\alpha,\beta}(t)$ with α, β as above, then the last proposition gives

$$\omega_{\mathtt{U}} \le \lambda_1 \max\{\psi_{\alpha,\beta}(t), \omega_{f^{1/n}}(t)\},$$

where $\lambda_1 > 0$ depends on λ and $\mathrm{diam}(\Omega)$.

2.4.2 Construction of barriers

In this subsection, we will construct a subsolution to the Dirichlet problem with the boundary value φ and estimate its modulus of continuity.

Proposition 2.4.4. *Let $\Omega \subset \mathbb{C}^n$ be a bounded SHL domain. Assume that $\varphi \in \mathcal{C}(\partial\Omega)$ and $0 \le f \in \mathcal{C}(\bar{\Omega})$, then there exists a subsolution $v \in \mathcal{V}(\Omega, \varphi, f)$ such that $v = \varphi$ on $\partial\Omega$ and the modulus of continuity of v satisfies the following inequality*

$$\omega_v(t) \le \lambda(1 + \|f\|^{1/n}_{L^\infty(\bar{\Omega})}) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\},$$

where $\lambda > 0$ depends on Ω.

Observe that we do not assume any smoothness on $\partial\Omega$.

Proof. First of all, fix $\xi \in \partial\Omega$. We claim that there exists $v_\xi \in \mathcal{V}(\Omega,\varphi,f)$ such that $v_\xi(\xi) = \varphi(\xi)$. It is sufficient to prove that there exists a constant $C > 0$ depending on Ω such that for every point $\xi \in \partial\Omega$ and $\varphi \in \mathcal{C}(\partial\Omega)$, there is a function $h_\xi \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega})$ satisfying
(1) $h_\xi(z) \le \varphi(z), \forall z \in \partial\Omega$,
(2) $h_\xi(\xi) = \varphi(\xi)$,
(3) $\omega_{h_\xi}(t) \le C\omega_\varphi(t^{1/2})$.
Assume this is true. We fix $z_0 \in \Omega$ and write $K_1 := \sup_{\bar{\Omega}} f^{1/n} \ge 0$. Hence

$$\Delta_H(K_1|z-z_0|^2) = K_1\Delta_H|z-z_0|^2 \ge f^{1/n}, \forall H \in H_n^+, \det H = n^{-n}.$$

We also set $K_2 := K_1|\xi - z_0|^2$. Then for the continuous function

$$\tilde{\varphi}(z) := \varphi(z) - K_1|z-z_0|^2 + K_2,$$

we have h_ξ such that (1)-(3) hold.
Then the desired function $v_\xi \in \mathcal{V}(\Omega,\varphi,f)$ is given by

$$v_\xi(z) = h_\xi(z) + K_1|z-z_0|^2 - K_2.$$

Thus $h_\xi(z) \le \tilde{\varphi}(z) = \varphi(z) - K_1|z-z_0|^2 + K_2$ on $\partial\Omega$, so $v_\xi(z) \le \varphi$ on $\partial\Omega$ and $v_\xi(\xi) = \varphi(\xi)$. Moreover, it is clear that

$$\Delta_H v_\xi = \Delta_H h_\xi + K_1\Delta_H(|z-z_0|^2) \ge f^{1/n}, \forall H \in H_n^+, \det H = n^{-n}.$$

Furthermore, using the hypothesis of h_ξ, we can control the modulus of continuity of v_ξ

$$\begin{aligned}\omega_{v_\xi}(t) = \sup_{|z-y|\le t}|v_\xi(z) - v_\xi(y)| &\le \omega_{h_\xi}(t) + K_1\omega_{|z-z_0|^2}(t)\\ &\le C\omega_{\tilde{\varphi}}(t^{1/2}) + 4d^{3/2}K_1t^{1/2}\\ &\le C\omega_\varphi(t^{1/2}) + 2dK_1(C+2d^{1/2})t^{1/2}\\ &\le (C+2d^{1/2})(1+2dK_1)\max\{\omega_\varphi(t^{1/2}), t^{1/2}\},\end{aligned}$$

where $d := \operatorname{diam}(\Omega)$. Hence, we conclude that

$$\omega_{v_\xi}(t) \le \lambda(1+K_1)\max\{\omega_\varphi(t^{1/2}), t^{1/2}\},$$

where $\lambda := (C+2d^{1/2})(1+2d)$ is a positive constant depending on Ω.

Now we will construct $h_\xi \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega})$ which satisfies the three conditions above. Let $B > 0$ be large enough such that the function

$$g(z) = B\rho(z) - |z-\xi|^2$$

is psh in Ω. Let $\bar{\omega}_\varphi$ be the minimal concave majorant of ω_φ and define

$$\chi(x) = -\bar{\omega}_\varphi((-x)^{1/2}),$$

which is a convex nondecreasing function on $[-d^2, 0]$. Now fix $r > 0$ so small that $|g(z)| \leq d^2$ in $B(\xi, r) \cap \Omega$ and define for $z \in B(\xi, r) \cap \bar{\Omega}$ the function

$$h(z) = \chi \circ g(z) + \varphi(\xi).$$

It is clear that h is a continuous psh function on $B(\xi, r) \cap \Omega$ and we see that $h(z) \leq \varphi(z)$ if $z \in B(\xi, r) \cap \partial\Omega$ and $h(\xi) = \varphi(\xi)$. Moreover by the subadditivity of $\bar{\omega}_\varphi$ and Lemma 2.4.1 we have

$$\begin{aligned}\omega_h(t) &= \sup_{|z-y|\leq t} |h(z) - h(y)| \\ &\leq \sup_{|z-y|\leq t} \bar{\omega}_\varphi \left[\left| |z-\xi|^2 - |y-\xi|^2 - B(\rho(z) - \rho(y)) \right|^{1/2} \right] \\ &\leq \sup_{|z-y|\leq t} \bar{\omega}_\varphi \left[(|z-y|(2d + B_1))^{1/2} \right] \\ &\leq C.\omega_\varphi(t^{1/2}),\end{aligned}$$

where $C := 1 + (2d + B_1)^{1/2}$ depends on Ω.

Recall that $\xi \in \partial\Omega$ and fix $0 < r_1 < r$ and $\gamma_1 \geq 1 + d/r_1$ such that

$$-\gamma_1 \bar{\omega}_\varphi \left[(|z-\xi|^2 - B\rho(z))^{1/2} \right] \leq \inf_{\partial\Omega} \varphi - \sup_{\partial\Omega} \varphi,$$

for $z \in \partial\Omega \cap \partial B(\xi, r_1)$. Set $\gamma_2 = \inf_{\partial\Omega} \varphi$. Then

$$\gamma_1(h(z) - \varphi(\xi)) + \varphi(\xi) \leq \gamma_2 \text{ for } z \in \partial B(\xi, r_1) \cap \bar{\Omega}.$$

Now set

$$h_\xi(z) = \begin{cases} \max[\gamma_1(h(z) - \varphi(\xi)) + \varphi(\xi), \gamma_2] & ; z \in \bar{\Omega} \cap B(\xi, r_1), \\ \gamma_2 & ; z \in \bar{\Omega} \setminus B(\xi, r_1), \end{cases}$$

which is a well defined psh function on Ω, continuous on $\bar{\Omega}$ and such that $h_\xi(z) \leq \varphi(z)$ for all $z \in \partial\Omega$. Indeed, on $\partial\Omega \cap B(\xi, r_1)$ we have

$$\gamma_1(h(z) - \varphi(\xi)) + \varphi(\xi) = -\gamma_1 \bar{\omega}_\varphi(|z-\xi|) + \varphi(\xi) \leq -\bar{\omega}_\varphi(|z-\xi|) + \varphi(\xi) \leq \varphi(z).$$

Hence it is clear that h_ξ satisfies the three conditions above.

We have just proved that for each $\xi \in \partial\Omega$, there is a function $v_\xi \in \mathcal{V}(\Omega, \varphi, f)$ with $v_\xi(\xi) = \varphi(\xi)$ and

$$\omega_{v_\xi}(t) \leq \lambda(1 + K_1) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\}.$$

Set

$$v(z) = \sup\{v_\xi(z); \xi \in \partial\Omega\}.$$

Since $0 \leq \omega_v(t) \leq \lambda(1+K_1)\max\{\omega_\varphi(t^{1/2}), t^{1/2}\}$, we see that $\omega_v(t)$ converges to zero when t converges to zero. Consequently, $v \in \mathcal{C}(\bar{\Omega})$ and $v = v^* \in PSH(\Omega)$. Thanks to Choquet's lemma, we can choose a nondecreasing sequence (v_j), where $v_j \in \mathcal{V}(\Omega, \varphi, f)$, converging to v almost everywhere. This implies that

$$\Delta_H v = \lim_{j\to\infty} \Delta_H v_j \geq f^{1/n}, \forall H \in H_n^+, \det H = n^{-n}.$$

It is clear that $v(\xi) = \varphi(\xi)$ for any $\xi \in \partial\Omega$. Finally, $v \in \mathcal{V}(\Omega, \varphi, f)$, $v = \varphi$ on $\partial\Omega$ and $\omega_v(t) \leq \lambda(1 + K_1)\max\{\omega_\varphi(t^{1/2}), t^{1/2}\}$. □

Remark 2.4.5. If we assume that Ω has a smooth boundary and φ is $\mathcal{C}^{1,\alpha}$-smooth for $0 < \alpha \leq 1$, then it is possible to construct a $(1+\alpha)/2$-Hölder continuous barrier v to the Dirichlet problem $Dir(\Omega, \varphi, f)$ (see [BT76, Theorem 6.2]). Here, for a bounded SHL domain, if $\varphi \in \mathcal{C}^{1,1}(\partial\Omega)$ we can find a Lipschitz barrier to $Dir(\Omega, \varphi, f)$. It is enough to take $v := A\rho + \tilde{\varphi}$ where $\tilde{\varphi}$ is an extension of φ to $\bar{\Omega}$ and $A \gg 1$.

Corollary 2.4.6. *Under the same assumption of Proposition 2.4.4, there exists a plurisuperharmonic function $\tilde{v} \in \mathcal{C}(\bar{\Omega})$ such that $\tilde{v} = \varphi$ on $\partial\Omega$ and*

$$\omega_{\tilde{v}}(t) \leq \lambda(1 + \|f\|^{1/n}_{L^\infty(\bar{\Omega})}) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\},$$

where $\lambda > 0$ depends on Ω.

Proof. We can perform the same construction as in the proof of Proposition 2.4.4 for the function $\varphi_1 = -\varphi \in \mathcal{C}(\partial\Omega)$; then we get $v_1 \in \mathcal{V}(\Omega, \varphi_1, f)$ such that $v_1 = \varphi_1$ on $\partial\Omega$ and $\omega_{v_1}(t) \leq \lambda(1 + \|f\|^{1/n}_{L^\infty(\bar{\Omega})}) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\}$. Hence, we set $\tilde{v} = -v_1$ which is a plurisuperharmonic function on Ω, continuous on $\bar{\Omega}$ and satisfying $\tilde{v} = \varphi$ on $\partial\Omega$ and

$$\omega_{\tilde{v}}(t) \leq \lambda(1 + \|f\|^{1/n}_{L^\infty(\bar{\Omega})}) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\},$$

where $\lambda > 0$ is a constant depending on Ω. □

2.5 Proof of main results

2.5.1 Proof of Theorem 2.1.1

Thanks to Proposition 2.4.4, we have a subsolution $v \in \mathcal{V}(\Omega, \varphi, f)$ with $v = \varphi$ on $\partial\Omega$ and

$$\omega_v(t) \leq \lambda(1 + \|f\|^{1/n}_{L^\infty(\bar{\Omega})}) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\}.$$

From Corollary 2.4.6, we get $w \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega})$ such that $w = -\varphi$ on $\partial\Omega$ and

$$\omega_w(t) \leq \lambda(1 + \|f\|^{1/n}_{L^\infty(\bar{\Omega})}) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\},$$

where $\lambda > 0$ is a constant. Applying the Proposition 2.4.2 we obtain the required result, that is

$$\omega_{\mathtt{U}}(t) \leq \eta(1 + \|f\|^{1/n}_{L^\infty(\bar{\Omega})}) \max\{\omega_\varphi(t^{1/2}), \omega_{f^{1/n}}(t), t^{1/2}\},$$

where $\eta > 0$ depends on Ω. □

Corollary 2.5.1. *Let Ω be a bounded SHL domain in $\mathbb{C}^n$. Let $\varphi \in \mathcal{C}^{0,\alpha}(\partial\Omega)$ and $0 \leq f^{1/n} \in \mathcal{C}^{0,\beta}(\bar{\Omega})$, $0 < \alpha, \beta \leq 1$. Then the solution $\mathtt{U}$ to the Dirichlet problem $Dir(\Omega, \varphi, f)$ belongs to $\mathcal{C}^{0,\gamma}(\bar{\Omega})$ for $\gamma = \min\{\beta, \alpha/2\}$.*

The following example illustrates that the estimate of $\omega_{\mathtt{U}}$ in Theorem 2.1.1 is optimal.

Example 2.5.2. Let ψ be a concave modulus of continuity on $[0,1]$ and

$$\varphi(z) = -\psi[\sqrt{(1+Rez_1)/2}], \text{ for } z = (z_1, z_2, ..., z_n) \in \partial\mathbb{B} \subset \mathbb{C}^n.$$

It is easy to show that $\varphi \in \mathcal{C}(\partial\mathbb{B})$ with modulus of continuity

$$\omega_\varphi(t) \leq C\psi(t),$$

for some $C > 0$.
Let $v(z) = -(1+Rez_1)/2 \in PSH(\mathbb{B}) \cap \mathcal{C}(\bar{\mathbb{B}})$ and $\chi(\lambda) = -\psi(\sqrt{-\lambda})$ be a convex increasing function on $[-1,0]$. Hence we see that

$$u(z) = \chi \circ v(z) \in PSH(\mathbb{B}) \cap \mathcal{C}(\bar{\mathbb{B}}),$$

and satisfies $(dd^c u)^n = 0$ in $\mathbb{B}$ and $u = \varphi$ on $\partial\mathbb{B}$. The modulus of continuity of $\mathtt{U}$, $\omega_{\mathtt{U}}(t)$, has the estimate

$$C_1\psi(t^{1/2}) \leq \omega_{\mathtt{U}}(t) \leq C_2\psi(t^{1/2}),$$

for $C_1, C_2 > 0$. Indeed, let $z_0 = (-1, 0, ..., 0)$ and $z = (z_1, 0, ..., 0) \in \mathbb{B}$ where $z_1 = -1 + 2t$ and $0 \leq t \leq 1$. Hence, by Lemma 2.4.1, we see that

$$\psi(t^{1/2}) = \psi[\sqrt{|z - z_0|/2}] = \psi[\sqrt{(1+Rez_1)/2}] = |\mathtt{U}(z) - \mathtt{U}(z_0)| \leq 3\omega_{\mathtt{U}}(t).$$

Finally, it is natural to try to relate the modulus of continuity of $\mathtt{U} := \mathtt{U}(\Omega, \varphi, f)$ to the modulus of continuity of $\mathtt{U}_0 := \mathtt{U}(\Omega, \varphi, 0)$ the solution to Bremermann problem in a bounded SHL domain.

Proposition 2.5.3. *Let Ω be a bounded SHL domain in $\mathbb{C}^n$, $0 \leq f \in \mathcal{C}(\bar{\Omega})$ and $\varphi \in \mathcal{C}(\partial\Omega)$. Then there exists a positive constant $C = C(\Omega)$ such that*

$$\omega_{\mathtt{U}}(t) \leq C(1 + \|f\|^{1/n}_{L^\infty(\bar{\Omega})}) \max\{\omega_{\mathtt{U}_0}(t), \omega_{f^{1/n}}(t)\}.$$

Proof. First, we search for a subsolution $v \in \mathcal{V}(\Omega, \varphi, f)$ such that $v|_{\partial\Omega} = \varphi$ and estimate its modulus of continuity. Since Ω is a bounded SHL domain, there exists a Lipschitz defining function ρ on $\bar{\Omega}$. Define the function

$$v(z) = \mathtt{U}_0(z) + A\rho(z),$$

where $A := \|f\|^{1/n}_{L^\infty}/c$ and $c > 0$ is as in Definition 2.2.1. It is clear that $v \in \mathcal{V}(\Omega, \varphi, f)$, $v = \varphi$ on $\partial\Omega$ and

$$\omega_v(t) \leq \tilde{C}\omega_{\mathtt{U}_0}(t),$$

where $\tilde{C} := \gamma(1 + \|f\|^{1/n}_{L^\infty(\bar{\Omega})})$ and $\gamma \geq 1$ depends on Ω.

On the other hand, by the comparison principle we get that $\mathtt{U} \leq \mathtt{U}_0$. So,

$$v \leq \mathtt{U} \leq \mathtt{U}_0 \text{ in } \Omega \text{ and } v = \mathtt{U} = \mathtt{U}_0 = \varphi \text{ on } \partial\Omega.$$

Thanks to Proposition 2.4.2, there exists $\lambda > 0$ depending on Ω such that

$$\omega_{\mathtt{U}}(t) \leq \lambda \max\{\omega_v(t), \omega_{\mathtt{U}_0}(t), \omega_{f^{1/n}}(t)\}.$$

Hence, for some $C > 0$ depending on Ω,

$$\omega_{\mathtt{U}}(t) \leq C(1 + \|f\|^{1/n}_{L^\infty(\bar{\Omega})}) \max\{\omega_{\mathtt{U}_0}(t), \omega_{f^{1/n}}(t)\}.$$

□

2.5.2 Estimate of the ψ-norm of the solution

Definition 2.5.4. Let ψ be a modulus of continuity, $E \subset \mathbb{C}^n$ be a bounded set and $g \in \mathcal{C} \cap L^\infty(E)$. We define the *norm of g with respect to ψ* (briefly, *ψ-norm*) as follows:

$$\|g\|_\psi := \sup_{z\in E} |g(z)| + \sup_{z\neq y\in E} \frac{|g(z)-g(y)|}{\psi(|z-y|)}.$$

Proposition 2.5.5. *Let $\Omega \subset \mathbb{C}^n$ be a bounded SHL domain, $\varphi \in \mathcal{C}(\partial\Omega)$ with modulus of continuity ψ_1 and $f^{1/n} \in \mathcal{C}(\bar{\Omega})$ with modulus of continuity ψ_2. Then there exists a constant $C > 0$ depending on Ω such that*

$$\|\mathtt{U}\|_\psi \leq C(1 + \|f\|^{1/n}_{L^\infty(\bar{\Omega})}) \max\{\|\varphi\|_{\psi_1}, \|f^{1/n}\|_{\psi_2}\},$$

where $\psi(t) = \max\{\psi_1(t^{1/2}), \psi_2(t)\}$.

Proof. By hypothesis, we see that $\|\varphi\|_{\psi_1} < \infty$ and $\|f^{1/n}\|_{\psi_2} < \infty$. Let $z \neq y \in \bar{\Omega}$. By Theorem 2.1.1, we get

$$\begin{aligned}|\mathtt{U}(z) - \mathtt{U}(y)| &\leq \eta(1 + \|f\|^{1/n}_{L^\infty(\bar{\Omega})}) \max\{\omega_\varphi(|z-y|^{1/2}), \omega_{f^{1/n}}(|z-y|)\} \\ &\leq \eta(1 + \|f\|^{1/n}_{L^\infty(\bar{\Omega})}) \max\{\|\varphi\|_{\psi_1}, \|f^{1/n}\|_{\psi_2}\}\psi(|z-y|),\end{aligned}$$

where $\psi(|z-y|) = \max\{\psi_1(|z-y|^{1/2}), \psi_2(|z-y|)\}$. Hence

$$\sup_{z\neq y\in\bar{\Omega}} \frac{|\mathtt{U}(z) - \mathtt{U}(y)|}{\psi(|z-y|)} \leq \eta(1 + \|f\|^{1/n}_{L^\infty(\bar{\Omega})}) \max\{\|\varphi\|_{\psi_1}, \|f^{1/n}\|_{\psi_2}\},$$

where $\eta \geq d^2 + 1$ and $d = \operatorname{diam}(\Omega)$ (see Proposition 2.4.2). From Proposition 2.3.9, we note that

$$\|\mathtt{U}\|_{L^\infty(\bar{\Omega})} \leq d^2 \|f\|^{1/n}_{L^\infty(\bar{\Omega})} + \|\varphi\|_{L^\infty(\partial\Omega)} \leq \eta \max\{\|\varphi\|_{\psi_1}, \|f^{1/n}\|_{\psi_2}\}.$$

Then we can conclude that

$$\|\mathtt{U}\|_\psi \leq 2\eta(1 + \|f\|^{1/n}_{L^\infty(\bar{\Omega})}) \max\{\|\varphi\|_{\psi_1}, \|f^{1/n}\|_{\psi_2}\}.$$

□

Chapter 3

Hölder continuity of solutions for general measures

3.1 Introduction

In this chapter, we are interested in studying the regularity of solutions to the following Dirichlet problem:

$$Dir(\Omega, \varphi, fd\mu): \begin{cases} u \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega}), & \\ (dd^c u)^n = fd\mu & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where μ is a nonnegative finite Borel measure on a bounded SHL domain Ω, $0 \leq f \in L^p(\Omega, \mu)$ for $p > 1$, and $\varphi \in \mathcal{C}(\partial\Omega)$.

Kołodziej demonstrated [Ko98, Ko99] the existence of a weak continuous solution to this problem as soon as μ is dominated by a suitable function of capacity on a bounded strongly pseudoconvex domain with smooth boundary.

We consider in this thesis the class of measures satisfying (3.3.1) and ensure Kołodziej's existence theorem in a bounded SHL domain. More precisely, we prove the following.

Theorem 3.1.1. *Let μ be a measure satisfying Condition $\mathcal{H}(\tau)$ for some $\tau > 0$ on a bounded SHL domain $\Omega \subset \mathbb{C}^n$ and $\varphi \in \mathcal{C}(\partial\Omega)$. Then there exists a unique continuous solution to $Dir(\Omega, \varphi, d\mu)$.*

Then we investigate the Hölder continuity of the solution in several cases.

In the case of the Lebesgue measure, we have estimated in Chapter 2 the modulus of continuity of the solution in terms of the modulus of continuity of the boundary data φ and the density f in a bounded SHL domain.

Guedj, Kołodziej and Zeriahi proved [GKZ08] that the solution to $Dir(\Omega, \varphi, fdV_{2n})$ is Hölder continuous on $\bar{\Omega}$ when $f \in L^p(\Omega)$, $p > 1$, is bounded near the boundary of strongly pseudoconvex domain and $\varphi \in \mathcal{C}^{1,1}(\partial\Omega)$. Recently, N. C. Nguyen [N14] proved the Hölder continuity when the density satisfies a growth condition near the boundary.

Here, we deal the case of L^p-density without assuming any condition near the boundary.

Theorem 3.1.2. *Let $\Omega \subset \mathbb{C}^n$ be a bounded SHL domain. Assume that $\varphi \in \mathcal{C}^{1,1}(\partial\Omega)$ and $f \in L^p(\Omega)$ for some $p > 1$. Then the unique solution $\mathtt{U}$ to $Dir(\Omega, \varphi, fdV_{2n})$ is γ-Hölder continuous on $\bar{\Omega}$ for any $0 < \gamma < 1/(nq+1)$ where $1/p + 1/q = 1$.*

Moreover, if $p \geq 2$, then the solution $\mathtt{U}$ is Hölder continuous on $\bar{\Omega}$ of exponent less than $\min\{1/2, 2/(nq+1)\}$.

In the case of singular measures with respect to the Lebesgue measure, there is no study about the regularity of solution in a bounded domain in $\mathbb{C}^n$ (see [Ph10] for regularity of solutions in the compact case). We will consider the case of measures having densities in L^p, for $p > 1$, with respect to Hausdorff-Riesz measures which are defined in (3.5.5).

We prove the Hölder continuity of the solution while the boundary data belongs to $\mathcal{C}^{1,1}(\partial\Omega)$.

Theorem 3.1.3. *Let Ω be a bounded SHL domain in $\mathbb{C}^n$ and μ be a Hausdorff-Riesz measure of order $2n-2+\epsilon$ for $0 < \epsilon \leq 2$. Suppose that $\varphi \in \mathcal{C}^{1,1}(\partial\Omega)$ and $0 \leq f \in L^p(\Omega, \mu)$ for some $p > 1$, then the unique solution to $Dir(\Omega, \varphi, fd\mu)$ is Hölder continuous on $\bar{\Omega}$ of exponent $\epsilon\gamma/2$ for any $0 < \gamma < 1/(nq+1)$ and $1/p + 1/q = 1$.*

This result generalizes the one proved in [GKZ08, Ch15a] from which the main idea of our proof originates.

When the boundary data is merely Hölder continuous we state the regularity of the solution using the previous theorem.

Theorem 3.1.4. *Let Ω be a bounded SHL domain in $\mathbb{C}^n$ and μ be a Hausdorff-Riesz measure of order $2n-2+\epsilon$ for $0 < \epsilon \leq 2$. Suppose that $\varphi \in \mathcal{C}^{0,\alpha}(\partial\Omega)$, $0 < \alpha \leq 1$ and $0 \leq f \in L^p(\Omega, \mu)$ for some $p > 1$, then the unique solution to $Dir(\Omega, \varphi, fd\mu)$ is Hölder continuous on $\bar{\Omega}$ of exponent $\frac{\epsilon}{\epsilon+6}\min\{\alpha, \epsilon\gamma\}$ for any $0 < \gamma < 1/(nq+1)$ and $1/p + 1/q = 1$.*

Moreover, when Ω is a smooth strongly pseudoconvex domain the Hölder exponent will be $\frac{\epsilon}{\epsilon+2}\min\{\alpha, \epsilon\gamma\}$, for any $0 < \gamma < 1/(nq+1)$.

In the case of the Lebesgue measure, i.e. $\epsilon = 2$, in a smooth strongly pseudoconvex domain we get the Hölder exponent $\min\{\alpha/2, \gamma\}$ which is better than the one obtained in [BKPZ15].

Our final purpose concerns how to get the Hölder continuity of the solution to the Dirichlet problem $Dir(\Omega, \varphi, fd\mu)$ by means of the Hölder continuity of a subsolution to $Dir(\Omega, \varphi, d\mu)$ for some special measure on Ω.

Theorem 3.1.5. *Let μ be a finite Borel measure on a bounded SHL domain Ω satisfying Condition $\mathcal{H}(\infty)$ mentioned below. Let also $\varphi \in \mathcal{C}^{0,\alpha}(\partial\Omega)$, $0 < \alpha \leq 1$ and $0 \leq f \in L^p(\Omega, \mu)$, $p > 1$. Assume that there exists a λ-Hölder continuous plurisubharmonic function w in Ω such that $(dd^c w)^n \geq \mu$. If, near the boundary, μ is Hausdorff-Riesz of order $2n-2+\epsilon$ for some $0 < \epsilon \leq 2$, then the solution $\mathtt{U}$ to $Dir(\Omega, \varphi, fd\mu)$ is Hölder continuous on $\bar{\Omega}$.*

Such a problem is still open for measures without any condition near the boundary of a bounded domain in $\mathbb{C}^n$.

Most of the content of this chapter will be found in my papers [Ch15a] and [Ch15b].

3.2 Stability theorem

Definition 3.2.1. A nonnegative finite Borel measure μ on Ω is said to satisfy Condition $\mathcal{H}(\infty)$ if for any $\tau > 0$ there exists a positive constant A depending on τ such that

$$\mu(K) \leq A\mathrm{Cap}(K, \Omega)^{1+\tau},$$

for any Borel subset K of Ω.

Before announcing the stability theorem, let us prove some useful lemmas.

Lemma 3.2.2. *Let $v_1, v_2 \in PSH(\Omega) \cap L^\infty(\Omega)$ be such that $\liminf_{z\to\partial\Omega}(v_1 - v_2)(z) \geq 0$. Then for all $t, s > 0$, we have*

$$t^n\, Cap(\{v_1 - v_2 < -s - t\}, \Omega) \leq \int_{\{v_1 - v_2 < -s\}} (dd^c v_1)^n.$$

Proof. Fix $v \in PSH(\Omega)$ such that $-1 \leq v \leq 0$. Then for any $t, s > 0$, we have

$$\{v_1 - v_2 < -s - t\} \subset \{v_1 - v_2 < -s + tv\} \subset \{v_1 - v_2 < -s\} \Subset \Omega.$$

The comparison principle yields that

$$\begin{aligned} t^n \int_{\{v_1 - v_2 < -s - t\}} (dd^c v)^n &\leq \int_{\{v_1 - v_2 < -s - t\}} (dd^c(v_2 + tv))^n \\ &\leq \int_{\{v_1 - v_2 < -s + tv\}} (dd^c(v_2 + tv))^n \\ &= \int_{\{v_1 < -s + v_2 + tv\}} (dd^c(-s + v_2 + tv))^n \\ &\leq \int_{\{v_1 < -s + v_2 + tv\}} (dd^c v_1)^n \\ &\leq \int_{\{v_1 - v_2 < -s\}} (dd^c v_1)^n. \end{aligned}$$

Taking the supremum over all such functions v gives the required result. □

Lemma 3.2.3. *Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be a decreasing right continuous function. Assume that there exist $\tau, B > 0$ such that*

$$tg(s + t) \leq B[g(s)]^{1+\tau}, \text{ for all } s, t > 0. \tag{3.2.1}$$

Then $g(s) = 0$ for all $s \geq s_\infty$, where $s_\infty := \frac{2B[g(0)]^\tau}{1-2^{-\tau}}$.

Proof. We define by induction an increasing sequence $(s_j) \in \mathbb{R}_+^{\mathbb{N}}$ as follows.

$$s_0 := 0,$$

$$s_j := \sup\{s > s_{j-1} : g(s) > g(s_{j-1})/2\}, \forall j \geq 1.$$

It is clear that for any $s > s_j$ we have $g(s) \le g(s_{j-1})/2$. As g is right continuous, we conclude that $g(s_j) \le g(s_{j-1})/2$. Hence, we infer

$$g(s_j) \le \frac{g(0)}{2^j}. \tag{3.2.2}$$

Let us set $M := 2B[g(0)]^\tau > 0$ and $M_j := 2^{-j\tau}M$ for $j \ge 1$. We apply (3.2.1) for s_j and M_j, then it follows from (3.2.2) that

$$g(s_j + M_j) \le \frac{B}{M_j} g(s_j)^{1+\tau} \le g(s_j)/2.$$

Consequently, we get $s_{j+1} \le s_j + M_j$ since $g(s) > g(s_j)/2$ for any $s \in (s_j, s_{j+1})$. In the same way we can see that $s_1 \le M$. Thus the sequence (s_j) is bounded from above with limit

$$\sum_{j\ge 0}(s_{j+1} - s_j) \le M + \sum_{j\ge 1} M_j = \frac{M}{1-2^{-\tau}} =: s_\infty.$$

Then the lemma follows. □

The following weak stability estimate, proved in [GKZ08] for the Lebesgue measure, plays an important role in our work. A similar, but weaker, estimate was established by Kołodziej [Ko02] and in the compact setting it was proved by Eyssidieux, Guedj and Zeriahi [EGZ09]. Here we show that this estimate is still true for any measure μ satisfying Condition $\mathcal{H}(\infty)$.

Theorem 3.2.4. *Let μ satisfy Condition $\mathcal{H}(\infty)$ on a bounded domain $\Omega \subset \mathbb{C}^n$ and $0 \le f \in L^p(\Omega, \mu)$, $p > 1$. Suppose that v_1, v_2 are two bounded psh functions in Ω such that $\liminf_{z\to\partial\Omega}(v_1 - v_2)(z) \ge 0$ and $(dd^c v_1)^n = f d\mu$. Fix $r \ge 1$ and $0 < \gamma < r/(nq + r)$, $1/p + 1/q = 1$. Then there exists a constant $C = C(r, \gamma, n, q) > 0$ such that*

$$\sup_\Omega (v_2 - v_1) \le C(1 + \|f\|^\eta_{L^p(\Omega,\mu)}) ||(v_2 - v_1)_+||^\gamma_{L^r(\Omega,\mu)}, \tag{3.2.3}$$

where $(v_2 - v_1)_+ = \max\{v_2 - v_1, 0\}$ and $\eta = \frac{1}{n} + \frac{\gamma q}{r-\gamma(r+nq)}$.

In order to prove this theorem we need the following proposition.

Proposition 3.2.5. *Under the same assumption of Theorem 3.2.4 and for any $\alpha > 0$, there exists a positive constant $C_1 = C_1(n, q, \alpha)$ such that for all $\epsilon > 0$,*

$$\sup_\Omega (v_2 - v_1) \le \epsilon + C_1 \|f\|^{1/n}_{L^p(\Omega,\mu)} [Cap(\{v_1 - v_2 < -\epsilon\}, \Omega)]^\alpha.$$

Proof. Let us set $g(s) := \mathrm{Cap}(\{v_1 - v_2 < -s - \epsilon\}, \Omega)^{1/n}$. By applying Lemma 3.2.2 we conclude that

$$\begin{aligned} t^n \mathrm{Cap}(\{v_1 - v_2 < -\epsilon - s - t\}, \Omega) &\le \int_{\{v_1 - v_2 < -\epsilon - s\}} (dd^c v_1)^n \\ &\le \int_{\{v_1 - v_2 < -\epsilon - s\}} f d\mu \\ &\le \|f\|_{L^p(\Omega,\mu)} \mu(\{v_1 - v_2 < -\epsilon - s\})^{1/q} \end{aligned}$$

Since μ satisfies Condition $\mathcal{H}(\infty)$, we find a positive constant $\tilde{C}$ depending on n, q and α such that

$$t^n \mathrm{Cap}(\{v_1 - v_2 < -\epsilon - s - t\}, \Omega) \leq \tilde{C}\|f\|_{L^p(\Omega,\mu)}[\mathrm{Cap}(\{v_1 - v_2 < -\epsilon - s\}, \Omega)]^{1+\alpha n}.$$

Therefore, this yields that

$$tg(s+t) \leq B[g(s)]^{1+\alpha n},$$

where $B := \tilde{C}^{1/n}\|f\|^{1/n}_{L^p(\Omega,\mu)}$.

Now, it follows from Lemma 3.2.3 that $\mathrm{Cap}(\{v_1 - v_2 < -\epsilon - s_\infty\}, \Omega) = 0$. Hence $v_2 - v_1 \leq \epsilon + s_\infty$ almost everywhere and then the inequality holds everywhere in Ω. Consequently, we have

$$\sup_\Omega (v_2 - v_1) \leq \epsilon + C_1\|f\|^{1/n}_{L^p(\Omega,\mu)}[\mathrm{Cap}(\{v_1 - v_2 < -\epsilon\}, \Omega)]^\alpha,$$

where C_1 depends only on n, q and α. □

Proof of Theorem 3.2.4. Applying Lemma 3.2.2 with $s = t = \epsilon$ and using Hölder inequality, we infer

$$\begin{aligned}\mathrm{Cap}(\{v_1 - v_2 < -2\epsilon\}, \Omega) &\leq \epsilon^{-n}\int_{\{v_1 - v_2 < -\epsilon\}} f d\mu \\ &\leq \epsilon^{-n-r/q}\int_\Omega (v_2 - v_1)_+^{r/q} f d\mu \\ &\leq \epsilon^{-n-r/q}\|f\|_{L^p(\Omega,\mu)}\|(v_2 - v_1)_+\|^{r/q}_{L^r(\Omega,\mu)}.\end{aligned}$$

Fix $\alpha > 0$ to be chosen later and apply Proposition 3.2.5 to get

$$\sup_\Omega (v_2 - v_1) \leq 2\epsilon + C_1\epsilon^{-\alpha(n+r/q)}\|f\|^{\alpha+1/n}_{L^p(\Omega,\mu)}\|(v_2 - v_1)_+\|^{\alpha r/q}_{L^r(\Omega,\mu)}.$$

We set $\epsilon := \|(v_2 - v_1)_+\|^\gamma$, where $0 < \gamma < r/(nq + r)$ is fixed and

$$\alpha := \frac{\gamma q}{r - \gamma(r + nq)}.$$

Then we get

$$\sup_\Omega (v_2 - v_1) \leq C(1 + \|f\|^{\alpha+1/n}_{L^p(\Omega,\mu)})||(v_2 - v_1)_+||^\gamma_{L^r(\Omega,\mu)},$$

where $C > 0$ depends on n, q, γ and r. □

Remark 3.2.6. When μ satisfies only the condition in Definition 3.3.1 below, we can get some stability estimate.

Suppose that v_1, v_2 are two bounded psh functions in Ω such that $\liminf_{z\to\partial\Omega}(v_1 - v_2)(z) \geq 0$ and $(dd^c v_1)^n = d\mu$. Fix $r \geq 1$, then there exists a constant $C = C(r, \tau, n) > 0$ such that

$$\sup_\Omega (v_2 - v_1) \leq C||(v_2 - v_1)_+||^\gamma_{L^r(\Omega,\mu)}, \tag{3.2.4}$$

where $(v_2 - v_1)_+ = \max\{v_2 - v_1, 0\}$ and $\gamma := \frac{\tau r}{n + \tau(n + r)}$.

3.3 Existence of solutions

This section is devoted to explain the existence of continuous solutions to the Dirichlet problem $Dir(\Omega, \varphi, \mu)$ for measures μ dominated by Bedford-Taylor's capacity, as in (3.3.1) below, on a bounded SHL domain.

Definition 3.3.1. A finite Borel measure μ on Ω is said to satisfy Condition $\mathcal{H}(\tau)$ for some fixed $\tau > 0$ if there exists a positive constant A such that

$$\mu(K) \leq A\mathrm{Cap}(K, \Omega)^{1+\tau}, \tag{3.3.1}$$

for any Borel subset K of Ω.

Kołodziej [Ko98] demonstrated the existence of a continuous solution to $Dir(\Omega, \varphi, \mu)$ when μ verifies (3.3.1) and some local extra condition in a bounded strongly pseudoconvex domain with smooth boundary. Furthermore, he disposed of the extra condition in [Ko99] using Cegrell's result [Ce98] about the existence of a solution in the energy class $\mathcal{F}_1$.

Here, the existence of continuous solutions to $Dir(\Omega, \varphi, \mu)$ in a bounded SHL domain follows from the lines of Kołodziej and Cegrell's arguments in [Ko98, Ce98].

First of all, we prove the existence of continuous solutions to the Dirichlet problem for measures having densities in $L^p(\Omega)$ with respect to the Lebesgue measure.

Theorem 3.3.2. *Let $\Omega \subset \mathbb{C}^n$ be a bounded SHL domain, $\varphi \in \mathcal{C}(\partial\Omega)$ and $0 \leq f \in L^p(\Omega)$, for some $p > 1$. Then there exists a unique solution $\mathtt{U}$ to the Dirichlet problem $Dir(\Omega, \varphi, f dV_{2n})$.*

Proof. Let (f_j) be a sequence of smooth functions on $\bar{\Omega}$ which converges to f in $L^p(\Omega)$. Thanks to Theorem 2.3.2, there exists a function $\mathtt{U}_j \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega})$ such that $\mathtt{U}_j = \varphi$ on $\partial\Omega$ and $(dd^c\mathtt{U}_j)^n = f_j dV_{2n}$ in Ω. We claim that

$$\|\mathtt{U}_k - \mathtt{U}_j\|_{L^\infty(\bar{\Omega})} \leq A(1 + \|f_k\|^\eta_{L^p(\Omega)})(1 + \|f_j\|^\eta_{L^p(\Omega)})\|f_k - f_j\|^{\gamma/n}_{L^1(\Omega)}, \tag{3.3.2}$$

where $0 \leq \gamma < 1/(q+1)$ is fixed, $\eta := \frac{1}{n} + \frac{\gamma q}{n - \gamma n(1+q)}$, $1/p + 1/q = 1$ and $A = A(\gamma, n, q, \mathrm{diam}(\Omega))$.

Indeed, by the stability theorem 3.2.4 and for $r = n$, we get that

$$\sup_\Omega(\mathtt{U}_k - \mathtt{U}_j) \leq C(1 + \|f_j\|^\eta_{L^p(\Omega)})\|(\mathtt{U}_k - \mathtt{U}_j)_+\|^\gamma_{L^n(\Omega)} \leq C(1 + \|f_j\|^\eta_{L^p(\Omega)})\|\mathtt{U}_k - \mathtt{U}_j\|^\gamma_{L^n(\Omega)},$$

where $0 \leq \gamma < 1/(q+1)$ is fixed and $C = C(\gamma, n, q) > 0$.
Hence by the $L^n - L^1$ stability theorem in [Bł93] (see our Remark 2.3.10),

$$\|\mathtt{U}_k - \mathtt{U}_j\|_{L^n(\Omega)} \leq \tilde{C}\|f_k - f_j\|^{1/n}_{L^1(\Omega)},$$

where $\tilde{C}$ depends on $\mathrm{diam}(\Omega)$.
Then, from the last two inequalities and reversing the role of $\mathtt{U}_j$ and $\mathtt{U}_k$, we deduce

$$\|\mathtt{U}_k - \mathtt{U}_j\|_{L^\infty(\Omega)} \leq C\tilde{C}^\gamma(1 + \|f_k\|^\eta_{L^p(\Omega)})(1 + \|f_j\|^\eta_{L^p(\Omega)})\|f_k - f_j\|^{\gamma/n}_{L^1(\Omega)}.$$

Since $\mathtt{U}_k = \mathtt{U}_j = \varphi$ on $\partial\Omega$, the inequality (3.3.2) holds.

As f_j converges to f in $L^p(\Omega)$, there is a uniform constant $B > 0$ such that

$$\|\mathtt{U}_k - \mathtt{U}_j\|_{L^\infty(\bar{\Omega})} \leq B\|f_k - f_j\|_{L^1(\Omega)}^{\gamma/n}.$$

This implies that the sequence $\mathtt{U}_j$ converges uniformly in $\bar{\Omega}$. Set

$$\mathtt{U} = \lim_{j\to+\infty} \mathtt{U}_j.$$

It is clear that $\mathtt{U} \in PSH(\Omega)\cap\mathcal{C}(\bar{\Omega})$, $\mathtt{U} = \varphi$ on $\partial\Omega$. Moreover, $(dd^c\mathtt{U}_j)^n$ converges to $(dd^c\mathtt{U})^n$ in the sense of currents, thus $(dd^c\mathtt{U})^n = f dV_{2n}$ in Ω. The uniqueness of the solution follows from the comparison principle. □

We will summarize the steps of the proof of Theorem 3.1.1.

- We approximate μ by non-negative measures μ_s having bounded denstities with respect to the Lebesgue measure and preserving the total mass on Ω.
- We find solutions $\mathtt{U}_s$ to $Dir(\Omega, \varphi, \mu_s)$ in a bounded SHL domain Ω using Theorem 3.3.2.
- We prove that the measures μ_s are uniformly dominated by capacity. Then, we can ensure that the solutions $\mathtt{U}_s$ are uniformly bounded on $\bar{\Omega}$.
- We set $\mathtt{U} := (\limsup \mathtt{U}_s)^*$ which is a candidate to be the solution of $Dir(\Omega, \varphi, \mu)$.
- The delicate point is then to show that $(dd^c\mathtt{U}_s)^n$ converges to $(dd^c\mathtt{U})^n$ in the weak sense of measures. For this purpose, we invoke Cegrell's techniques [Ce98] to ensure that
$$\int_\Omega \mathtt{U}_s\, d\mu \to \int_\Omega \mathtt{U}\, d\mu,$$
and
$$\int_\Omega |\mathtt{U}_s - \mathtt{U}|\, d\mu_s \to 0,$$
when $s \to +\infty$.
- Finally, we assert the continuity of this solution in $\bar{\Omega}$.

Suppose first that μ has compact support in Ω. Let us consider a subdivision I^s of suppμ consisting of 3^{2ns} congruent semi-open cubes I_j^s with side $d_s = d/3^s$, where $d := \text{diam}(\Omega)$ and $1 \leq j \leq 3^{2ns}$. Thanks to Theorem 3.3.2, one can find $\mathtt{U}_s \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega})$ such that

$$\mathtt{U}_s = \varphi \text{ on } \partial\Omega,$$

and

$$(dd^c\mathtt{U}_s)^n = \mu_s := \sum_j \frac{\mu(I_j^s)}{d_s^{2n}} \chi_{I_j^s} dV_{2n} \text{ in } \Omega.$$

We will control the L^∞-norm of $\mathtt{U}_s$. For this end, we first prove that μ_s are uniformly dominated by Bedford-Taylor's capacity.

The following lemma is due to S. Kołodziej [Ko96].

Lemma 3.3.3. *Let $E \Subset \Omega$ be a Borel set. Then for any $D > 0$ there exists $t_0 > 0$ such that*

$$Cap(K_y, \Omega) \leq D\, Cap(K, \Omega),\ |y| < t_0,$$

where $K \subset E$ and $K_y := \{x;\ x - y \in K\}$.

Proof. Without loss of generality we can assume that K is compact and $K \Subset E$. We define $w_y := u^*_{K_y}(x + y)$, where u_{K_y} is the extremal function of K_y defined by

$$u_{K_y} := \sup\{v \in PSH(\Omega) : v \leq 0 \text{ on } \Omega, v \leq -1 \text{ on } K_y\}.$$

For any $0 < c < 1/2$, we set $\Omega_c := \{u^*_E < -c\}$. Let $A \gg 1$ be such that $A\rho \leq u_E$ in Ω. Since $\rho \leq -c/(2A)$ for any $x \in \Omega_{c/2}$, we can find $t_0 := t_0(E, \Omega)$ such that $x + y \in \Omega$ for any $|y| < t_0$. Therefore,

$$g(x) := \begin{cases} \max\{w_y(x) - c, (1 + 2c)u^*_E(x)\} & ; x \in \Omega_{c/2}, \\ (1 + 2c)u^*_E(x) & ; x \in \Omega \setminus \Omega_{c/2}, \end{cases}$$

is a well defined bounded psh function in Ω.
Since $K \Subset E$ and $u^*_E = -1$ on a neighborhood of K, we infer that $w_y - c \geq (1 + 2c)u^*_E$ there. Hence, we have

$$\begin{aligned} \mathrm{Cap}(K, \Omega) &\geq (1 + 2c)^{-n} \int_K (dd^c g)^n = (1 + 2c)^{-n} \int_K (dd^c w_y)^n \\ &= (1 + 2c)^{-n} \int_{K_y} (dd^c u^*_{K_y})^n = (1 + 2c)^{-n} \mathrm{Cap}(K_y, \Omega). \end{aligned}$$

Consequently, we obtain

$$\mathrm{Cap}(K_y, \Omega) \leq (1 + 2c)^n \mathrm{Cap}(K, \Omega),$$

for any $|y| < t_0$. □

Lemma 3.3.4. *Let Ω be a bounded SHL domain and μ be a compactly supported measure satisfying Condition $\mathcal{H}(\tau)$ for some $\tau > 0$. Then there exist $s_0 > 0$ and $B = B(n, \tau) > 0$ such that for all $s > s_0$ the measures μ_s, defined above, satisfy*

$$\mu_s(K) \leq B\, Cap(K, \Omega)^{1+\tau},$$

for all Borel subsets K of Ω.

Proof. Let us set $\delta_s := \operatorname{diam} I^s_j$. We define for large $s \gg 1$ a regularizing sequence of measures

$$\tilde{\mu}_s = \mu * \rho_s,$$

where $\rho_s \in \mathcal{C}^\infty_0(B(0, 2\delta_s))$ is a radially symmetric non-negative function such that

$$\rho_s = \frac{1}{2\mathrm{Vol}(B(0, \delta_s))} \text{ on } B(0, \delta_s),$$

and

$$\int_{B(0,2\delta_s)} \rho_s dV_{2n} = 1.$$

For all Borel subsets $K \subset \Omega$, we get

$$\begin{aligned}
\tilde{\mu}_s(K) &= \sum_j \int_{K\cap I_j^s} \left(\int_{B(x,2\delta_s)} \rho_s(x-y)d\mu(y) \right) dV_{2n} \\
&\geq \sum_j \int_{K\cap I_j^s} \left(\int_{B(x,\delta_s)} \rho_s(x-y)d\mu(y) \right) dV_{2n} \\
&\geq \sum_j \int_{K\cap I_j^s} \left(\frac{\mu(B(x,\delta_s))}{2\mathrm{Vol}(B(x,\delta_s))} \right) dV_{2n} \\
&\geq \frac{1}{2(2n)^n \tau_{2n}} \sum_j \int_K \left(\frac{\mu(I_j^s)}{d_s^{2n}} \chi_{I_j^s} \right) dV_{2n} \\
&= \frac{\mu_s(K)}{2(2n)^n \tau_{2n}},
\end{aligned}$$

where τ_{2n} is the volume of the unit ball in $\mathbb{C}^n$.
We set $K_y := \{x; x-y \in K\}$, for $y \in \mathbb{C}^n$. Then, by Lemma 3.3.3, we find $t_0 > 0$ and $s_0 > 1/t_0$ such that

$$\mathrm{Cap}(K_y,\Omega) \leq 2\mathrm{Cap}(K,\Omega),\ |y| < t_0,$$

for any Borel set $K \subset \cup_{s>s_0}\mathrm{supp}\mu_s \Subset \Omega$.
We infer for all $s > s_0$ and $K \subset \Omega$, that

$$\tilde{\mu}_s(K) \leq \sup_{|y|<1/s} \mu(K_y) \leq A \sup_{|y|<1/s} \mathrm{Cap}(K_y,\Omega)^{1+\tau} \leq 2^{1+\tau} A\mathrm{Cap}(K,\Omega)^{1+\tau}.$$

This completes the proof. □

Proposition 3.3.5. *There exists a uniform constant $C > 0$ such that*

$$\|\mathtt{U}_s\|_{L^\infty(\bar{\Omega})} \leq C,$$

for all $s > s_0$, where s_0 is as in Lemma 3.3.4.

Proof. We owe the idea of the proof to Benelkourchi, Guedj and Zeriahi [BGZ08] in a slightly different context. Without loss of generality we can assume $\varphi = 0$ in $Dir(\Omega,\varphi,\mu)$ and $\mu(\Omega) \leq 1$.
Let us fix $s > s_0$. It follows from Lemma 3.3.4 that there exists a uniform constant $B = B(n,\tau) > 0$ so that the following inequality holds for all Borel sets $K \subset \Omega$,

$$\mu_s(K) \leq B\mathrm{Cap}(K,\Omega)^{1+\tau}.$$

We define for $k > 0$,

$$g(k) := -\frac{1}{n} \ln(\mathrm{Cap}\{\mathtt{U}_s < -k\}).$$

This function is increasing on $[0,+\infty]$ and $g(+\infty) = +\infty$. We claim that

$$\ln t + (1+\tau)g(k) - \ln B/n \leq g(k+t), \tag{3.3.3}$$

for all $t, k > 0$. Indeed, Lemma 3.2.2 yields that

$$t^n \mathrm{Cap}(\{\mathtt{U}_s < -k-t\}) \leq \mu_s(\{\mathtt{U}_s < -k\}) \leq B\mathrm{Cap}(\{\mathtt{U}_s < -k\})^{1+\tau}. \tag{3.3.4}$$

Now we define an increasing sequence (k_j) as follows

$$k_{j+1} := k_j + B^{1/n} e^{1-\tau g(k_j)}, \text{ for all } j \in \mathbb{N},$$

where $k_0 = 2$.
We claim that $g(k_0) \geq 0$. To get this end, we apply the inequality (3.3.4) for $t = k = 1$, then we get

$$\mathrm{Cap}(\{\mathtt{U}_s < -2\}) \leq \mu_s(\{\mathtt{U}_s < -1\}) \leq \mu(\Omega) \leq 1.$$

We apply (3.3.3) with $t = t_j = k_{j+1} - k_j$ and $k = k_j$ to get that

$$g(k_j) \geq j + g(k_0) \geq j.$$

Thus $g(k_j)$ goes to $+\infty$ as j goes to $+\infty$.
Let us set $k_\infty := \lim_{N\to+\infty} k_N$. Then $g(k_\infty) = +\infty$. We claim that k_∞ is bounded by an absolute constant independent of $\mathtt{U}_s$.

$$\begin{aligned}
k_\infty &= \lim_{N\to+\infty} \sum_0^{N-1} (k_{j+1} - k_j) + 2 \\
&= \lim_{N\to+\infty} \sum_0^{N-1} (B^{1/n} e^{1-\tau g(k_j)}) + 2 \\
&\leq \lim_{N\to+\infty} e B^{1/n} \sum_0^{N-1} e^{-\tau j} + 2 \\
&\leq e B^{1/n}/(1 - e^{-\tau}) + 2 =: M(n, \tau).
\end{aligned}$$

For any $k \geq k_\infty$, we conclude that $g(k) = +\infty$, hence

$$\mathrm{Cap}(\{\mathtt{U}_s < -k\}) = 0 \text{ for all } k \geq k_\infty.$$

This means that for any $s > s_0$ the function $\mathtt{U}_s$ is bounded from below by an absolute constant $-k_\infty \geq -M(n, \tau)$. □

Thanks to Proposition 3.3.5, the sequence $(\mathtt{U}_s)$ is uniformly bounded. Passing to a subsequence we can assume that $\mathtt{U}_s$ converges in $L^1_{\mathrm{loc}}(\Omega)$ (see Theorem 4.1.9 in [H83]). Let us set $\mathtt{U} := (\limsup \mathtt{U}_s)^* \in PSH \cap L^\infty(\Omega)$. Hence $\mathtt{U}_s$ converges to $\mathtt{U}$ almost everywhere in Ω with respect to the Lebesgue measure dV_{2n}.

Lemma 3.3.6. *Let μ be a finite Borel measure on Ω. Suppose that $\mathtt{U}_s \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega})$ converges to $\mathtt{U} \in PSH(\Omega) \cap L^\infty(\Omega)$ almost everywhere with respect to the Lebesgue measure and $\|\mathtt{U}_s\|_{L^\infty(\bar{\Omega})} \leq C$, for some uniform constant $C > 0$. Then, we have*

$$\lim_{s\to+\infty} \int_\Omega \mathtt{U}_s d\mu = \int_\Omega \mathtt{U} d\mu, \tag{3.3.5}$$

and

$$\lim_{s\to+\infty} \int_\Omega |\mathtt{U}_s - \mathtt{U}| (dd^c \mathtt{U}_s)^n = 0. \tag{3.3.6}$$

Proof. Since $\mathtt{U}_s$ is uniformly bounded in $L^2(\Omega, d\mu)$, there exists a subsequence, for which we keep the same notation, $(\mathtt{U}_s)$ converges weakly to v_1 in $L^2(\Omega, d\mu)$. In particular, $\mathtt{U}_s$ converges to v_1 almost everywhere with respect to $d\mu$ and

$$\int_\Omega \mathtt{U}_s d\mu \to \int_\Omega v_1 d\mu.$$

By Banach-Saks' Theorem there exists a subsequence $\mathtt{U}_s$ such that $(1/M)\sum_{s=1}^M \mathtt{U}_s$ converges to v_2 in $L^2(\Omega, d\mu)$ and hence there exists a subsequence such that $f_M = (1/M)\sum_{s=1}^M \mathtt{U}_s$ converges to v_2 almost everywhere with respect to $d\mu$, when $M \to +\infty$. Hence $v_1 = v_2$ almost everywhere with respect to $d\mu$ and we have

$$\int_\Omega (\sup_{N\geq M} f_M)^* d\mu = \int_\Omega \sup_{N\geq M} f_M d\mu \to \int_\Omega v_2 d\mu = \int_\Omega v_1 d\mu.$$

On the other hand, $f_M \to \mathtt{U}$ in $L^2(\Omega, dV_{2n})$ and so $(\sup_{N\geq M} f_M)^* \searrow \mathtt{U}$ everywhere in Ω and thus

$$\int_\Omega (\sup_{N\geq M} f_M)^* d\mu \to \int_\Omega \mathtt{U} d\mu.$$

Then we get

$$\lim_{s\to+\infty} \int_\Omega \mathtt{U}_s d\mu = \int_\Omega v_1 d\mu = \int_\Omega v_2 d\mu = \lim_{M\to+\infty} \int_\Omega (\sup_{N\geq M} f_M)^* d\mu = \int_\Omega \mathtt{U} d\mu.$$

So as to prove (3.3.6), we define

$$v_s(x) = \frac{1}{\tau_{2n}(2nd_s)^{2n}} \int_{|\xi|\leq 2nd_s} |\mathtt{U}(x+\xi) - \mathtt{U}_s(x+\xi)| dV_{2n},$$

where τ_{2n} is the volume of the unit ball in $\mathbb{C}^n$ and $d_s = \operatorname{diam}(\Omega)/3^s$.
Then we see that

$$\begin{aligned}
\int_\Omega |\mathtt{U}_s - \mathtt{U}|(dd^c\mathtt{U}_s)^n &= \sum_j \frac{\mu(I_j^s)}{d_s^{2n}} \int_{I_j^s} |\mathtt{U} - \mathtt{U}_s| dV_{2n} \\
&\leq \sum_j \tau_{2n}(2n)^{2n} \int_{I_j^s} v_s(x) d\mu(x) \\
&\leq \tau_{2n}(2n)^{2n} \int_\Omega v_s(x) d\mu(x).
\end{aligned}$$

We claim that $\int_\Omega v_s(x)d\mu(x) \to 0$ as $s \to +\infty$. Indeed, we note that

$$\begin{aligned}
v_s(x) &= \frac{1}{\tau_{2n}(2nd_s)^{2n}} \int_{|\xi|\leq 2nd_s} |\mathtt{U}(x+\xi) - \sup_{j\geq s} \mathtt{U}_j(x+\xi) + \sup_{j\geq s} \mathtt{U}_j(x+\xi) - \mathtt{U}_s(x+\xi)| dV_{2n} \\
&\leq \frac{1}{\tau_{2n}(2nd_s)^{2n}} \int_{|\xi|\leq 2nd_s} (\sup_{j\geq s} \mathtt{U}_j(x+\xi) - \mathtt{U}(x+\xi)) dV_{2n} \\
&\quad + \frac{1}{\tau_{2n}(2nd_s)^{2n}} \int_{|\xi|\leq 2nd_s} \sup_{j\geq s} \mathtt{U}_j(x+\xi) dV_{2n} - \frac{1}{\tau_{2n}(2nd_s)^{2n}} \int_{|\xi|\leq 2nd_s} \mathtt{U}_s(x+\xi) dV_{2n} \\
&\leq \frac{2}{\tau_{2n}(2nd_s)^{2n}} \int_{|\xi|\leq 2nd_s} (\sup_{j\geq s} \mathtt{U}_j(x+\xi))^* dV_{2n} - \mathtt{U}(x) - \mathtt{U}_s(x).
\end{aligned}$$

It stems from the monotone convergence theorem and (3.3.5) that

$$\int_\Omega v_s(x)d\mu(x) \to 0,\ s \to +\infty.$$

□

Proof of Theorem 3.1.1. We can assume, by passing to a subsequence in (3.3.6), that $\int_\Omega |\mathtt{U}_s - \mathtt{U}|(dd^c\mathtt{U}_s)^n \le 1/s^2$. Consider

$$\tilde{\mathtt{U}}_s := \max\{\mathtt{U}_s, \mathtt{U} - 1/s\} \in PSH(\Omega) \cap L^\infty(\bar{\Omega}).$$

It follows from Hartogs' lemma that $\tilde{\mathtt{U}}_s \to \mathtt{U}$ in Bedford-Taylor's capacity. In fact, we prove that for any Borel set $K \subset \Omega$ such that $\mathtt{U}|_K$ is continuous we have $\tilde{\mathtt{U}}_s$ converges uniformly to $\mathtt{U}$ on K. Since $\tilde{\mathtt{U}}_s \to \mathtt{U}$ in $L^1_{loc}(\Omega)$ and by Theorem 4.1.9 in [H83] we get

$$\lim_{s\to+\infty} \sup_K (\tilde{\mathtt{U}}_s - \mathtt{U}) = 0.$$

Thereby, we conclude that

$$\|\tilde{\mathtt{U}}_s - \mathtt{U}\|_{L^\infty(K)} \to 0,\ \text{as } s \to +\infty.$$

Thus the convergence in capacity of $\tilde{\mathtt{U}}_s$ to $\mathtt{U}$ comes immediately from the quasicontinuity of $\mathtt{U}$. Now, since $\tilde{\mathtt{U}}_s$ is uniformly bounded for all $s > s_0$ as in Proposition 3.3.5, we get by Theorem 1.2.3 that $(dd^c\tilde{\mathtt{U}}_s)^n$ converges to $(dd^c\mathtt{U})^n$ in the weak sense of currents.
We need now to compare $(dd^c\tilde{\mathtt{U}}_s)^n$ and $(dd^c\mathtt{U}_s)^n$ following [GZ07]. It is known that

$$(dd^c\tilde{\mathtt{U}}_s)^n \ge \mathbf{1}_{\{\mathtt{U}_s \ge \mathtt{U}-1/s\}}(dd^c\mathtt{U}_s)^n.$$

Our assumption implies that $\mathbf{1}_{\{\mathtt{U}_s < \mathtt{U}-1/s\}}(dd^c\mathtt{U}_s)^n \to 0$. Indeed,

$$0 \le \int_{\{\mathtt{U}_s < \mathtt{U}-1/s\}} (dd^c\mathtt{U}_s)^n \le s\int_\Omega |\mathtt{U}_s - \mathtt{U}|(dd^c\mathtt{U}_s)^n \le 1/s.$$

Therefore, $0 \le (dd^c\mathtt{U}_s)^n \le (dd^c\tilde{\mathtt{U}}_s)^n + o(1)$, hence we get by letting $s \to +\infty$ that

$$(dd^c\mathtt{U})^n \ge d\mu.$$

Now, we prove that

$$\int_\Omega (dd^c\mathtt{U})^n = \int_\Omega d\mu.$$

Actually, let v be the continuous solution to the Dirichlet problem for the homogeneous Monge-Ampère equation with the boundary data φ. From the comparison principle we get $\mathtt{U}_s \le v$ for all $s > 0$ and so $\mathtt{U} \le v$ in Ω. Since the continuous function $v - \mathtt{U}_s$ equals to zero on $\partial\Omega$, we find a neighborhood of $\partial\Omega$ such that $v - \mathtt{U}_s < 1/s$ there. Hence, $\mathtt{U} - 1/s \le v - 1/s < \mathtt{U}_s$ in this neighborhood and so that $\tilde{\mathtt{U}}_s = \mathtt{U}_s$ there. Now, we get by Stokes' theorem

$$\int_\Omega (dd^c\tilde{\mathtt{U}}_s)^n = \int_\Omega (dd^c\mathtt{U}_s)^n = \int_\Omega d\mu.$$

By the weak convergence of measures, we obtain

$$\int_\Omega (dd^c \mathtt{U})^n \le \int_\Omega d\mu.$$

This complete the proof of Theorem 3.1.1 when μ has compact support in Ω.

For the general case, when μ is only satisfying Condition $\mathcal{H}(\tau)$. Let χ_j is a nondecreasing sequence of smooth cut-off function, $\chi_j \nearrow 1$ in Ω, we can do the same argument and get solutions $\mathtt{U}_j$ to the Dirichlet problem for the measures $\chi_j\mu$. By Lemma 3.3.5, the solutions $\mathtt{U}_j$ are uniformly bounded. We set $\mathtt{U} := (\limsup \mathtt{U}_j)^* \in PSH(\Omega) \cap L^\infty(\bar\Omega)$ and the last argument yields that $\mathtt{U}$ is the required bounded solution to $Dir(\Omega, \varphi, \mu)$.

It remains to prove the continuity of the solution $\mathtt{U}$ in $\bar\Omega$. It is clear that

$$\lim_{z\to\xi} \mathtt{U}(z) = \varphi(\xi),\ \forall \xi \in \partial\Omega. \tag{3.3.7}$$

Let us fix $K \subset \Omega$ and let u_j be the standard regularization of $\mathtt{U}$. We extend φ to a continuous function on $\bar\Omega$. For all small $d > 0$ we can find by (3.3.7) an open set $K_d \supset K$ and $j_0 > 0$ such that

$$\varphi < \mathtt{U} + d/2 \text{ and } u_j < \varphi + d/2 \text{ in a neighborhood of } \partial K_d, \forall j \ge j_0.$$

Hence $u_j < \mathtt{U} + d$ in a neighborhood of ∂K_d for all $j \ge j_0$ and then

$$\liminf_{z\to\zeta} (\mathtt{U}(z) + d - u_j(z)) \ge 0,$$

for all $\zeta \in \partial K_d$.

We claim that the set $\{u_j - \mathtt{U} > 2d\}$ is empty for any $j \ge j_0$. Otherwise, we will get a contradiction following similar techniques to those in Lemma 3.2.3 and Lemma 3.3.5 as follows. Let us set $v_1 := \mathtt{U} + d$ and $v_2 := u_j$. We define for $s \ge 0$ the function $g(s) := \mathrm{Cap}(\{v_1 - v_2 < -s\})$ and an increasing sequence (k_m) such that $k_0 := 0$ and

$$k_m := \sup\{k > k_{m-1}; g(k) > g(k_{m-1})/e\}.$$

Hence we get $g(k_m) \le g(k_{m-1})/e$. Let N be an integer so that $k_N \le d$ and

$$g(d) \ge g(k_N)/e.$$

By Lemma 3.2.2 we obtain

$$(d - k_N)^n g(d) \le \mu(\{v_1 - v_2 < -k_N\}) \le Ae^{1+\tau} g(d)^{1+\tau}.$$

Then we get

$$d - k_N \le A^{1/n} e^{(1+\tau)/n} g(d)^{\tau/n}. \tag{3.3.8}$$

Now, let $t := k - k_{m-1}$ where $0 < k_{m-1} < k \le d$ such that $g(k) > g(k_{m-1})/e$. We infer again by Lemma 3.2.2 that

$$t^n g(k) \le \mu(\{v_1 - v_2 < -k_{m-1}\} \le Aeg(k) g(k_{m-1})^\tau.$$

Hence,

$$t \le (Ae)^{1/n} g(k_{m-1})^{\tau/n}.$$

Letting $k \to k_m^-$, we get

$$t_m := k_m - k_{m-1} \le (Ae)^{1/n} g(k_{m-1})^{\tau/n}.$$

Then we have

$$k_N = \sum_{m=1}^{m=N} t_m \le (Ae)^{1/n} \sum_{m=1}^{m=N} g(k_{m-1})^{\tau/n} \le (Ae)^{1/n} N g(0)^{\tau/n}.$$

By the definition of convergence by capacity, we get for $j \ge j_0$ that $g(0)$ is very small so that $k_N \le d/2$. Then (3.3.8) yields that

$$d/2 \le A^{1/n} e^{(1+\tau)/n} g(d)^{\tau/n}.$$

Since $d > 0$ is fixed and $g(d) = \mathrm{Cap}(\{u_j - \mathtt{U} > 2d\})$ goes to zero when j goes to $+\infty$, we obtain a contradiction in the last inequality. □

3.4 Hölder continuity of solutions

We introduce in this section the basic ingredients of proofs of main theorems. Let μ be a measure satisfying Condition $\mathcal{H}(\infty)$, $0 \le f \in L^p(\Omega, \mu)$, $p > 1$ and $\varphi \in \mathcal{C}(\partial\Omega)$. Thanks to Theorem 3.1.1, we denote by $\mathtt{U}$ the continuous solution to $Dir(\Omega, \varphi, f d\mu)$ and consider

$$\mathtt{U}_\delta(z) := \sup_{|\zeta| \le \delta} \mathtt{U}(z + \zeta), \ z \in \Omega_\delta,$$

where $\Omega_\delta := \{z \in \Omega; \mathrm{dist}(z, \partial\Omega) > \delta\}$.

To ensure the Hölder continuity of the solution in Ω, we need to control the L^∞-norm of $\mathtt{U}_\delta - \mathtt{U}$ in Ω_δ.

It will be shown in Lemma 3.4.3 that the Hölder norm of the solution $\mathtt{U}$ can be estimated by using either $\sup_{\Omega_\delta}(\mathtt{U}_\delta - \mathtt{U})$ or $\sup_{\Omega_\delta}(\hat{\mathtt{U}}_\delta - \mathtt{U})$, where

$$\hat{\mathtt{U}}_\delta(z) := \frac{1}{\tau_{2n}\delta^{2n}} \int_{|\zeta - z| \le \delta} \mathtt{U}(\zeta) dV_{2n}(\zeta), \quad z \in \Omega_\delta,$$

and τ_{2n} is the volume of the unit ball in $\mathbb{C}^n$.

It is clear that $\hat{\mathtt{U}}_\delta$ is not globally defined in Ω, so we extend it with a good control near the boundary $\partial\Omega$. To this end, we assume the existence of ν-Hölder continuous function v such that $v \le \mathtt{U}$ in Ω and $v = \mathtt{U}$ on $\partial\Omega$. Then, we present later the construction of such a function.

Lemma 3.4.1. *Let Ω be a bounded SHL domain and $\varphi \in \mathcal{C}^{0,\alpha}(\partial\Omega)$, $0 < \alpha \le 1$. Assume that there is a function $v \in \mathcal{C}^{0,\nu}(\bar{\Omega})$ for $0 < \nu \le 1$, such that $v \le \mathtt{U}$ in Ω and $v = \varphi$ on $\partial\Omega$. Then there exist $\delta_0 > 0$ small enough and $c_0 > 0$, depending on $\Omega, \|\varphi\|_{\mathcal{C}^{0,\alpha}(\partial\Omega)}$ and $\|v\|_{\mathcal{C}^{0,\nu}(\bar{\Omega})}$, such that for any $0 < \delta_1 \le \delta < \delta_0$ the function*

$$\tilde{\mathtt{U}}_{\delta_1} = \begin{cases} \max\{\hat{\mathtt{U}}_{\delta_1}, \mathtt{U} + c_0\delta^{\nu_1}\} & in \ \ \Omega_\delta, \\ \mathtt{U} + c_0\delta^{\nu_1} & in \ \ \Omega \setminus \Omega_\delta, \end{cases}$$

is plurisubharmonic in Ω and continuous on $\bar{\Omega}$, where $\nu_1 = \min\{\nu, \alpha/2\}$.

Proof. If we prove that $\hat{\mathtt{U}}_{\delta_1} \leq \mathtt{U} + c_0\delta^{\nu_1}$ on $\partial\Omega_\delta$, then the required result can be obtained by the standard gluing procedure.
Thanks to Corollary 2.4.6, we find a plurisuperharmonic function $\tilde{v} \in \mathcal{C}^{0,\alpha/2}(\bar{\Omega})$ such that $\tilde{v} = \varphi$ on $\partial\Omega$ and

$$\|\tilde{v}\|_{\mathcal{C}^{0,\alpha/2}(\bar{\Omega})} \leq C\|\varphi\|_{\mathcal{C}^{0,\alpha}(\partial\Omega)},$$

where C depends on Ω. From the maximum principle we see that $\mathtt{U} \leq \tilde{v}$ in Ω and $\tilde{v} = \varphi$ on $\partial\Omega$.
Fix $z \in \partial\Omega_\delta$, there exists $\zeta \in \mathbb{C}^n$ with $\|\zeta\| = \delta_1$ such that $\hat{\mathtt{U}}_{\delta_1}(z) \leq \mathtt{U}(z+\zeta)$. Hence, we obtain

$$\hat{\mathtt{U}}_{\delta_1}(z) - \mathtt{U}(z) \leq \mathtt{U}(z+\zeta) - \mathtt{U}(z) \leq \tilde{v}(z+\zeta) - v(z).$$

We choose $\zeta_0 \in \mathbb{C}^n$, with $\|\zeta_0\| = \delta$, so that $z + \zeta_0 \in \partial\Omega$. Since $\tilde{v}(z+\zeta_0) = v(z+\zeta_0)$, we infer

$$\begin{aligned}\tilde{v}(z+\zeta) - v(z) &\leq [\tilde{v}(z+\zeta) - \tilde{v}(z+\zeta_0)] + [v(z+\zeta_0) - v(z)]\\ &\leq 2\|\tilde{v}\|_{\mathcal{C}^{0,\alpha/2}(\bar{\Omega})}\delta^{\alpha/2} + \|v\|_{\mathcal{C}^{0,\nu}(\bar{\Omega})}\delta^{\nu}\\ &\leq c_0\delta^{\nu_1},\end{aligned}$$

where $c_0 := 2C\|\varphi\|_{\mathcal{C}^{0,\alpha}(\partial\Omega)} + \|v\|_{\mathcal{C}^{0,\nu}(\bar{\Omega})}$. □

Moreover, we can conclude from the last argument that

$$|\mathtt{U}(z_1) - \mathtt{U}(z_2)| \leq 2c_0\delta^{\nu_1}, \tag{3.4.1}$$

for all $z_1, z_2 \in \bar{\Omega} \setminus \Omega_\delta$ such that $|z_1 - z_2| \leq \delta$.

Remark 3.4.2. When $\varphi \in \mathcal{C}^{1,1}(\partial\Omega)$, the last lemma holds for $\nu_1 = \nu$. Indeed, let $\tilde{\varphi}$ be a $\mathcal{C}^{1,1}$-extension of φ to $\bar{\Omega}$. We define the plurisuperharmonic Lipschitz function $\tilde{v} := -A\rho + \tilde{\varphi}$, where $A \gg 1$ and ρ is the defining function of Ω. Hence, the constant c_0 in Lemma 3.4.1 will depend only on Ω, $\|\varphi\|_{\mathcal{C}^{1,1}(\partial\Omega)}$ and $\|v\|_{\mathcal{C}^{0,\nu}(\bar{\Omega})}$.

Lemma 3.4.3. *Given* $0 < \alpha < 1$*, the following conditions are equivalent.*

1. *There exist* $\delta', A > 0$ *such that for any* $0 < \delta \leq \delta'$*,*

$$\mathtt{U}_\delta - \mathtt{U} \leq A\delta^\alpha \text{ on } \Omega_\delta.$$

2. *There exist* $\delta'', B > 0$ *such that for any* $0 < \delta \leq \delta''$*,*

$$\hat{\mathtt{U}}_\delta - \mathtt{U} \leq B\delta^\alpha \text{ on } \Omega_\delta.$$

Proof. Since $\hat{\mathtt{U}}_\delta \leq \mathtt{U}_\delta$, we get immediately the implication (1) ⇒ (2). In order to prove (2) ⇒ (1) we need to show that there exist $\delta', A > 0$ such that

$$\omega(\delta) := \sup_{z\in\Omega_\delta}[(\mathtt{U}_\delta - \mathtt{U})(z)] \leq A\delta^\alpha.$$

Fix $\delta_\Omega > 0$ small enough so that $\Omega_\delta \neq \emptyset$ for $\delta \leq \tilde{\delta}_\Omega := (C+2)\delta_\Omega$ where $C > 0$ is a constant to be chosen later. Since $\mathtt{U}$ is uniformly continuous on $\bar{\Omega}$, we have for any fixed $0 < \delta < \tilde{\delta}_\Omega$,

$$\nu(\delta) := \sup_{\delta < t \leq \tilde{\delta}_\Omega} \omega(t)t^{-\alpha} < +\infty.$$

We claim that there exists $\delta' > 0$ small enough such that for any $0 < \delta \leq \delta'$, we have

$$\omega(\delta) \leq A\delta^\alpha \text{ with } A = 4c_0(C+3)^\alpha + e^4(C+1)^\alpha B + \nu(\delta_\Omega),$$

where c_0 is as in Lemma 3.4.1. Assume that this is not the case. Then there exists $\delta \leq \delta_\Omega$ such that

$$\omega(\delta) > A\delta^\alpha.$$

Let us set $\delta := \sup\{t < \delta_\Omega;\ \omega(t) > At^\alpha\}$. Then

$$\frac{\omega(\delta)}{\delta^\alpha} \geq A \geq \frac{\omega(t)}{t^\alpha} \text{ for all } t \in [\delta, \tilde{\delta}_\Omega]. \tag{3.4.2}$$

Since $\mathtt{U}$ is continuous on $\bar{\Omega}$, we find $z_0 \in \bar{\Omega}_\delta$, $\zeta_0 \in \bar{\Omega}$ such that $|z_0 - \zeta_0| \leq \delta$ and

$$\omega(\delta) = \mathtt{U}(\zeta_0) - \mathtt{U}(z_0).$$

We assert that $\text{dist}(z_0, \partial\Omega) > (C+2)\delta$. In fact, if $\text{dist}(z_0, \partial\Omega) \leq (C+2)\delta$ and $z_1 \in \partial\Omega$ such that $\text{dist}(z_0, z_1) = \text{dist}(z_0, \partial\Omega)$, then we have by (3.4.1) that

$$\omega(\delta) = \mathtt{U}(\zeta_0) - \mathtt{U}(z_1) + \mathtt{U}(z_1) - \mathtt{U}(z_0) \leq 4c_0(C+3)^\alpha\delta^\alpha < A\delta^\alpha.$$

This is a contradiction.
Now we apply (3.4.2) for $t = (C+2)\delta$ and hence we get

$$\mathtt{U}(\zeta_0) - \mathtt{U}(z) \leq (C+2)^\alpha\omega(\delta) \text{ for all } z \in B(z_0, (C+1)\delta).$$

As $B_1 := B(\zeta_0, C\delta) \subset B_2 := B(z_0, (C+1)\delta)$, we can write

(3.4.3)

$$\begin{aligned}
\hat{\mathtt{U}}_{(C+1)\delta}(z_0) &= \frac{1}{\tau_{2n}(C+1)^{2n}\delta^{2n}}\int_{B_2}\mathtt{U}(z)dV_{2n}(z)\\
&= \left(\frac{C}{C+1}\right)^{2n}\frac{1}{\tau_{2n}C^{2n}\delta^{2n}}\int_{B_1}\mathtt{U}(z)dV_{2n}(z) + \frac{1}{\tau_{2n}(C+1)^{2n}\delta^{2n}}\int_{B_2\setminus B_1}\mathtt{U}(z)dV_{2n}(z)\\
&\geq \left(\frac{C}{C+1}\right)^{2n}\mathtt{U}(\zeta_0) + [\mathtt{U}(\zeta_0) - (C+2)^\alpha\omega(\delta)]\left(1 - \left(\frac{C}{C+1}\right)^{2n}\right)\\
&= \mathtt{U}(\zeta_0) - (C+2)^\alpha\left(1 - \left(\frac{C}{C+1}\right)^{2n}\right)\omega(\delta)\\
&= \mathtt{U}(z_0) + D\omega(\delta),
\end{aligned}$$

where $D := 1 - (C+2)^\alpha\left(1 - \left(\frac{C}{C+1}\right)^{2n}\right)$. We have $D \geq e^{-4}$ if

$$\alpha \leq \frac{1}{\log(C+2)}\log\left(\frac{1-e^{-4}}{1-\left(\frac{C}{C+1}\right)^{2n}}\right) =: \tilde{\alpha}.$$

Hence, we infer

$$\hat{\mathtt{U}}_{(C+1)\delta}(z_0) \geq \mathtt{U}(z_0) + e^{-4}\omega(\delta).$$

By (2), the last inequality is equivalent to

$$\omega(\delta) \leq e^4B(C+1)^\alpha\delta^\alpha < A\delta^\alpha.$$

This is a contradiction and hence our claim is true. It remains to show that for any fixed $0 < \alpha < 1$ we can find $C > 0$ such that $\tilde{\alpha} > \alpha$. For this end, we choose $C := n/x$ with $0 < x < 1$ and note that

$$\left(\frac{n/x}{n/x+1}\right)^{2n} \geq e^{-2x} \text{ for all } n \in \mathbb{N}.$$

Hence this yields that

$$\tilde{\alpha} \geq \frac{1}{\log(n/x+2)} \log\left(\frac{1-e^{-4}}{1-e^{-2x}}\right).$$

Since the function

$$g(x) := \frac{\log(1-e^{-4}) - \log(1-e^{-2x})}{\log(n/x+2)}$$

is continuous on $]0,1[$ and $\lim_{x\to 0} g(x) = 1$, we can find $x > 0$ small enough such that $g(x) \geq \alpha$.
This completes the proof. □

Theorem 3.4.4. *Let Ω be a bounded SHL domain and let μ be a finite Borel measure on Ω satisfying Condition $\mathcal{H}(\infty)$. Suppose that $\varphi \in \mathcal{C}^{0,\alpha}(\partial\Omega)$, $0 < \alpha \leq 1$, and $0 \leq f \in L^p(\Omega, \mu)$ for $p > 1$. Then the solution $\mathtt{U}$ to $Dir(\Omega, \varphi, f d\mu)$ is Hölder continuous on $\bar{\Omega}$ of exponent $\frac{1}{\lambda}\min\{\nu, \alpha/2, \tau\gamma\}$, for any $\gamma < 1/(nq+1)$ and $1/p + 1/q = 1$, if the two following conditions hold:*
(i) there exists $v \in \mathcal{C}^{0,\nu}(\bar{\Omega})$, for $0 < \nu \leq 1$, such that $v \leq \mathtt{U}$ in Ω and $v = \varphi$ on $\partial\Omega$,
(ii) and $\|\hat{\mathtt{U}}_{\delta_1} - \mathtt{U}\|_{L^1(\Omega_\delta,\mu)} \leq c\delta^\tau$, where $c, \tau > 0$ and $0 < \delta_1 = \delta^\lambda$, for some $\lambda \geq 1$.
Moreover, if $\varphi \in \mathcal{C}^{1,1}(\partial\Omega)$ then the Hölder exponent of $\mathtt{U}$ will be $\frac{1}{\lambda}\min\{\nu, \tau\gamma\}$.

Proof. It follows from Lemma 3.4.1 that there exist $c_0 > 0$ and $\delta_0 > 0$ so that

$$\tilde{\mathtt{U}}_{\delta_1} = \begin{cases} \max\{\hat{\mathtt{U}}_{\delta_1}, \mathtt{U} + c_0\delta^{\nu_1}\} & \text{in } \Omega_\delta, \\ \mathtt{U} + c_0\delta^{\nu_1} & \text{in } \Omega \setminus \Omega_\delta, \end{cases}$$

belongs to $PSH(\Omega) \cap \mathcal{C}(\bar{\Omega})$, for $0 < \delta_1 \leq \delta < \delta_0$ and $\nu_1 = \min\{\nu, \alpha/2\}$.
By applying Theorem 3.2.4 with $v_1 := \mathtt{U} + c_0\delta^{\nu_1}$ and $v_2 := \tilde{\mathtt{U}}_{\delta_1}$, we infer that

$$\sup_{\Omega_\delta}(\hat{\mathtt{U}}_{\delta_1} - \mathtt{U} - c_0\delta^{\nu_1}) \leq \sup_{\Omega}(\tilde{\mathtt{U}}_{\delta_1} - \mathtt{U} - c_0\delta^{\nu_1}) \leq c_1(1 + \|f\|^\eta_{L^p(\Omega,\mu)})\|(\tilde{\mathtt{U}}_{\delta_1} - \mathtt{U} - c_0\delta^{\nu_1})_+\|^\gamma_{L^1(\Omega,\mu)},$$

where $\eta := 1/n + \gamma q/[1 - \gamma(1+nq)]$, $c_1 = c_1(n, q, \gamma)$ and $0 < \gamma < 1/(nq+1)$ is fixed.

Since $\tilde{\mathtt{U}}_{\delta_1} = \mathtt{U} + c_0\delta^{\nu_1}$ in $\Omega \setminus \Omega_\delta$ and

$$\|(\tilde{\mathtt{U}}_{\delta_1} - \mathtt{U} - c_0\delta^{\nu_1})_+\|_{L^1(\Omega,\mu)} \leq \|\hat{\mathtt{U}}_{\delta_1} - \mathtt{U}\|_{L^1(\Omega_\delta,\mu)}.$$

We conclude that

$$\sup_{\Omega_\delta}(\hat{\mathtt{U}}_{\delta_1} - \mathtt{U}) \leq c_0\delta^{\nu_1} + c_1(1 + \|f\|^\eta_{L^p(\Omega,\mu)})\|\hat{\mathtt{U}}_{\delta_1} - \mathtt{U}\|^\gamma_{L^1(\Omega_\delta,\mu)}.$$

By hypotheses we have

$$\sup_{\Omega_\delta}(\hat{\mathtt{U}}_{\delta_1} - \mathtt{U}) \leq c_0\delta^{\nu_1} + c_1 c^\gamma(1 + \|f\|^\eta_{L^p(\Omega,\mu)})\delta^{\tau\gamma}.$$

Let us set $c_2 := (c_0 + c_1 c^\gamma)(1 + \|f\|^\eta_{L^p(\Omega,\mu)})$. We derive from the last inequality that

$$\sup_{\Omega_\delta}(\hat{\mathtt{U}}_{\delta_1} - \mathtt{U}) \le c_2 \delta^{\min\{\nu_1, \tau\gamma\}}.$$

This means that

$$\hat{\mathtt{U}}_\delta - \mathtt{U} \le c_2 \delta^{\frac{1}{\lambda}\min\{\nu_1,\tau\gamma\}} \text{ in } \Omega_{\delta^{1/\lambda}}.$$

Hence, by Lemma 3.4.3, there exists $c_3, \tilde{\delta}_0 > 0$ such that for all $0 < \delta < \tilde{\delta}_0$ we have

$$\mathtt{U}_\delta - \mathtt{U} \le c_3 \delta^{\frac{1}{\lambda}\min\{\nu_1,\tau\gamma\}} \text{ in } \Omega_{\delta^{1/\lambda}}. \tag{3.4.4}$$

Thus, (3.4.4) and (3.4.1) yield the Hölder continuity of $\mathtt{U}$ on $\bar{\Omega}$ of exponent $\frac{1}{\lambda}\min\{\nu, \alpha/2, \tau\gamma\}$, for any $\gamma < 1/(nq+1)$ and $1/p + 1/q = 1$.

Finally, if $\varphi \in \mathcal{C}^{1,1}(\partial\Omega)$, we get that the Hölder exponent is $\frac{1}{\lambda}\min\{\nu, \tau\gamma\}$, since $\nu_1 = \nu$ (see Remark 3.4.2). □

We prove in the following proposition that the total mass of Laplacian of the solution is finite when the boundary data is $\mathcal{C}^{1,1}$-smooth.

Proposition 3.4.5. *Let μ be a finite Borel measure satisfying Condition $\mathcal{H}(\tau)$ on Ω and $\varphi \in \mathcal{C}^{1,1}(\partial\Omega)$. Then the solution $\mathtt{U}$ to $Dir(\Omega, \varphi, d\mu)$ has the property that*

$$\int_\Omega \Delta \mathtt{U} \le C,$$

where $C > 0$ depends on n, Ω, $\|\varphi\|_{\mathcal{C}^{1,1}(\partial\Omega)}$ and $\mu(\Omega)$.

Proof. Let $\mathtt{U}_0$ be the solution to the Dirichlet problem $Dir(\Omega, 0, d\mu)$. We first claim that the total mass of $\Delta\mathtt{U}_0$ is finite in Ω. Indeed, let ρ be the defining function of Ω. Then by Corollary 1.3.25 we get

$$\begin{aligned} \int_\Omega dd^c\mathtt{U}_0 \wedge (dd^c\rho)^{n-1} &\le \Big(\int_\Omega (dd^c\mathtt{U}_0)^n\Big)^{1/n} \Big(\int_\Omega (dd^c\rho)^n\Big)^{(n-1)/n} \\ &\le \mu(\Omega)^{1/n}\Big(\int_\Omega (dd^c\rho)^n\Big)^{(n-1)/n}. \end{aligned} \tag{3.4.5}$$

Since Ω is a bounded SHL domain, there exists a constant $c > 0$ such that $dd^c\rho \ge c\beta$ in Ω. Hence, (3.4.5) yields

$$\begin{aligned} \int_\Omega dd^c\mathtt{U}_0 \wedge \beta^{n-1} &\le \frac{1}{c^{n-1}} \int_\Omega dd^c\mathtt{U}_0 \wedge (dd^c\rho)^{n-1} \\ &\le \frac{\mu(\Omega)^{1/n}}{c^{n-1}}\Big(\int_\Omega (dd^c\rho)^n\Big)^{(n-1)/n}. \end{aligned}$$

Now we note that the total mass of complex Monge-Ampère measure of ρ is finite in Ω by the Chern-Levine-Nirenberg inequality and since ρ is psh and bounded in a neighborhood of $\bar{\Omega}$. Therefore, the total mass of $\Delta\mathtt{U}_0$ is finite in Ω.
Let $\tilde{\varphi}$ be a $\mathcal{C}^{1,1}$-extension of φ to $\bar{\Omega}$ such that $\|\tilde{\varphi}\|_{\mathcal{C}^{1,1}(\bar{\Omega})} \le C\|\varphi\|_{\mathcal{C}^{1,1}(\partial\Omega)}$ for some $C > 0$. Now, let $v = A\rho + \tilde{\varphi} + \mathtt{U}_0$ where $A > 0$ is big enough such that $A\rho + \tilde{\varphi} \in PSH(\Omega)$. By the comparison principle we see that $v \le \mathtt{U}$ in Ω and $v = \mathtt{U} = \varphi$ on $\partial\Omega$. Since ρ is psh in a neighborhood of $\bar{\Omega}$ and $\|\Delta\mathtt{U}_0\|_\Omega < +\infty$, we deduce that $\|\Delta v\|_\Omega < +\infty$. Then the following lemma completes the proof. □

Lemma 3.4.6. *Let Ω be a bounded domain in $\mathbb{C}^n$. Suppose that v_1, v_2 are continuous subharmonic function in Ω such that $v_1 \leq v_2$ in Ω and $v_1 = v_2$ on $\partial\Omega$, then we have*

$$\int_\Omega dd^c v_2 \wedge \beta^{n-1} \leq \int_\Omega dd^c v_1 \wedge \beta^{n-1}.$$

Proof. First assume that $v_1 = v_2$ in a neighborhood of $\partial\Omega$. Then Stokes' theorem yields that

$$\int_\Omega dd^c v_2 \wedge \beta^{n-1} = \int_\Omega dd^c v_1 \wedge \beta^{n-1}.$$

For the general case, we define the function $v_\epsilon := \max\{v_2 - \epsilon, v_1\}$. Hence we see that $v_1 \leq v_\epsilon$ in Ω and $v_\epsilon = v_1$ near the boundary $\partial\Omega$. Therefore, we get

$$\int_\Omega dd^c v_\epsilon \wedge \beta^{n-1} = \int_\Omega dd^c v_1 \wedge \beta^{n-1}.$$

Since $v_1 \leq v_2$ in Ω, we get that $v_\epsilon \nearrow v_2$ in Ω. Hence $dd^c v_\epsilon \wedge \beta^{n-1}$ converges to $dd^c v_1 \wedge \beta^{n-1}$ in the weak sense of measures and we conclude that

$$\int_\Omega dd^c v_2 \wedge \beta^{n-1} \leq \liminf_{\epsilon \to 0} \int_\Omega dd^c v_\epsilon \wedge \beta^{n-1} = \int_\Omega dd^c v_1 \wedge \beta^{n-1}.$$

□

3.5 Proof of main results

Our first aim is to prove Theorem 3.1.2 by applying Theorem 3.4.4. It is well known that the Lebesgue measure dV_{2n} satisfies Condition $\mathcal{H}(\infty)$ (see [Z01]). We first estimate the L^1-norm of $\hat{\mathtt{U}}_\delta - \mathtt{U}$ with respect to the Lebesgue measure as in [GKZ08].

Lemma 3.5.1. *([GKZ08]). Let $\varphi \in \mathcal{C}^{1,1}(\partial\Omega)$ and $f \in L^p(\Omega)$, $p > 1$. Then the solution $\mathtt{U}$ to the Dirichlet problem satisfies*

$$\int_{\Omega_\delta} [\hat{\mathtt{U}}_\delta(z) - \mathtt{U}(z)] dV_{2n}(z) \leq C\delta^2,$$

where C is a positive constant depending on n, Ω and $\|f\|_{L^p(\Omega)}$.

Proof. Let us denote by σ_{2n-1} the surface measure of the unit sphere. It follows from the Poisson-Jensen formula, for $z \in \Omega_\delta$ and $0 < r < \delta$, that

$$\frac{1}{\sigma_{2n-1} r^{2n-1}} \int_{\partial B(z,r)} \mathtt{U}(\xi) d\sigma(\xi) - \mathtt{U}(z) = c_n \int_0^r t^{1-2n} \left(\int_{B(z,t)} \Delta \mathtt{U}(\xi) \right) dt.$$

Using polar coordinates we obtain for $z \in \Omega_\delta$,

$$\hat{\mathtt{U}}_\delta(z) - \mathtt{U}(z) = \frac{c_n}{\delta^{2n}} \int_0^\delta r^{2n-1} dr \int_0^r t^{1-2n} dt \left(\int_{B(z,t)} \Delta \mathtt{U}(\xi) \right).$$

Now we integrate on Ω_δ with respect to dV_{2n} and use Fubini's theorem

$$\begin{aligned}\int_{\Omega_\delta}[\hat{\mathtt{U}}_\delta(z)-\mathtt{U}(z)]dV_{2n}(z) &= \frac{c_n}{\delta^{2n}}\int_{\Omega_\delta}\int_0^\delta r^{2n-1}dr\int_0^r t^{1-2n}dt\left(\int_{|\xi-z|\le t}\Delta\mathtt{U}(\xi)\right)dV_{2n}(z)\\ &= \frac{c_n}{\delta^{2n}}\int_0^\delta r^{2n-1}dr\int_0^r t^{1-2n}dt\int_{\Omega_\delta}\left(\int_{B(z,t)}\Delta\mathtt{U}(\xi)\right)dV_{2n}(z)\\ &\le \frac{c_n}{\delta^{2n}}\int_0^\delta r^{2n-1}dr\int_0^r t^{1-2n}dt\int_{\Omega}\left(\int_{B(\xi,t)}dV_{2n}(z)\right)\Delta\mathtt{U}(\xi)\\ &\le c_n\int_\Omega \Delta\mathtt{U}\delta^2\end{aligned}$$

Proposition 3.4.5 yields that the total mass of $\Delta\mathtt{U}$ is finite in Ω and this completes the proof. □

We will introduce here the interplay between the real and complex Monge-Ampère measures which really goes back to Cheng-Yau and was first explained in Bedford's survey [Be88] (see also [CP92]). This relation will be useful in the proof of Theorem 3.1.2.

We recall that if u is a locally convex smooth function in Ω, its real Monge-Ampère measure is defined by

$$Mu := det\left(\frac{\partial^2 u}{\partial x_j\partial x_k}\right)dV_{2n}.$$

When u is only convex, then Mu can be defined following Alexandrov [A55] by means of the gradient image as a nonnegative Borel measure on Ω (see [Gut01], [RT77], [Gav77]).

We recall the theorem of existence of convex solution to the Dirichlet problem for the real Monge-Ampère equation, this theorem is due to Rauch and Taylor.

Theorem 3.5.2. *([RT77]). Let Ω be a strictly convex domain. Assume that $\varphi\in\mathcal{C}(\partial\Omega)$ and μ is a nonnegative Borel measure on Ω with $\mu(\Omega)<\infty$. Then there is a unique convex $u\in\mathcal{C}(\bar\Omega)$ such that $Mu=\mu$ in Ω and $u=\varphi$ on $\partial\Omega$.*

Proposition 3.5.3. *Let $0\le f\in L^p(\Omega)$, $p\ge 2$ and u be a locally convex function in Ω and continuous on $\bar\Omega$. If the real Monge-Ampère measure $Mu\ge f^2dV_{2n}$ then the complex Monge-Ampère measure satisfies the inequality $(dd^cu)^n\ge fdV_{2n}$ in the weak sense of measures in Ω.*

Proof. For a smooth function u, we have

$$|\det(\partial^2u/\partial z_j\partial\bar z_k)|^2\ge\det(\partial^2u/\partial x_j\partial x_k). \tag{3.5.1}$$

Hence, we immediately get that $(dd^cu)^n\ge fdV_{2n}$ (see [CP92]).
Moreover, it is well known for smooth convex function that

$$(Mu)^{1/n}=\inf\Delta_H u,\ \text{where}\ \Delta_H u:=\sum_{j,k}h_{jk}\frac{\partial^2u}{\partial x_j\partial x_k}, \tag{3.5.2}$$

for any symmetric positive definite matrix $H=(h_{jk})$ with $\det H=n^{-n}$ (see [Gav77], [Bł97]). In general case, we will prove that $(dd^cu)^n\ge f\beta^n$ weakly in Ω. Indeed, the problem being local, we can assume that u is defined and convex in a neighborhood of a

ball $\bar{B} \subset \Omega$. For $\delta > 0$, we set $\mu_\delta := Mu * \rho_\delta$ then $\mu_\delta \geq g_\delta$, where $g_\delta := f^2 * \rho_\delta$ (without loss of generality we assume $g_\delta > 0$). We may assume that u and μ_δ are defined in this neighborhood of $\bar{B}$. Let φ_δ be a sequence of smooth functions on ∂B converging uniformly to u there. Let u^δ be a smooth convex function such that $Mu^\delta = \mu_\delta$ in B and $u^\delta = \varphi_\delta$ on ∂B. Let $\tilde{u} \in \mathcal{C}(\bar{B})$ be a convex function such that $M\tilde{u} = 0$ and $\tilde{u} = \varphi_\delta$ on ∂B. Moreover, let $v^\delta \in \mathcal{C}(\bar{B})$ be a convex function such that $Mv^\delta = \mu_\delta$ and $v^\delta = 0$ on ∂B.
From the comparison principle for the real Monge-Ampère operator (see [RT77]), we can infer that

$$\tilde{u} + v^\delta \leq u^\delta \leq \tilde{u} - v^\delta. \tag{3.5.3}$$

It follows from Lemma 3.5 in [RT77] that

$$(-v^\delta(x))^{2n} \leq c_n(\operatorname{diam}(B))^{2n-1} \operatorname{dist}(x, \partial B) Mv^\delta(B), \quad x \in B. \tag{3.5.4}$$

Then we conclude that $\{u^\delta\}$ is uniformly bounded sequence of convex functions, hence there exists a subsequence $\{u^{\delta_j}\}$ converging locally uniformly on B.
Moreover, (3.5.3) and (3.5.4) imply that $\{u^{\delta_j}\}$ is uniformly convergent on $\bar{B}$. From the comparison principle it follows that u^{δ_j} converges uniformly to u. Since $u^{\delta_j} \in \mathcal{C}^\infty(\bar{B})$ and $Mu^{\delta_j} \geq f^2 * \rho_{\delta_j} dV_{2n}$, we get that

$$(dd^c u^{\delta_j})^n \geq (f^2 * \rho_{\delta_j})^{1/2} dV_{2n}.$$

Finally, as u^{δ_j} converges uniformly to u, we conclude by Bedford and Taylor's convergence theorem that

$$(dd^c u)^n \geq f dV_{2n}.$$

□

We prove now Hölder continuity of the solution to the Dirichlet problem $Dir(\Omega, \varphi, f dV_{2n})$ with $0 \leq f \in L^p(\Omega)$.

Proof of Theorem 3.1.2. We first suppose that $f = 0$ near the boundary of Ω, that is, there exists a compact $K \Subset \Omega$ such that $f = 0$ in $\Omega \setminus K$. To apply Theorem 3.4.4, we establish a Hölder continuous function v such that $v \leq \mathtt{U}$ in Ω and $v = \varphi$ on $\partial\Omega$. Let ρ be the defining function of Ω given by Definition 2.2.1 and $\tilde{\varphi}$ be a $\mathcal{C}^{1,1}$-extension of φ to $\bar{\Omega}$ such that $\|\tilde{\varphi}\|_{\mathcal{C}^{1,1}(\bar{\Omega})} \leq C\|\varphi\|_{\mathcal{C}^{1,1}(\partial\Omega)}$, for some $C > 0$. Now, we take $A > 0$ large enough such that $v := A\rho + \tilde{\varphi} \in PSH(\Omega) \cap \mathcal{C}^{0,1}(\bar{\Omega})$ and $v \leq \mathtt{U}$ in a neighborhood of K. By the comparison principle, we can find that $v \leq \mathtt{U}$ in $\Omega \setminus K$ and hence $v \leq \mathtt{U}$ in Ω and $v|_{\partial\Omega} = \mathtt{U}|_{\partial\Omega} = \varphi$. Hence, by this construction and Lemma 3.5.1, the two conditions in Theorem 3.4.4 are satisfied. This implies that the solution $\mathtt{U}$ is Hölder continuous in $\bar{\Omega}$ of exponent 2γ for any $\gamma < 1/(nq+1)$ and $1/p + 1/q = 1$.

For the general case, when $f \in L^p(\Omega)$, $p > 1$. Let us fix a large ball $B \subset \mathbb{C}^n$ so that $\Omega \Subset B \subset \mathbb{C}^n$. Let $\tilde{f}$ be the trivial extension of f to B. Since $\tilde{f} \in L^p(\Omega)$ is equal to zero near ∂B, the first case yields that the solution h_1 to the following Dirichlet problem

$$(dd^c h_1)^n = \tilde{f} dV_{2n} \text{ in } B, \text{ and } h_1 = 0 \text{ on } \partial B,$$

is Hölder continuous on $\bar{B}$ of exponent 2γ. Now, let h_2 denote the solution to the Dirichlet problem in Ω with boundary values $\varphi - h_1$ and the zero density. Thanks to Theorem 2.1.1, we infer that $h_2 \in \mathcal{C}^{0,\gamma}(\bar{\Omega})$. Therefore, the required barrier will be $v := h_1 + h_2$. It is clear that $v \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega})$, $v|_{\partial\Omega} = \varphi$ and $(dd^c v)^n \geq f dV_{2n}$ in the weak sense in Ω. Hence, by the comparison principle we get that $v \leq \mathtt{U}$ in Ω and $v = \mathtt{U} = \varphi$ on $\partial\Omega$. Moreover, we have $v \in \mathcal{C}^{0,\gamma}(\bar{\Omega})$, for any $\gamma < 1/(nq+1)$. By applying Theorem 3.4.4, we conclude that the solution $\mathtt{U}$ belongs to $\mathcal{C}^{0,\gamma}(\bar{\Omega})$.

In the special case when $f \in L^p(\Omega)$, $p \geq 2$. We can improve the Hölder exponent of $\mathtt{U}$ by using the relation between the real and complex Monge-Ampère measures. Let us set $\mu := \tilde{f}^2 dV_{2n}$ which is a nonnegative Borel measure on B with $\mu(B) < \infty$. Thanks to Theorem 3.5.2 there exists a unique convex function $u \in \mathcal{C}(\bar{B})$ such that $Mu = \mu$ in B and $u = 0$ on ∂B. Hence u is Lipschitz continuous in $\bar{\Omega}$. By Proposition 3.5.3, we have $(dd^c u)^n \geq f dV_{2n}$ in Ω.
We will construct the required barrier as follows. Let $h_{\varphi-u}$ be the solution to the Dirichlet problem with zero density and $\varphi - u$ boundary data. Then $h_{\varphi-u}$ is Hölder continuous of exponent $1/2$ in $\bar{\Omega}$ by Theorem 2.1.1. Now, it is easy to check that $v := u + h_{\varphi-u}$ is psh in Ω and satisfies $v = \varphi$ in $\partial\Omega$ and $(dd^c v)^n \geq f dV_{2n}$ in Ω. So, by the comparison principle, we have $v \leq \mathtt{U}$ in Ω. By Theorem 3.4.4 and Lemma 3.5.1, our solution $\mathtt{U}$ will be Hölder continuous of exponent $\min\{1/2, 2\gamma\}$, for any $\gamma < 1/(nq+1)$. □

Remark 3.5.4. It is shown in [GKZ08] that we cannot expect a better Hölder exponent than $2/(nq)$ (see also [Pl05]).

We introduce an important class of Borel measures on Ω containing Riesz measures and closely related to Hausdorff measures which play an important role in geometric measure theory [Ma95]. We call such measures Hausdorff-Riesz measures.

Definition 3.5.5. A finite Borel measure on Ω is called a Hausdorff-Riesz measure of order $2n - 2 + \epsilon$, for $0 < \epsilon \leq 2$ if it satisfies the following condition :

$$\mu(B(z,r) \cap \Omega) \leq C r^{2n-2+\epsilon}, \ \ \forall z \in \bar{\Omega}, \ \forall 0 < r < 1, \tag{3.5.5}$$

for some positive constant C.

We give some interesting examples of Hausdorff-Riesz measures.

Example 3.5.6.

1. The Lebesgue measure dV_{2n} on Ω, for $\epsilon = 2$.
2. The surface measure of a compact real hypersurface, for $\epsilon = 1$.
3. Measures of the type $dd^c v \wedge \beta^{n-1}$, where v is a α-Hölder continuous subharmonic function in a neighborhood of $\bar{\Omega}$, for $\epsilon = \alpha$.
4. The measure $\mathbf{1}_E \mathcal{H}^{2n-2+\epsilon}$, where $\mathcal{H}^{2n-2+\epsilon}$ is the Hausdorff measure and E is a Borel set such that $\mathcal{H}^{2n-2+\epsilon}(E) < +\infty$.
5. If μ is a Hausdorff-Riesz measure of order $2n-2+\epsilon$, then $f d\mu$ is Hausdorff-Riesz of order $2n-2+\epsilon'$, with $\epsilon' := \epsilon - (2n-2+\epsilon)/p$, for any $f \in L^p(\Omega, \mu)$, $p > (2n-2+\epsilon)/\epsilon$.

The existence of continuous solutions to $Dir(\Omega, \varphi, f d\mu)$ for such measures follows immediately from Theorem 3.1.1 and the following lemma.

Lemma 3.5.7. *Let Ω be a bounded SHL domain and μ be a Hausdorff-Riesz measure of order $2n-2+\epsilon$, for $0<\epsilon\leq 2$. Assume that $0\leq f\in L^p(\Omega,\mu)$ for $p>1$, then for all $\tau>0$ there exists $D>0$ depending on τ,ϵ,q and* $\operatorname{diam}(\Omega)$ *such that for any Borel set $K\subset\Omega$,*

$$\int_K f d\mu \leq D\|f\|_{L^p(\Omega,\mu)}[Cap(K,\Omega)]^{1+\tau}. \tag{3.5.6}$$

Proof. By the Hölder inequality we have

$$\int_K f d\mu \leq \|f\|_{L^p(\Omega,\mu)}\mu(K)^{1/q}.$$

Let $z_0\in\Omega$ be a fixed point and $R:=2\operatorname{diam}(\Omega)$. Hence, $\Omega\Subset B:=B(z_0,R)$. For any Borel set $K\subset\Omega$ we get, by Corollary 5.2 in [Z04] and Alexander-Taylor's inequality, that

$$\mu(K)\leq C(T_R(K))^{\epsilon/2}\leq C\exp(-\epsilon/2\ \mathrm{Cap}(K,B)^{-1/n})\leq C\exp(-\epsilon/2\ \mathrm{Cap}(K,\Omega)^{-1/n}),$$

where $C>0$ depends on ϵ and $\operatorname{diam}(\Omega)$.
Now, for any $\tau>0$, we can find $D>0$ depending on τ,ϵ,q and $\operatorname{diam}(\Omega)$ such that

$$\int_K f d\mu \leq D\|f\|_{L^p(\Omega,\mu)}[\mathrm{Cap}(K,\Omega)]^{1+\tau}.$$

□

The first step in the proof of Theorem 3.1.3 is to estimate $\|\hat{\mathtt{U}}_\delta-\mathtt{U}\|_{L^1(\Omega_\delta,\mu)}$, so we present the following lemma.

Lemma 3.5.8. *Let $\Omega\subset\mathbb{C}^n$ be a SHL domain and μ be a Hausdorff-Riesz measure of order $2n-2+\epsilon$ on Ω, for $0<\epsilon\leq 2$. Suppose that $0\leq f\in L^p(\Omega,\mu)$, $p>1$ and $\varphi\in\mathcal{C}^{1,1}(\partial\Omega)$. Then the solution $\mathtt{U}$ to $Dir(\Omega,\varphi,fd\mu)$ satisfies*

$$\int_{\Omega_\delta}[\hat{\mathtt{U}}_\delta(z)-\mathtt{U}(z)]d\mu(z)\leq C\delta^\epsilon,$$

where C is a positive constant depending on n, ϵ, Ω, $\|f\|_{L^p(\Omega,\mu)}$ and $\mu(\Omega)$.

Proof. Following a slight modification in the proof of Lemma 3.5.1, we can get the required inequality. □

When φ is not $\mathcal{C}^{1,1}$-smooth, the measure $\Delta\mathtt{U}$ may have infinite mass on Ω. Fortunately, we can estimate $\|\hat{\mathtt{U}}_{\delta_1}-\mathtt{U}\|_{L^1(\Omega_\delta,\mu)}$ for some $\delta_1<\delta\leq 1$.

We need the following property of a bounded SHL domain.

Lemma 3.5.9. *Let Ω be a bounded SHL domain. Then there exist a function $\tilde{\rho}\in PSH(\Omega)\cap\mathcal{C}^{0,1}(\bar{\Omega})$ such that near $\partial\Omega$ we have*

$$c_1\operatorname{dist}(z,\partial\Omega)\geq-\tilde{\rho}(z)\geq c_2\operatorname{dist}(z,\partial\Omega)^2, \tag{3.5.7}$$

for some $c_1,c_2>0$ depending on Ω.
Moreover, $dd^c\tilde{\rho}\geq c_2\beta$ in the weak sense of currents on Ω.

Proof. Since Ω is a strongly hyperconvex Lipschitz domain, there exist a constant $c > 0$ and a defining function ρ such that $dd^c\rho \geq c\beta$ in the weak sense of currents on Ω. Let us fix $\xi \in \partial\Omega$, then the function defined by $\tilde{\rho}_\xi(z) := \rho(z) - c/2|z-\xi|^2$ is Lipschitz continuous in $\bar{\Omega}$ and satisfies $dd^c\tilde{\rho}_\xi \geq (c/2)\beta$ in the weak sense of currents on Ω. Hence, $\tilde{\rho}_\xi \in PSH(\Omega) \cap \mathcal{C}^{0,1}(\bar{\Omega})$. Set

$$\tilde{\rho} := \sup\{\tilde{\rho}_\xi : \xi \in \partial\Omega\}.$$

It is clear that $\tilde{\rho} \in \mathcal{C}^{0,1}(\bar{\Omega}) \cap PSH(\Omega)$ and thus the first inequality in (3.5.7) holds. For any $\xi \in \partial\Omega$ we have $-\tilde{\rho}_\xi(z) \geq (c/2)|z-\xi|^2$, so we infer that

$$-\tilde{\rho}(z) \geq (c/2)\,\mathrm{dist}(z, \partial\Omega)^2,$$

for any z near $\partial\Omega$.

The last statement follows from the fact that for any $\xi \in \partial\Omega$, $dd^c\tilde{\rho}_\xi \geq (c/2)\beta$ in the weak sense of currents on Ω. □

Remark 3.5.10. When Ω is a smooth strongly pseudoconvex domain, we know that the defining function ρ satisfies near the boundary,

$$-\rho \approx \mathrm{dist}(., \partial\Omega).$$

Lemma 3.5.11. *Let $\Omega \subset \mathbb{C}^n$ be a bounded SHL domain and μ be a Hausdorff-Riesz measure of order $2n-2+\epsilon$ on Ω, for $0 < \epsilon \leq 2$. Suppose that $0 \leq f \in L^p(\Omega, \mu)$, $p > 1$ and $\varphi \in \mathcal{C}^{0,\alpha}(\partial\Omega)$, $\alpha \leq 1$. Then for any small $\epsilon_1 > 0$, we have the following inequality*

$$\int_{\Omega_\delta} [\hat{\mathtt{U}}_{\delta_1}(z) - \mathtt{U}(z)] d\mu(z) \leq C\delta^{\epsilon/2-\epsilon_1},$$

where $\delta_1 = (1/2)\delta^{1/2+3/\epsilon}$ and C is a positive constant depending on n, Ω, ϵ, ϵ_1 and $\|u\|_{L^\infty(\bar{\Omega})}$.

Proof. One sees as in the proof of Lemma 3.5.1 that

$$\hat{\mathtt{U}}_{\delta_1}(z) - \mathtt{U}(z) = \frac{c_n}{\delta_1^{2n}} \int_0^{\delta_1} r^{2n-1} dr \int_0^r t^{1-2n} dt \left(\int_{B(z,t)} \Delta\mathtt{U}(\xi) \right).$$

Then, we integrate on Ω_δ with respect to μ and use Fubini's Theorem

$$\begin{aligned}
\int_{\Omega_\delta} [\hat{\mathtt{U}}_{\delta_1}(z) - \mathtt{U}(z)] d\mu(z) &\leq \frac{c_n}{\delta_1^{2n}} \int_0^{\delta_1} r^{2n-1} dr \int_0^r t^{1-2n} dt \int_{\Omega_{\delta-t}} \left(\int_{B(\xi,t)} d\mu(z) \right) \Delta\mathtt{U}(\xi) \\
&\leq \frac{c_n}{\delta_1^{2n}} \int_0^{\delta_1} r^{2n-1} dr \int_0^r t^{-1+\epsilon} dt \int_{\Omega_{\delta-t}} \Delta\mathtt{U}(\xi) \\
&\leq \frac{c_n}{\delta_1^{2n}} \sup_{\Omega_{\delta-\delta_1}} (-\tilde{\rho})^{-(3+\epsilon_1)/2} \int_0^{\delta_1} r^{2n-1} dr \int_0^r t^{-1+\epsilon} dt \int_{\Omega_{\delta-t}} (-\tilde{\rho})^{(3+\epsilon_1)/2} \Delta\mathtt{U}(\xi) \\
&\leq \frac{c_n}{\delta_1^{2n}} \sup_{\Omega_{\delta/2}} (-\tilde{\rho})^{-(3+\epsilon_1)/2} \|(-\tilde{\rho})^{(3+\epsilon_1)/2} \Delta\mathtt{U}\|_\Omega \int_0^{\delta_1} r^{2n-1} dr \int_0^r t^{-1+\epsilon} dt \\
&\leq \frac{c_n \delta^{-3-\epsilon_1}}{\epsilon(2n+\epsilon)} \delta_1^\epsilon \|(-\tilde{\rho})^{(3+\epsilon_1)/2} \Delta\mathtt{U}\|_\Omega \\
&\leq C_1 \delta^{\epsilon/2-\epsilon_1} \|(-\tilde{\rho})^{(3+\epsilon_1)/2} \Delta\mathtt{U}\|_\Omega,
\end{aligned}$$

where $\tilde{\rho}$ is as in Lemma 3.5.9 and $C_1 > 0$ is a positive constant depending on ϵ and n.

To complete the proof we demonstrate that the mass $\|(-\tilde{\rho})^{(3+\epsilon_1)/2}\Delta \mathtt{U}\|_\Omega$ is finite. The following idea is due to [BKPZ15] with some appropriate modifications. We set for simplification $\theta := (3+\epsilon_1)/2$. Let ρ_η be the standard regularizing kernels with supp $\rho_\eta \subset B(0,\eta)$ and $\int_{B(0,\eta)} \rho_\eta dV_{2n} = 1$. Hence, $u_\eta = \mathtt{U} * \rho_\eta \in \mathcal{C}^\infty \cap PSH(\Omega_\eta)$ decreases to $\mathtt{U}$ in Ω. It is clear that $\|u_\eta\|_{L^\infty(\Omega_\eta)} \leq \|\mathtt{U}\|_{L^\infty(\Omega)}$ and the first derivatives of u_η have L^∞-norms less than $\|\mathtt{U}\|_{L^\infty(\Omega)}/\eta$. We denote by χ_{Ω_η} the characteristic function of Ω_η. Since $u_\eta \searrow \mathtt{U}$ in Ω, we have $\chi_{\Omega_\eta}(-\tilde{\rho})^\theta \Delta u_\eta$ converges to $(-\tilde{\rho})^\theta \Delta \mathtt{U}$ in the weak sense of measures.

It is sufficient to show that

$$I := \int_{\Omega_\eta} (-\tilde{\rho})^\theta dd^c u_\eta \wedge \beta^{n-1},$$

is bounded by an absolute constant independent of η. We have by Stokes' theorem

$$I = \int_{\partial\Omega_\eta} (-\tilde{\rho})^\theta d^c u_\eta \wedge \beta^{n-1} + \theta \int_{\Omega_\eta} (-\tilde{\rho})^{\theta-1} d\tilde{\rho} \wedge d^c u_\eta \wedge \beta^{n-1}.$$

Note that

$$\begin{aligned}
\int_{\partial\Omega_\eta} (-\tilde{\rho})^{\theta-1} u_\eta d^c\tilde{\rho} \wedge \beta^{n-1} = & \int_{\Omega_\eta} (-\tilde{\rho})^{\theta-1} du_\eta \wedge d^c\tilde{\rho} \wedge \beta^{n-1} + \\
& + \int_{\Omega_\eta} (-\tilde{\rho})^{\theta-1} u_\eta dd^c\tilde{\rho} \wedge \beta^{n-1} \\
& - (\theta-1) \int_{\Omega_\eta} (-\tilde{\rho})^{\theta-2} u_\eta d\tilde{\rho} \wedge d^c\tilde{\rho} \wedge \beta^{n-1}.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
I = & \int_{\partial\Omega_\eta} (-\tilde{\rho})^\theta d^c u_\eta \wedge \beta^{n-1} + \theta \int_{\partial\Omega_\eta} (-\tilde{\rho})^{\theta-1} u_\eta d^c\tilde{\rho} \wedge \beta^{n-1} \\
& - \theta \int_{\Omega_\eta} (-\tilde{\rho})^{\theta-1} u_\eta dd^c\tilde{\rho} \wedge \beta^{n-1} + \theta(\theta-1) \int_{\Omega_\eta} (-\tilde{\rho})^{\theta-2} u_\eta d\tilde{\rho} \wedge d^c\tilde{\rho} \wedge \beta^{n-1} \\
\leq & \, C\|u\|_{L^\infty(\bar{\Omega})} \left(\int_{\partial\Omega_\eta} d\sigma + \int_{\Omega_\eta} dd^c\tilde{\rho} \wedge \beta^{n-1} + \int_{\Omega_\eta} (-\tilde{\rho})^{\theta-2} \beta^n \right), \\
\leq & \, C\|u\|_{L^\infty(\bar{\Omega})} \left(\int_{\partial\Omega_\eta} d\sigma + \int_{\Omega} dd^c\rho \wedge \beta^{n-1} + \int_{\Omega} (-\tilde{\rho})^{(-1+\epsilon_1)/2} \beta^n \right),
\end{aligned}$$

where $d\sigma = d^c\rho \wedge (dd^c\rho)^{n-1}$ and ρ is the defining function of Ω. Since ρ is psh in a neighborhood of $\bar{\Omega}$, the second integral in the last inequality is finite. Thanks to Lemma 3.5.9, we have $-\tilde{\rho} \geq c_2 \operatorname{dist}(., \partial\Omega)^2$ near $\partial\Omega$ and so the third integral will be finite since $\epsilon_1 > 0$ small enough. Consequently, we infer that I is bounded by a constant independent of η and then this proves our claim. □

Corollary 3.5.12. *When Ω is a smooth strongly pseudoconvex domain, then Lemma 3.5.11 holds also for $\delta_1 = (1/2)\delta^{1/2+1/\epsilon}$.*

Proof. Let ρ be the defining function of Ω. In view of Remark 3.5.10 and the last argument, we can estimate $\|(-\rho)^{1+\epsilon_1}\Delta \mathtt{U}\|_\Omega$, for $\epsilon_1 > 0$ small enough, and ensure that this mass is finite. So the proof of Lemma 3.5.11 is still true for more better $\delta_1 := (1/2)\delta^{1/2+1/\epsilon}$. □

We are in a position to prove the Hölder continuity of the solution to $Dir(\Omega, \varphi, fd\mu)$ where μ is a Hausdorff-Riesz measure of order $2n-2+\epsilon$ and $\varphi \in \mathcal{C}^{1,1}(\partial\Omega)$.

Proof of Theorem 3.1.3. We first assume that f equals to zero near the boundary $\partial\Omega$, then there exists a compact $K \Subset \Omega$ such that $f = 0$ on $\Omega \setminus K$. Since $\varphi \in \mathcal{C}^{1,1}(\partial\Omega)$, we extend it to $\tilde{\varphi} \in \mathcal{C}^{1,1}(\bar{\Omega})$ such that $\|\tilde{\varphi}\|_{\mathcal{C}^{1,1}(\bar{\Omega})} \leq C\|\varphi\|_{\mathcal{C}^{1,1}(\partial\Omega)}$ for some constant C. Let ρ be the defining function of Ω and let $A \gg 1$ be so that $v := A\rho + \tilde{\varphi} \in PSH(\Omega)$ and $v \leq \mathtt{U}$ in a neighborhood of K. Moreover, by the comparison principle, we see that $v \leq \mathtt{U}$ in $\Omega \setminus K$. Consequently, $v \in PSH(\Omega) \cap \mathcal{C}^{0,1}(\bar{\Omega})$ and satisfies $v \leq \mathtt{U}$ on $\bar{\Omega}$ and $v = \mathtt{U} = \varphi$ on $\partial\Omega$. It follows from Theorem 3.4.4 and Lemma 3.5.8 that $\mathtt{U} \in \mathcal{C}^{0,\epsilon\gamma}(\bar{\Omega})$, for any $0 < \gamma < 1/(nq+1)$.

In the general case, fix a large ball $B \subset \mathbb{C}^n$ containing Ω and define a function $\tilde{f} \in L^p(B, \mu)$ so that $\tilde{f} := f$ in Ω and $\tilde{f} := 0$ in $B \setminus \Omega$. Hence, the solution to the following Dirichlet problem

$$\begin{cases} v_1 \in PSH(B) \cap \mathcal{C}(\bar{B}), & \\ (dd^c v_1)^n = \tilde{f} d\mu & \text{in } B, \\ v_1 = 0 & \text{on } \partial B, \end{cases}$$

belongs to $\mathcal{C}^{0,\gamma'}(\bar{B})$, with $\gamma' = \epsilon\gamma$ for any $\gamma < 1/(nq+1)$.
Let $h_{\varphi - v_1}$ be the continuous solution to $Dir(\Omega, \varphi - v_1, 0)$. Then, Theorem 2.1.1 implies that $h_{\varphi - v_1}$ belongs to $\mathcal{C}^{0,\gamma'/2}(\bar{\Omega})$. This enables us to construct a Hölder barrier for our problem. We take $v_2 = v_1 + h_{\varphi - v_1}$. It is clear that $v_2 \in PSH(\Omega) \cap \mathcal{C}^{0,\gamma'/2}(\bar{\Omega})$ and $v_2 \leq \mathtt{U}$ on $\bar{\Omega}$ by the comparison principle. Hence, Theorem 3.4.4 and Lemma 3.5.8 imply that the solution $\mathtt{U}$ to $Dir(\Omega, \varphi, fd\mu)$ is Hölder continuous on $\bar{\Omega}$ of exponent $\epsilon\gamma/2$ for any $0 < \gamma < 1/(nq+1)$. □

In the case when φ is only Hölder continuous, we prove the Hölder regularity of the solution.

Proof of Theorem 3.1.4. Let also v_1 be as in the proof of Theorem 3.1.3 and $h_{\varphi - v_1}$ be the solution to $Dir(\Omega, \varphi - v_1, 0)$. In order to apply Theorem 3.4.4, we set $v = v_1 + h_{\varphi - v_1}$. Hence, $v \in PSH(\Omega) \cap \in \mathcal{C}(\bar{\Omega})$, $v = \varphi$ on $\partial\Omega$ and $(dd^c v)^n \geq fd\mu$ in Ω. The comparison principle yields $v \leq \mathtt{U}$ in Ω. Moreover, by Theorem 2.1.1, we have $h_{\varphi - v_1} \in \mathcal{C}^{0,\gamma''}(\bar{\Omega})$ with $\gamma'' = 1/2 \min\{\alpha, \epsilon\gamma\}$. Hence, it stems from Theorem 3.4.4 and Lemma 3.5.11 that the solution $\mathtt{U}$ is Hölder continuous on $\bar{\Omega}$ of exponent $\frac{\epsilon}{\epsilon+6} \min\{\alpha, \epsilon\gamma\}$, for any $0 < \gamma < 1/(nq+1)$.

Moreover, when Ω is a smooth strongly pseudoconvex domain and by Corollary 3.5.12 we get more better Hölder exponent $\frac{\epsilon}{\epsilon+2} \min\{\alpha, \epsilon\gamma\}$, for any $0 < \gamma < 1/(nq+1)$. □

Corollary 3.5.13. *Let Ω be a finite intersection of strongly pseudoconvex domains in $\mathbb{C}^n$. Assume that $\varphi \in \mathcal{C}^{0,\alpha}(\partial\Omega)$, $0 < \alpha \leq 1$, and $0 \leq f \in L^p(\Omega)$ for some $p > 1$. Then the solution $\mathtt{U}$ to the Dirichlet problem $Dir(\Omega, \varphi, fdV_{2n})$ belongs to $\mathcal{C}^{0,\alpha'}(\bar{\Omega})$ with $\alpha' = \min\{\alpha/2, \gamma\}$ for any $0 < \gamma < 1/(nq+1)$. Moreover, if $\varphi \in \mathcal{C}^{1,1}(\partial\Omega)$ the solution $\mathtt{U}$ is γ-Hölder continuous on $\bar{\Omega}$.*

The first part of this corollary was proved in Theorem 1.2 in [BKPZ15] with the Hölder exponent $\min\{2\gamma, \alpha\}\gamma$ and the second part was proved in [GKZ08] and [Ch15a] (see also [N14, Ch14] for the complex Hessian equation).

Our final purpose concerns how to get the Hölder continuity of the solution to the Dirichlet problem $Dir(\Omega, \varphi, fd\mu)$, by means of the Hölder continuity of a subsolution to

$Dir(\Omega, \varphi, d\mu)$ for some special measure μ on Ω. We suppose here that μ is less than the Monge-Ampère measure of a Hölder continuous psh function and has the behavior of some Hausdorff-Riesz measure near the boundary.

Proof of Theorem 3.1.5. Let $\Omega_1 \Subset \Omega$ be an open set such that μ is a Hausdorff-Riesz measure on $\Omega \setminus \Omega_1$. Let also $\Omega_2 \Subset \Omega$ be a a neighborhood of $\bar{\Omega}_1$. We claim that

$$\int_{\Omega_1} (\hat{\mathtt{U}}_\delta - \mathtt{U}) d\mu \leq \int_{\Omega_1} (\hat{\mathtt{U}}_\delta - \mathtt{U})(dd^c w)^n \leq C \|\Delta \mathtt{U}\|_{\Omega_2} \delta^{\frac{2\lambda}{\lambda+2n}}, \tag{3.5.8}$$

where C depends on Ω_1 and Ω_2. This estimate was proved in [DDGHKZ14]. We can assume without loss of generality that $\Omega_1 := \mathbb{B}_1$, $\Omega_2 := \mathbb{B}_2$ and $-2 \leq w \leq -1$ in Ω. This implies that $h(z) := |z|^2 - 4 < w$ on $\mathbb{B}_1$, while $w < h$ on $\mathbb{B}_2 \setminus \mathbb{B}_{r_0}$ for some $1 < r_0 < 2$.
Replacing w by $\max\{w, h\}$, we can assume that $w = h$ on $\mathbb{B}_2 \setminus \mathbb{B}_{r_0}$. Fix $\chi \in \mathcal{C}_0^\infty(\mathbb{C}^n)$ such that $\chi \geq 0$, $\chi(z) := \chi(|z|)$, $\operatorname{supp} \chi \subset \mathbb{B}_1$ and $\int_{\mathbb{B}_1} \chi dV_{2n} = 1$. Let us set

$$w_\delta(z) := \frac{1}{\delta^{2n}} \int_{\mathbb{B}(z,\delta)} w(y) \chi(\frac{z-y}{\delta}) dV_{2n}(y).$$

Since $w \in PSH(\Omega) \cap \mathcal{C}^{0,\lambda}(\Omega)$, we obtain that

$$w_\delta(z) - w(z) \leq C_1 \delta^\lambda.$$

Observe that

$$\left| \frac{\partial^2 w_\delta}{\partial z_j \partial \bar{z}_k} \right| \leq \frac{C_2 \|w\|_{L^\infty(\Omega)}}{\delta^2}. \tag{3.5.9}$$

We choose $\phi \in \mathcal{C}_0^\infty(\mathbb{C}^n)$ such that $0 \leq \phi \leq 1$, $\phi = 1$ on $\mathbb{B}_{r_1}$ and $\operatorname{supp} \phi \subset \mathbb{B}_{r_2}$, where $r_0 < r_1 < r_2 < 2$. We define

$$\tilde{w}_\delta(z) = \int_{\mathbb{B}_1} w\big(z - \delta\phi(z)y\big) \chi(y)\, dV_{2n}(y).$$

Note that

$$\tilde{w}_\delta(z) - w(z) = \int_{\mathbb{B}_1} [w\big(z - \delta\phi(z)y\big) - w(z)] \chi(y)\, dV_{2n}(y) \leq C_1 \delta^\lambda,$$

and

$$\tilde{w}_\delta(z) = w_\delta(z) \text{ on } \mathbb{B}_{r_1}, \qquad \tilde{w}_\delta(z) = w(z) \text{ on } \mathbb{B}_2 \backslash \mathbb{B}_{r_2}.$$

Fix now any $z \in \mathbb{B}_2 \backslash \overline{\mathbb{B}}_{r_0}$. Since $w = h$ there, we have for any $\delta < \delta_0$,

$$\begin{aligned}
\frac{\partial^2 \tilde{w}_\delta}{\partial z_j \partial \bar{z}_k}(z) &= \int_{\mathbb{B}_1} \frac{\partial^2}{\partial z_j \partial \bar{z}_k} h\big(z - \delta\phi(z)y\big) \chi(y)\, dV_{2n}(y) \\
&= \int_{\mathbb{B}_1} [\delta_{jk} + \delta O(1)] \chi(y)\, dV_{2n}(y) \\
&= \delta_{jk} + \delta O(1).
\end{aligned}$$

If δ is small enough, we conclude that $\tilde{w}_\delta \in PSH(\mathbb{B}_2 \backslash \overline{\mathbb{B}}_{r_0})$, $\forall \delta < \delta_0$. Hence $\tilde{w}_\delta$ is actually plurisubharmonic in all of $\mathbb{B}_2$. Set

$$T := \sum_{j=0}^{n-1} (dd^c w)^j \wedge (dd^c \tilde{w}_{\delta^\varepsilon})^{n-1-j},$$

where $\epsilon > 0$ to be chosen later. From (3.5.9), Lemma 3.5.1 and Stokes' formula we get

$$\begin{aligned}
\int_{\Omega_1} (\hat{\mathtt{U}}_\delta - \mathtt{U})(dd^c w)^n &\leq \int_{\mathbb{B}_2} (\hat{\mathtt{U}}_\delta - \mathtt{U})(dd^c w)^n \\
&= \int_{\mathbb{B}_2} (\hat{\mathtt{U}}_\delta - \mathtt{U})[(dd^c w)^n - (dd^c \tilde{w}_{\delta^\varepsilon})^n] + \int_{\mathbb{B}_2} (\hat{\mathtt{U}}_\delta - \mathtt{U})(dd^c \tilde{w}_{\delta^\varepsilon})^n \\
&\leq \int_{\mathbb{B}_2} (\hat{\mathtt{U}}_\delta - \mathtt{U}) dd^c (w - \tilde{w}_{\delta^\varepsilon}) \wedge T + \frac{C_3}{\delta^{2n\varepsilon}} \int_{\mathbb{B}_2} (\hat{\mathtt{U}}_\delta - \mathtt{U})\, dV_{2n} \\
&\leq \int_{\mathbb{B}_2} (\tilde{w}_{\delta^\varepsilon} - w) dd^c (\mathtt{U} - \hat{\mathtt{U}}_\delta) \wedge T + \frac{C_3}{\delta^{2n\varepsilon}} \int_{\mathbb{B}_2} \Delta \mathtt{U} \delta^2 \\
&\leq \int_{\mathbb{B}_2} (\tilde{w}_{\delta^\varepsilon} - w) dd^c \mathtt{U} \wedge T + C_3 \int_{\mathbb{B}_2} \Delta \mathtt{U} \delta^{2(1-n\varepsilon)} \\
&\leq C_1 \delta^{\varepsilon\lambda} \int_{\mathbb{B}_{r_2}} dd^c \mathtt{U} \wedge T + C_3 \delta^{2(1-n\varepsilon)} \int_{\mathbb{B}_2} \Delta \mathtt{U} \\
&\leq C_4 \int_{\mathbb{B}_2} \Delta \mathtt{U} [\delta^{\varepsilon\lambda} \|w\|^{n-1}_{L^\infty(\Omega)} + \delta^{2(1-n\varepsilon)}] \\
&\leq C_4 \int_{\mathbb{B}_2} \Delta \mathtt{U}\ \delta^\tau,
\end{aligned}$$

where $\varepsilon = \frac{2}{\lambda + 2n}$ and $\tau = \frac{2\lambda}{\lambda+2n}$.

Now, let $\tilde{\mu}$ be a Hausdorff-Riesz measure on Ω of order $2n - 2 + \epsilon$ so that $\tilde{\mu}$ equals μ in $\Omega \setminus \Omega_1$. As φ is not $\mathcal{C}^{1,1}$-smooth, we estimate $\|\hat{\mathtt{U}}_{\delta_1} - \mathtt{U}\|_{L^1(\Omega_\delta, \mu)}$ with $\delta_1 := (1/2)\delta^{1/2+3/\epsilon}$. Then, we have

$$\int_{\Omega_\delta} (\hat{\mathtt{U}}_{\delta_1} - \mathtt{U}) d\mu \leq \int_{\Omega_1} (\hat{\mathtt{U}}_{\delta_1} - \mathtt{U}) d\mu + \int_{\Omega_\delta} (\hat{\mathtt{U}}_{\delta_1} - \mathtt{U}) d\tilde{\mu}.$$

Fix $\epsilon_1 > 0$ small enough. Then, it follows from (3.5.8) and Lemma 3.5.11 that

$$\begin{aligned}
\int_{\Omega_\delta} (\hat{\mathtt{U}}_{\delta_1} - \mathtt{U}) d\mu &\leq \int_{\Omega_1} (\hat{\mathtt{U}}_{\delta_1} - \mathtt{U})(dd^c w)^n + \int_{\Omega_\delta} (\hat{\mathtt{U}}_{\delta_1} - \mathtt{U}) d\tilde{\mu} \\
&\leq C \|\Delta \mathtt{U}\|_{\Omega_2} \delta_1^{\frac{2\lambda}{\lambda+2n}} + C' \delta^{\epsilon/2 - \epsilon_1},
\end{aligned}$$

where $C = C(\Omega_1, \Omega_2, \|w\|_{L^\infty})$ is a positive constant and C' depends on n, Ω, ϵ, ϵ_1 and $\|\mathtt{U}\|_{L^\infty(\bar{\Omega})}$. Since the mass of $\Delta \mathtt{U}$ is locally bounded, there exists a constant $C'' > 0$ such that

$$\int_{\Omega_\delta} (\hat{\mathtt{U}}_{\delta_1} - \mathtt{U}) d\mu \leq C'' \delta^\tau,$$

where $\tau = \min\{\frac{\epsilon}{2} - \epsilon_1, \frac{\lambda(\epsilon+6)}{\epsilon(\lambda+2n)}\}$.

The last requirement to apply Theorem 3.4.4 is to construct a function $v \in \mathcal{C}^{0,\nu}(\bar{\Omega})$ for $0 < \nu \leq 1$ such that $v \leq \mathtt{U}$ in Ω and $v = \varphi$ on $\partial\Omega$. Let us denote by w_1 the solution to

$Dir(\Omega, 0, fd\tilde{\mu})$ and h_φ the solution to $Dir(\Omega, \varphi, 0)$. Now, set $v = w_1 + h_\varphi + A\rho$ with $A \gg 1$ so that $v \leq \mathtt{U}$ in a neighborhood of $\bar{\Omega}_1$. It is clear that $v \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega})$, $v = \varphi$ on $\partial\Omega$ and $v \leq \mathtt{U}$ in Ω by the comparison principle. Moreover, by Theorem 2.1.1, we infer that $v \in \mathcal{C}^{0,\nu}(\bar{\Omega})$, for $\nu = 1/2 \min\{\epsilon\gamma, \alpha\}$ and any $\gamma < 1/(nq+1)$. Finally, we get from Theorem 3.4.4 that $\mathtt{U}$ is Hölder continuous on $\bar{\Omega}$ of exponent $\frac{\epsilon}{\epsilon+6} \min\{\alpha, \epsilon\gamma, \frac{2\lambda\gamma(\epsilon+6)}{\epsilon(\lambda+2n)}\}$. □

The following are nice applications of Theorem 3.1.5.

Corollary 3.5.14. *Let $\Omega \subset \mathbb{C}^n$ be a bounded SHL domain and μ be a finite Borel measure with compact support on Ω. Let also $\varphi \in \mathcal{C}^{0,\alpha}(\partial\Omega)$, $0 < \alpha \leq 1$ and $0 \leq f \in L^p(\Omega, \mu)$, $p > 1$. Assume that there exists a λ-Hölder continuous psh function w in Ω such that $(dd^c w)^n \geq \mu$. Then the solution to the Dirichlet problem $Dir(\Omega, \varphi, fd\mu)$ is Hölder continuous on $\bar{\Omega}$ of exponent* $\min\{\alpha/2, \frac{2\lambda\gamma}{\lambda+2n}\}$, *for any $\gamma < 1/(nq+1)$ and $1/p + 1/q = 1$.*

Example 3.5.15. Let μ be a finite Borel measure with compact support on a bounded SHL domain Ω. Let also $\varphi \in \mathcal{C}^{0,\alpha}(\partial\Omega)$, $0 < \alpha \leq 1$ and $0 \leq f \in L^p(\Omega, \mu)$, $p > 1$. Suppose that $\mu \leq dV_n$, where dV_n is the Lebesgue measure of the totally real part $\mathbb{R}^n$ of $\mathbb{C}^n$, then the solution to the Dirichlet problem $Dir(\Omega, \varphi, fd\mu)$ is Hölder continuous on $\bar{\Omega}$ of exponent $\min\{\alpha/2, \frac{2\gamma}{1+2n}\}$, for any $\gamma < 1/(nq+1)$ and $1/p + 1/q = 1$.

Proof. Since $\mathbb{R}^n = \{Imz_j = 0, j = 1, ..., n\}$, one can present the Lebesgue measure of the totaly real part $\mathbb{R}^n$ of $\mathbb{C}^n$ in the form

$$\left(dd^c \sum_{j=1}^{n} (Imz_j)_+ \right)^n .$$

Let us set $w = \sum_{j=1}^{n} (Imz_j)_+$. It is clear that $w \in PSH(\Omega) \cap \mathcal{C}^{0,1}(\bar{\Omega})$ and $\mu \leq (dd^c w)^n$ on Ω. Corollary 3.5.14 yields that the solution $\mathtt{U}$ belongs to $\mathcal{C}^{0,\alpha'}(\bar{\Omega})$ with $\alpha' = \min\{\alpha/2, \frac{2\gamma}{1+2n}\}$, for any $\gamma < 1/(nq+1)$. □

At the end, we note that a slight modification in the proof of Theorem 3.1.5 enables us to estimate the modulus of continuity of the solution in terms of the modulus of continuity of a subsolution.

Remark 3.5.16. Let μ be a measure satisfying the Condition $\mathcal{H}(\infty)$ on a bounded SHL domain Ω. Let also $\varphi \in \mathcal{C}^{0,\alpha}(\partial\Omega)$, $0 < \alpha \leq 1$ and $0 \leq f \in L^p(\Omega, \mu)$, $p > 1$. Assume that there exists a continuous plurisubharmonic function w in Ω such that $(dd^c w)^n \geq \mu$. If the measure μ is Hausdorff-Riesz of order $2n - 2 + \epsilon$ in $\Omega \setminus \Omega_1$ for some $0 < \epsilon \leq 2$, where $\Omega_1 \Subset \Omega$, then the solution $\mathtt{U}$ to $Dir(\Omega, \varphi, fd\mu)$ has the following modulus of continuity

$$\omega_{\mathtt{U}}(\delta) \leq C \max\{\delta^\nu, \omega_w^\gamma(\delta^{\frac{2-\tau}{2n}})\},$$

where $\nu = \min\{\frac{\alpha\epsilon}{\epsilon+6}, \frac{\epsilon^2\gamma}{\epsilon+6}, \tau\gamma\}$, $0 < \tau < 2$ is an arbitrary constant and C is a positive constant depends on Ω, Ω_1, n, ϵ, $\|\mathtt{U}\|_{L^\infty(\Omega)}$, $\|w\|_{L^\infty(\Omega)}$.

3.6 Open questions

- Let $\varphi \in \mathcal{C}^{0,\alpha}(\partial\Omega)$, $0 < \alpha \le 1$ and let μ be a finite Borel measure on Ω satisfying Condition $\mathcal{H}(\infty)$. Suppose that the Dirichlet problem $Dir(\Omega, \varphi, d\mu)$ has a Hölder continuous subsolution in Ω. Is the solution to this problem Hölder continuous in $\bar{\Omega}$? We have shown in Theorem 3.1.5 an affirmative answer when μ satisfies some nice condition near $\partial\Omega$.

- Suppose that μ is a finite Borel measure on Ω and it is strongly dominated by capacity, that is, there exist $A, B > 0$ so that for any Borel set $K \subset \Omega$,

$$\mu(K) \le Ae^{-B\mathrm{Cap}(K,\Omega)^{-1/n}}.$$

  Suppose that $\varphi \in \mathcal{C}^{0,\alpha}(\partial\Omega)$, $0 < \alpha \le 1$. Does the solution to $Dir(\Omega, \varphi, d\mu)$ belong to $\mathcal{C}^{0,\alpha'}(\bar{\Omega})$ for some $0 < \alpha' < 1$?

Chapter 4

The Dirichlet problem for complex Hessian equations

4.1 Introduction

Let Ω be a bounded domain in $\mathbb{C}^n$ with smooth boundary and let m be an integer such that $1 \leq m \leq n$. Given $\varphi \in \mathcal{C}(\partial\Omega)$ and $0 \leq f \in \mathcal{C}(\bar{\Omega})$, we consider the Dirichlet problem for the complex Hessian equation:

$$\begin{cases} u \in SH_m(\Omega) \cap \mathcal{C}(\bar{\Omega}), & \\ (dd^c u)^m \wedge \beta^{n-m} = f\beta^n & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{4.1.1}$$

where $SH_m(\Omega)$ denotes the set of all m-subharmonic functions in Ω and $\beta := dd^c|z|^2$ is the standard Kähler form in $\mathbb{C}^n$.

In the case $m = 1$, this equation corresponds to the Poisson equation which is classical (see [GT01]). The case $m = n$ corresponds to the complex Monge-Ampère equation which was intensively studied these last decades by several authors (see [BT76], [CP92], [CK94], [Ko98]).

The complex Hessian equation is a new subject and is much more difficult to handle than the complex Monge-Ampère equation (e.g. the m-subharmonic functions are not invariant under holomorphic change of variables, for $m < n$). Despite these difficulties, the pluripotential theory developed in ([BT82] , [De89], [Ko98]) for the complex Monge-Ampère equation, can be adapted to the complex Hessian equation.

The Dirichlet problem (4.1.1) was considered by S.Y. Li in [Li04]. He proved that if Ω is a bounded strongly m-pseudoconvex domain with smooth boundary (see the definition below), $\varphi \in \mathcal{C}^\infty(\partial\Omega)$ and $0 < f \in \mathcal{C}^\infty(\bar{\Omega})$ then there exists a unique smooth solution to (4.1.1).
The existence of continuous solution for the homogeneous Dirichlet problem in the unit ball was proved by Z. Błocki [Bł05].
Recently, S. Dinew and S. Kołodziej proved in [DK14] that there exists a unique continuous solution to (4.1.1) when $0 \leq f \in L^p(\Omega)$, $p > n/m$.

A potential theory for the complex Hessian equation was independently developed by Sadullaev and Abdullaev in [SA12] and H.C. Lu in [Lu12].
H.C. Lu developed [Lu13b] a viscosity approach to the following Dirichlet problem for the complex Hessian equation.

$$\begin{cases} u \in SH_m(\Omega) \cap \mathcal{C}(\bar{\Omega}), \\ (dd^c u)^m \wedge \beta^{n-m} = F(z,u)\beta^n & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{4.1.2}$$

where $F : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}^+$ is a continuous function and nondecreasing in the second variable.

Our first main result in this chapter gives a sharp estimate for the modulus of continuity of the solution to the Dirichlet problem (4.1.2). More precisely, we will prove the following theorem.

Theorem 4.1.1. *Let Ω be a smoothly bounded strongly m-pseudoconvex domain in $\mathbb{C}^n$, $\varphi \in \mathcal{C}(\partial\Omega)$ and $0 \leq F \in \mathcal{C}(\bar{\Omega} \times \mathbb{R})$ be a nondecreasing function in the second variable. Then the modulus of continuity $\omega_{\mathtt{U}}$ of the solution $\mathtt{U}$ satisfies the following estimate*

$$\omega_{\mathtt{U}}(t) \leq \gamma(1 + \|F\|^{1/m}_{L^\infty(K)}) \ \max\{\omega_\varphi(t^{1/2}), \omega_{F^{1/m}}(t), t^{1/2}\},$$

where γ is a positive constant depending only on Ω, $K = \bar{\Omega} \times \{a\}$, $a = \sup_{\partial\Omega} |\varphi|$ and $\omega_{F^{1/m}}(t)$ is given by

$$\omega_{F^{1/m}}(t) := \sup_{y \in [-M,M]} \sup_{|z_1 - z_2| \leq t} |F^{1/m}(z_1, y) - F^{1/m}(z_2, y)|,$$

with $M := a + 2\,\mathrm{diam}(\Omega)^2 \sup_{\bar{\Omega}} F^{1/m}(., -a)$.

H.C. Lu proved in [Lu13b] that the solution to (4.1.2) is Hölder continuous on a smooth bounded strongly pseudoconvex domain Ω under conditions of Hölder continuity of φ and F.

In the case of the complex Monge-Ampère equation, Y. Wang gave a control on the modulus of continuity of the solution assuming the existence of a subsolution and a supersolution with the given boundary data ([Wan12]).

Here we do not assume the existence of a subsolution and a supersolution. Actually the main argument in our proof consists in constructing adequate barriers for the Dirichlet problem for the complex Hessian equation (4.1.2) in a strongly m-pseudoconvex domain.

In the case when the density $f \in L^p(\Omega)$ with $p > n/m$, N.C. Nguyen [N14] proved the Hölder continuity of the solution to (4.1.1) when the boundary data is in $\mathcal{C}^{1,1}(\partial\Omega)$ and the density f satisfies a growth condition near the boundary of Ω.

In the case $m = n$, the author recently proved [Ch15a] that the solution to the Dirichlet problem (4.1.1) is Hölder continuous on $\bar{\Omega}$ without assuming any condition near the boundary. Using the same idea we can prove a similar result for the complex Hessian equation. Accurately, we have the following theorem.

Theorem 4.1.2. *Let $\Omega \subset \mathbb{C}^n$ be a bounded strongly m-pseudoconvex domain with smooth boundary, $\varphi \in \mathcal{C}^{1,1}(\partial\Omega)$ and $0 \leq f \in L^p(\Omega)$, for some $p > n/m$. Then the solution to (4.1.1) $\mathtt{U} \in \mathcal{C}^{0,\alpha}(\bar{\Omega})$ for any $0 < \alpha < \gamma_1$, where γ_1 is a constant depending on m, n, p defined by (4.5.1).*

Moreover, if $p \geq 2n/m$ then the solution $\mathtt{U} \in \mathcal{C}^{0,\alpha}(\bar{\Omega})$, for any $0 < \alpha < \min\{\frac{1}{2}, 2\gamma_1\}$.

In the particular case when $f \in L^p(\Omega)$, for $p > n/m$, and satisfies some condition near the boundary $\partial\Omega$ we can get a better exponent.

Theorem 4.1.3. *Let $\Omega \subset \mathbb{C}^n$ be a strongly m-pseudoconvex bounded domain with smooth boundary. Suppose $\varphi \in \mathcal{C}^{1,1}(\partial\Omega)$ and $0 \leq f \in L^p(\Omega)$ for some $p > n/m$, such that*

$$f(z) \leq (h \circ \rho(z))^m \text{ near } \partial\Omega,$$

where ρ is the defining function on Ω and $0 \leq h \in L^2([-A, 0[)$, with $A \geq \sup_\Omega |\rho|$, is an increasing function. Then the solution $\mathtt{U}$ to (4.1.1) is Hölder continuous of exponent $\alpha < \min\{1/2, 2\gamma_1\}$, where γ_1 is a constant defined by (4.5.1).

Finally, we prove Hölder continuity of the radially symmetric solution with a better exponent which turns out to be optimal.

Theorem 4.1.4. *Let $f \in L^p(\mathbb{B})$ be a radial function, where $p > n/m$. Then the unique solution $\mathtt{U}$ to (4.1.1) with zero boundary values is given by the explicit formula*

$$\mathtt{U}(r) = -B \int_r^1 \frac{1}{t^{2n/m-1}} \left(\int_0^t \rho^{2n-1} f(\rho) d\rho \right)^{1/m} dt, \quad r = |z|, \tag{4.1.3}$$

where $B = \left(\frac{C_n^m}{2^{m+1}n}\right)^{-1/m}$. Moreover, $\mathtt{U} \in \mathcal{C}^{0,2-\frac{2n}{mp}}(\bar{\mathbb{B}})$ for $n/m < p < 2n/m$ and $\mathtt{U} \in Lip(\bar{\mathbb{B}})$ for $p \geq 2n/m$.

4.2 Preliminaries

We define the differential operator $L_\alpha : SH_m(\Omega) \cap L^\infty_{loc}(\Omega) \to \mathcal{D}'(\Omega)$ such that

$$dd^c u \wedge \alpha_1 \wedge ... \wedge \alpha_{m-1} \wedge \beta^{n-m} = L_\alpha u \beta^n,$$

where $\alpha_1, ..., \alpha_{m-1} \in \Sigma_m$. In appropriate complex coordinates this operator is the Laplace operator.

Example 4.2.1. Using the Gårding inequality (1.3.1), one can note that $L_\alpha(|z|^2) \geq 1$ for any $\alpha_i \in \Sigma_m, 1 \leq i \leq m-1$.

We will prove the following essential proposition by applying ideas from the viscosity theory developed in [EGZ11] for the complex Monge-Ampère equation and extended to the complex Hessian equation by H.C.Lu [Lu13b]. A similar result to the following proposition, but for $m = n$, was proved in [Bł96] (see also [Ch15a]).

Proposition 4.2.2. *Let $u \in SH_m(\Omega) \cap \mathcal{C}(\Omega)$ and $0 \leq F \in \mathcal{C}(\Omega \times \mathbb{R})$. The following conditions are equivalent:*
1) $L_\alpha u \geq F^{1/m}(z,u), \forall \alpha_1, ..., \alpha_{m-1} \in \Sigma_m$.
2) $(dd^c u)^m \wedge \beta^{n-m} \geq F(z,u)\beta^n$ in Ω.

Proof. First observe that if $u \in \mathcal{C}^2(\Omega)$, then by Lemma 1.3.2 we can see that (1) is equivalent to

$$\tilde{S}_m(\alpha)^{1/m} \geq F^{1/m}(z,u),$$

where $\alpha = dd^c u$ is a real (1,1)-form belongs to $\hat{\Gamma}_m$.
The last inequality corresponds to

$$(dd^c u)^m \wedge \beta^{n-m} \geq F(z,u)\beta^n \text{ in } \Omega.$$

($1 \Rightarrow 2$) We consider the standard regularization u_ϵ of u by convolution with smoothing kernel. We then get

$$L_\alpha u_\epsilon \geq (F^{1/m}(z,u))_\epsilon.$$

Since u_ϵ is smooth, we infer by the observation above that

$$(dd^c u_\epsilon)^m \wedge \beta^{n-m} \geq ((F^{1/m}(z,u))_\epsilon)^m \beta^n.$$

Letting $\epsilon \to 0$, by the convergence theorem for the Hessian operator under decreasing sequence, we get

$$(dd^c u)^m \wedge \beta^{n-m} \geq F(z,u)\beta^n \text{ in } \Omega.$$

($2 \Rightarrow 1$) Fix $x_0 \in \Omega$ and q is a $\mathcal{C}^2$-function in a neighborhood $V \Subset \Omega$ of x_0 such that $u \leq q$ in this neighborhood and $u(x_0) = q(x_0)$. We will prove that

$$(dd^c q)^m_{x_0} \wedge \beta^{n-m} \geq F(x_0, u(x_0))\beta^n.$$

First step: we claim that $dd^c q_{x_0} \in \hat{\Gamma}_m$.
If u is smooth, we note that x_0 is a local minimum point of $q - u$, then $dd^c(q-u)_{x_0} \geq 0$. Hence, we see that $(dd^c q)^k \wedge \beta^{n-k} \geq 0$ in x_0, for $1 \leq k \leq m$. This gives that $dd^c q_{x_0} \in \hat{\Gamma}_m$. If u is non-smooth, let u_ϵ be the standard smooth regularization of u. Then u_ϵ is m-sh, smooth and $u_\epsilon \searrow u$. Now let us fix $\delta > 0$ and $\epsilon_0 > 0$ such that the neighborhood of x_0, $V \subset \Omega_{\epsilon_0}$. For each $\epsilon < \epsilon_0$, let y_ϵ be the maximum point of $u_\epsilon - q - \delta|x - x_0|^2$ on $\bar{B} \Subset V$, where B is a small ball centered at x_0. Then we have

$$u_\epsilon(x) - q(x) - \delta|x - x_0|^2 \leq u_\epsilon(y_\epsilon) - q(y_\epsilon) - \delta|y_\epsilon - x_0|^2.$$

Assume that $y_\epsilon \to y \in \bar{B}$ and set $x = x_0$. By passing to the limit in the last inequality, we derive that

$$0 \leq u(y) - q(y) - \delta|y - x_0|^2,$$

but $q \geq u$ in V, then we can conclude that $y = x_0$.
Let us then define

$$\tilde{q} := q + \delta|x - x_0|^2 + u_\epsilon(y_\epsilon) - q(y_\epsilon) - \delta|y_\epsilon - x_0|^2,$$

which is a $\mathcal{C}^2$-function in B and satisfies $u_\epsilon(y_\epsilon) = \tilde{q}(y_\epsilon)$ and $\tilde{q} \geq u_\epsilon$ in B, then the following inequality holds in y_ϵ,

$$(dd^c\tilde{q})^k \wedge \beta^{n-k} \geq 0 \text{ for } 1 \leq k \leq m.$$

This means that

$$(dd^c q + \delta\beta)^k_{y_\epsilon} \wedge \beta^{n-k} \geq 0 \text{ for } 1 \leq k \leq m.$$

Letting ϵ tend to 0, we get

$$(dd^c q + \delta\beta)^k_{x_0} \wedge \beta^{n-k} \geq 0 \text{ for } 1 \leq k \leq m.$$

Since the last inequality holds for any $\delta > 0$, we can get that $dd^c q_{x_0} \in \hat{\Gamma}_m$.

Second step: assume that there exist a point $x_0 \in \Omega$ and a $\mathcal{C}^2$-function q satisfying $u \leq q$ in a neighborhood of x_0 and $u(x_0) = q(x_0)$ such that

$$(dd^c q)^m_{x_0} \wedge \beta^{n-m} < F(x_0, u(x_0))\beta^n.$$

Let us set

$$q^\epsilon(x) = q(x) + \epsilon(|x - x_0|^2 - \frac{r^2}{2}),$$

which is a $\mathcal{C}^2$-function and for $0 < \epsilon \ll 1$ small enough we have

$$0 < (dd^c q^\epsilon)^m_{x_0} \wedge \beta^{n-m} < F(x_0, u(x_0))\beta^n.$$

Since F is continuous on $\Omega \times \mathbb{R}$, there exists $r > 0$ such that

$$(dd^c q^\epsilon)^m \wedge \beta^{n-m} \leq F(x, u(x))\beta^n \text{ in } \mathbb{B}(x_0, r).$$

Hence, we get

$$(dd^c q^\epsilon)^m \wedge \beta^{n-m} \leq (dd^c u)^m \wedge \beta^{n-m} \text{ in } \mathbb{B}(x_0, r),$$

and $q^\epsilon = q + \epsilon r^2/2 > q \geq u$ on $\partial\mathbb{B}(x_0, r)$. It follows from the comparison principle (see [Bł05, Lu12]) that $q^\epsilon \geq u$ in $\mathbb{B}(x_0, r)$. But this contradicts that $q^\epsilon(x_0) = u(x_0) - \epsilon r^2/2 < u(x_0)$.

We have shown that for every point $x_0 \in \Omega$, and every $\mathcal{C}^2$-function q in a neighborhood of x_0 such that $u \leq q$ in this neighborhood and $u(x_0) = q(x_0)$, we have $(dd^c q)^m_{x_0} \wedge \beta^{n-m} \geq F(x_0, u(x_0))\beta^n$, hence we have $L_\alpha q_{x_0} \geq F^{1/m}(x_0, u(x_0))$.

Final step: assume that $F > 0$ is a smooth function. Then there exists a $\mathcal{C}^\infty$-function, say g such that $L_\alpha g = F^{1/m}(x, u)$. Hence Theorem 3.2.10' in [H94] implies that $\varphi = u - g$ is L_α-subharmonic, consequently $L_\alpha u \geq F^{1/m}(x, u)$.
In case $F > 0$ is only continuous, we note that

$$F(z, u) = \sup\{w \in \mathcal{C}^\infty, 0 < w \leq F\}.$$

Since $(dd^c u)^m \wedge \beta^{n-m} \geq F(x, u)\beta^n$, we get $(dd^c u)^m \wedge \beta^{n-m} \geq w\beta^n$. As $w > 0$ is smooth, we see that $L_\alpha u \geq w^{1/m}$. Therefore, we conclude $L_\alpha u \geq F^{1/m}(x, u)$.
In the general case $0 \leq F \in \mathcal{C}(\Omega \times \mathbb{R})$, we observe that $u_\epsilon(z) = u(z) + \epsilon|z|^2$ satisfies

$$(dd^c u_\epsilon)^m \wedge \beta^{n-m} \geq (F(x, u) + \epsilon^m)\beta^n.$$

By the last step, we get $L_\alpha u_\epsilon \geq (F(x, u) + \epsilon^m)^{1/m}$, then the required result follows by letting ϵ tend to 0. □

Definition 4.2.3. Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded domain, we say that Ω is strongly m-pseudoconvex if there exist a defining function ρ of Ω (i.e. a smooth function in a neighborhood U of $\bar{\Omega}$ such that $\rho < 0$ on Ω, $\rho = 0$ and $d\rho \neq 0$ on $\partial\Omega$) and $c > 0$ such that

$$(dd^c\rho)^k \wedge \beta^{n-k} \geq c\beta^n \text{ in } U, \quad \text{for } 1 \leq k \leq m.$$

The existence of a solution $\mathtt{U}$ to the Dirichlet problem (4.1.1) was proved in [DK14]. This solution can be given by the upper envelope of subsolutions to the Dirichlet problem as in [BT76] for the complex Monge-Ampère equation.

$$\mathtt{U} = \sup\{v \in SH_m(\Omega) \cap \mathcal{C}(\bar{\Omega});\ v|_{\partial\Omega} \leq \varphi \text{ and } (dd^cv)^m \wedge \beta^{n-m} \geq F(z,v)\beta^n\}. \tag{4.2.1}$$

However, thanks to Lemma 4.2.2, we can describe the solution as the following

$$\mathtt{U} = \sup\{v \in \mathcal{V}_m(\Omega,\varphi,F)\}, \tag{4.2.2}$$

where the family $\mathcal{V}_m(\Omega,\varphi,F)$ is defined as

$$\mathcal{V}_m = \{v \in SH_m(\Omega) \cap \mathcal{C}(\bar{\Omega});\ v|_{\partial\Omega} \leq \varphi \text{ and } L_\alpha v \geq F(z,v)^{1/m}, \forall \alpha_i \in \Sigma_m, 1 \leq i \leq m-1\}.$$

This family is nonempty and stable under the operation of taking finite maximum. Observe that the description of the solution in formula (4.2.2) is more convenient, since it deals with subsolutions with respect to a family of linear elliptic operators.

4.3 Existence of solutions

At first, Li proved [Li04] that there exist smooth solutions to (4.1.1) for smooth positive densities and smooth boundary values. Moreover, it is well known that there exist continuous solutions to (4.1.1) for L^p-densities (see [DK14]). We can give an alternative proof to the existence of these solutions using an analogue method to the proof of Proposition 3.3.2.

In this section we study the existence of a continuous solution to (4.1.2) following Cegrell [Ce84] and using the Schauder-Tychonoff fixed point theorem.

Let u_1 be the continuous solution to (4.1.1) for the boundary values φ and the density $f = 0$ and let also u_2 be the continuous solution to (4.1.1) for the boundary values φ and the density $f = \max_K F(z,t)$ where $K := \bar{\Omega} \times \{\max_{\partial\Omega} |\varphi|\}$.

Let us set

$$\mathcal{A} := \{v \in SH_m(\Omega) \cap L^\infty(\Omega); u_2 \leq v \leq u_1\}.$$

This set is convex and compact in the weak topology. We define the operator $G : \mathcal{A} \to \mathcal{A}$ by taking $G(v)$ to be the continuous solution to the Dirichlet problem:

$$(dd^cw)^m \wedge \beta^{n-m} = F(z,v)\beta^n \text{ and } \lim_{z\to\partial\Omega} w(z) = \varphi,$$

which exists and is unique by [DK14]. We claim that this operator is continuous in the $L^1(\Omega)$-topology. Let $v_j \in \mathcal{A}$ converges to v in the $L^1(\Omega)$-topology. By passing to a subsequence, we can assume that v_j converges pointwise almost everywhere to v. We set

$$m_i(z) := \inf_{j\geq i} F(z,v_j) \text{ and } M_i(z) := \sup_{j\geq i} F(z,v_j).$$

It is clear $m_i(z) \leq F(z, v_i) \leq M_i(z)$. We take $\tilde{v}_i$ and $\hat{v}_i$ to be the solutions to (4.1.1) with densities m_i and M_i respectively. Thus, we conclude $\hat{v}_i \leq G(v_i) \leq \tilde{v}_i$. Hence, $(\tilde{v}_i)$ is decreasing sequence and $(\hat{v}_i)$ is increasing sequence. So, we put $\lim \tilde{v} := \lim \tilde{v}_i \in SH_m(\Omega)$ and $\hat{v} := (\lim \hat{v}_i)^* \in SH_m(\Omega)$. Hence, we infer

$$(dd^c\tilde{v})^m \wedge \beta^{n-m} = (dd^c\hat{v})^m \wedge \beta^{n-m} = F(z, v)\beta^n.$$

The comparison principle implies that $\tilde{v} = \hat{v}$. Finally, we get $\lim G(v_i) = \hat{v} = \tilde{v} = G(v)$ almost everywhere. Hence G is continuous in the weak topology.

It follows from the Schauder-Tychonoff fixed point theorem that there exists $v \in \mathcal{A}$ such that $G(v) = v$. So that we have a function $u \in SH_m(\Omega) \cap L^\infty(\Omega)$ such that $(dd^cu)^m \wedge \beta^{n-m} = F(z, u)\beta^n$ and $\lim_{z\to\xi} u(z) = \varphi(\xi), \forall\xi \in \partial\Omega$.

Since our solution is the unique solution to (4.1.1) for the bounded density $f = F(z, u)$, this implies that u is continuous on $\bar{\Omega}$.

The uniqueness of the solution to (4.1.2) is a consequence of the comparison principle. Indeed, suppose that there exist two continuous solutions u_1, u_2 such that the open set $V := \{u_1 < u_2\}$ is not empty. Since F is nondecreasing in the second variable, we get that $(dd^cu_1)^m \wedge \beta^{n-m} \leq (dd^cu_2)^m \wedge \beta^{n-m}$ in V and $u_1 = u_2$ on ∂V. By the comparison principle, we infer that $u_1 \geq u_2$ in V. This is a contradiction.

4.4 Modulus of continuity of the solution

Lemma 4.4.1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded strongly m-pseudoconvex domain with smooth boundary. Then for every point $\xi \in \partial\Omega$ and $\varphi \in \mathcal{C}(\partial\Omega)$, there exist a constant $C > 0$ depending only on Ω and a function $h_\xi \in SH_m(\Omega)\cap\mathcal{C}(\bar{\Omega})$ such that the following conditions hold:*
(1) $h_\xi(z) \leq \varphi(z), \forall z \in \partial\Omega$,
(2) $h_\xi(\xi) = \varphi(\xi)$,
(3) $\omega_{h_\xi}(t) \leq C\omega_\varphi(t^{1/2})$.

Proof. Since Ω is strongly m-pseudoconvex and its defining function ρ is smooth, we can choose $B > 0$ large enough such that the function

$$g(z) = B\rho(z) - |z - \xi|^2,$$

is m-subharmonic in Ω. Let $\bar{\omega}_\varphi$ be the minimal concave majorant of ω_φ and define

$$\chi(x) = -\bar{\omega}_\varphi((-x)^{1/2}),$$

which is a convex nondecreasing function on $[-d^2, 0]$. Now, fix $r > 0$ so small that $|g(z)| \leq d^2$ in $B(\xi, r) \cap \Omega$ and define for $z \in B(\xi, r) \cap \bar{\Omega}$ the function

$$h(z) = \chi \circ g(z) + \varphi(\xi).$$

It is clear that h is a continuous m-subharmonic function on $B(\xi, r)\cap\Omega$ and one can observe that $h(z) \leq \varphi(z)$ if $z \in B(\xi, r) \cap \partial\Omega$ and $h(\xi) = \varphi(\xi)$. Moreover, by the subadditivity of

$\bar{\omega}_\varphi$ and Lemma 2.4.1 we have

$$\begin{aligned}\omega_h(t) &= \sup_{|z-y|\le t} |h(z)-h(y)| \\ &\le \sup_{|z-y|\le t} \bar{\omega}_\varphi \left[\left| |z-\xi|^2 - |y-\xi|^2 - B(\rho(z)-\rho(y)) \right|^{1/2} \right] \\ &\le \sup_{|z-y|\le t} \bar{\omega}_\varphi \left[((2d+B_1)|z-y|)^{1/2} \right] \\ &\le \tilde{C}.\omega_\varphi(t^{1/2}),\end{aligned}$$

where $\tilde{C} := 1 + (2d+B_1)^{1/2}$ is a constant depending on Ω.

Recall that $\xi \in \partial\Omega$ and fix $0 < r_1 < r$ and $\gamma_1 \ge 1 + d/r_1$ such that

$$-\gamma_1 \bar{\omega}_\varphi \left[(|z-\xi|^2 - B\rho(z))^{1/2} \right] \le \inf_{\partial\Omega} \varphi - \sup_{\partial\Omega} \varphi,$$

for $z \in \partial\Omega \cap \partial B(\xi, r_1)$. Let us set $\gamma_2 = \inf_{\partial\Omega} \varphi$, it follows that

$$\gamma_1(h(z) - \varphi(\xi)) + \varphi(\xi) \le \gamma_2 \text{ for } z \in \partial B(\xi, r_1) \cap \bar{\Omega}.$$

Now set

$$h_\xi(z) = \begin{cases} \max\{\gamma_1(h(z)-\varphi(\xi)) + \varphi(\xi), \gamma_2\} & ; z \in \bar{\Omega} \cap (B(\xi, r_1), \\ \gamma_2 & ; z \in \bar{\Omega} \setminus B(\xi, r_1), \end{cases}$$

which is a well defined m-subharmonic function on Ω and continuous on $\bar{\Omega}$. Moreover, it satisfies $h_\xi(z) \le \varphi(z)$ for all $z \in \partial\Omega$. Indeed, on $\partial\Omega \cap B(\xi, r_1)$ we have

$$\gamma_1(h(z)-\varphi(\xi)) + \varphi(\xi) = -\gamma_1 \bar{\omega}_\varphi(|z-\xi|) + \varphi(\xi) \le -\bar{\omega}_\varphi(|z-\xi|) + \varphi(\xi) \le \varphi(z).$$

Furthermore, the modulus of continuity of h_ξ satisfies

$$\omega_{h_\xi}(t) \le C\omega_\varphi(t^{1/2}),$$

where $C := \gamma_1 \tilde{C}$ depends on Ω. Hence, h_ξ satisfies the conditions (1)-(3), and this completes the proof. □

In the following proposition, we establish a barrier to the problem (4.1.2) and estimate its modulus of continuity.

Proposition 4.4.2. *Let $\Omega \subset \mathbb{C}^n$ be a bounded strongly m-pseudoconvex domain with smooth boundary. Assume that ω_φ is the modulus of continuity of $\varphi \in \mathcal{C}(\partial\Omega)$ and $0 \le F \in \mathcal{C}(\bar{\Omega} \times \mathbb{R})$ is nondecreasing in the second variable. Then there exists a subsolution $v \in \mathcal{V}_m(\Omega, \varphi, F)$ such that $v = \varphi$ on $\partial\Omega$ and the modulus of continuity of v satisfies the following inequality*

$$\omega_v(t) \le \lambda \max\{\omega_\varphi(t^{1/2}), t^{1/2}\},$$

where $\lambda = \eta(1 + \|F\|^{1/m}_{L^\infty(K)})$, $K = \bar{\Omega} \times \{\sup_{\partial\Omega} |\varphi|\}$ and η is a positive constant depending on Ω.

Proof. First of all, fix $\xi \in \partial\Omega$. We will prove that there exists $v_\xi \in \mathcal{V}_m(\Omega, \varphi, F)$ such that $v_\xi(\xi) = \varphi(\xi)$.
We fix $z_0 \in \Omega$ and set $K_1 := \sup_K F^{1/m}$. Hence, we have

$$L_\alpha(K_1|z - z_0|^2) = K_1 L_\alpha |z - z_0|^2 \geq F^{1/m}(z, \sup_{\partial\Omega} |\varphi|),$$

for all $\alpha_i \in \Sigma_m, 1 \leq i \leq m-1$ and $z \in \bar{\Omega}$. We also set $K_2 := K_1|\xi - z_0|^2$ and define the continuous function

$$\tilde{\varphi}(z) := \varphi(z) - K_1|z - z_0|^2 + K_2.$$

we find, by Lemma 4.4.1, a constant $C > 0$ depending on Ω and a function h_ξ satisfying the following conditions:
1) $h_\xi(z) \leq \tilde{\varphi}(z), \forall z \in \partial\Omega$,
2) $h_\xi(\xi) = \tilde{\varphi}(\xi)$,
3) $\omega_{h_\xi}(t) \leq C\omega_{\tilde{\varphi}}(t^{1/2})$.
Then the required function $v_\xi \in \mathcal{V}_m(\Omega, \varphi, F)$ is given by

$$v_\xi(z) := h_\xi(z) + K_1|z - z_0|^2 - K_2.$$

It is obvious that $v_\xi \in SH_m(\Omega) \cap \mathcal{C}(\bar{\Omega})$. Since

$$h_\xi(z) \leq \tilde{\varphi}(z) = \varphi(z) - K_1|z - z_0|^2 + K_2 \text{ on } \partial\Omega,$$

we conclude $v_\xi(z) \leq \varphi(z)$ on $\partial\Omega$ and $v_\xi(\xi) = \varphi(\xi)$. Moreover, we have

$$L_\alpha v_\xi = L_\alpha h + K_1 L_\alpha |z - z_0|^2 \geq F^{1/m}(z, v_\xi) \text{ in } \Omega.$$

Furthermore, by the hypothesis on h_ξ , we can estimate the modulus of continuity of v_ξ:

$$\begin{aligned}
\omega_{v_\xi}(t) = \sup_{|z-y|\leq t} |v(z) - v(y)| &\leq \omega_h(t) + K_1\omega_{|z-z_0|^2}(t) \\
&\leq C\omega_{\tilde{\varphi}}(t^{1/2}) + 4d^{3/2}K_1 t^{1/2} \\
&\leq C\omega_\varphi(t^{1/2}) + 2dK_1(C + 2d^{1/2})t^{1/2} \\
&\leq (C + 2d^{1/2})(1 + 2dK_1)\max\{\omega_\varphi(t^{1/2}), t^{1/2}\},
\end{aligned}$$

where $d := \mathrm{diam}(\Omega)$. Hence, we have

$$\omega_{v_\xi}(t) \leq \eta(1 + K_1)\max\{\omega_\varphi(t^{1/2}), t^{1/2}\},$$

where $\eta := (C + 2d^{1/2})(1 + 2d)$ is a constant depending on Ω.

We have just proved that for each $\xi \in \partial\Omega$, there is a function $v_\xi \in \mathcal{V}_m(\Omega, \varphi, F)$ such that $v_\xi(\xi) = \varphi(\xi)$, and

$$\omega_{v_\xi}(t) \leq \eta(1 + K_1)\max\{\omega_\varphi(t^{1/2}), t^{1/2}\}.$$

Let us set

$$v(z) = \sup\{v_\xi(z); \xi \in \partial\Omega\}.$$

We have $0 \leq \omega_v(t) \leq \eta(1+K_1)\max\{\omega_\varphi(t^{1/2}), t^{1/2}\}$, thus $\omega_v(t)$ converges to zero when t converges to zero. Consequently, we get $v \in \mathcal{C}(\bar{\Omega})$ and $v = v^* \in SH_m(\Omega)$. Thanks to Choquet's lemma, we can choose a nondecreasing sequence (v_j), where $v_j \in \mathcal{V}_m(\Omega, \varphi, F)$, converging to v almost everywhere. So that

$$L_\alpha v = \lim_{j\to\infty} L_\alpha v_j \geq F^{1/m}(z, v), \forall \alpha_i \in \Sigma_m.$$

It is clear that $v(\xi) = \varphi(\xi)$ for any $\xi \in \partial\Omega$. Finally, we get $v \in \mathcal{V}_m(\Omega, \varphi, F)$ such that $v = \varphi$ on $\partial\Omega$ and $\omega_v(t) \leq \eta(1+K_1)\max\{\omega_\varphi(t^{1/2}), t^{1/2}\}$. ☐

Corollary 4.4.3. *Under the same assumption of Proposition 4.4.2. There exists a m-superharmonic function $\tilde{v} \in \mathcal{C}(\bar{\Omega})$ such that $\tilde{v} = \varphi$ on $\partial\Omega$ and*

$$\omega_{\tilde{v}}(t) \leq \lambda \max\{\omega_\varphi(t^{1/2}), t^{1/2}\},$$

where $\lambda > 0$ is as in Proposition 4.4.2.

Proof. We can do the same construction as in the proof of Proposition 4.4.2 for the function $\varphi_1 = -\varphi \in \mathcal{C}(\partial\Omega)$, then we get $v_1 \in \mathcal{V}_m(\Omega, \varphi_1, F)$ such that $v_1 = \varphi_1$ on $\partial\Omega$ and $\omega_{v_1}(t) \leq \lambda \max\{\omega_\varphi(t^{1/2}), t^{1/2}\}$. Hence, we set $\tilde{v} = -v_1$ which is a m-superharmonic function on Ω, continuous on $\bar{\Omega}$ and satisfying $\tilde{v} = \varphi$ on $\partial\Omega$ and $\omega_{\tilde{v}}(t) \leq \lambda \max\{\omega_\varphi(t^{1/2}), t^{1/2}\}$. ☐

Proof of Theorem 4.1.1. Thanks to Proposition 4.4.2, we obtain a subsolution $v \in \mathcal{V}_m(\Omega, \varphi, F)$ with $v = \varphi$ on $\partial\Omega$ and $\omega_v(t) \leq \lambda \max\{\omega_\varphi(t^{1/2}), t^{1/2}\}$. From Corollary 4.4.3, we construct a m-superharmonic function $\tilde{v} \in \mathcal{C}(\bar{\Omega})$ such that $\tilde{v} = \varphi$ on $\partial\Omega$ and $\omega_{\tilde{v}}(t) \leq \lambda \max\{\omega_\varphi(t^{1/2}), t^{1/2}\}$, where λ is as in Proposition 4.4.2.
Applying the maximum principle, we get that

$$v(z) \leq \mathtt{U}(z) \leq \tilde{v}(z) \text{ for all } z \in \bar{\Omega}.$$

We set $g(t) = \max\{\lambda \max\{\omega_\varphi(t^{1/2}), t^{1/2}\}, \omega_{F^{1/m}}(t)\}$ and $d := \operatorname{diam}(\Omega)$. Then

$$|\mathtt{U}(z) - \mathtt{U}(\xi)| \leq g(|z-\xi|); \forall z \in \Omega, \forall \xi \in \partial\Omega.$$

Let us fix a point $z_0 \in \Omega$, for any vector $\tau \in \mathbb{C}^n$ with small enough norm, we define

$$V(z,\tau) = \begin{cases} \mathtt{U}(z) & ; z+\tau \notin \Omega, z \in \bar{\Omega}, \\ \max\{\mathtt{U}(z), v_1(z)\} & ; z, z+\tau \in \Omega, \end{cases}$$

where $v_1(z) = \mathtt{U}(z+\tau) + g(|\tau|)|z-z_0|^2 - d^2 g(|\tau|) - g(|\tau|)$.
Observe that if $z \in \Omega, z+\tau \in \partial\Omega$, we have

$$v_1(z) - \mathtt{U}(z) \leq g(|\tau|) + g(|\tau|)|z-z_0|^2 - d^2 g(|\tau|) - g(|\tau|) \leq 0. \tag{4.4.1}$$

Then $v_1(z) \leq \mathtt{U}(z)$ for $z \in \Omega, z+\tau \in \partial\Omega$. In particular, $V(z,\tau)$ is well defined and belongs to $SH_m(\Omega) \cap \mathcal{C}(\bar{\Omega})$.
We claim that

$$F^{1/m}(z_1, \mathtt{U}(x)) - F^{1/m}(z_2, \mathtt{U}(x)) \leq \omega_{F^{1/m}}(|z_1 - z_2|),$$

for all $x, z_1, z_2 \in \bar{\Omega}$. Indeed, it is enough to show that

$$\|\mathtt{U}\|_{L^\infty(\bar{\Omega})} \leq M := a + 2d^2 \sup_{\bar{\Omega}} F^{1/m}(.,-a),$$

with $a := \sup_{\partial\Omega} |\varphi|$. By the maximum principle, we have $\mathtt{U} \leq a$. We set $b = \sup_{\bar{\Omega}} F^{1/m}(.,-a)$ and $\tilde{u} = b(|z-z_0|^2 - d^2) - a \in SH_m(\Omega) \cap \mathcal{C}(\bar{\Omega})$ where $z_0 \in \Omega$ is a fixed point. Hence, $\tilde{u} \leq \varphi$ on $\partial\Omega$. Since F is nondecreasing in the second variable, we get

$$(dd^c\tilde{u})^m \wedge \beta^{n-m} \geq F(z,-a)\beta^n \geq F(z,\tilde{u})\beta^n.$$

Consequently, $\tilde{u} \leq \mathtt{U}$ in Ω and then we get the required statement.

Now, we assert that $L_\alpha V \geq F^{1/m}(z, V)$, for all $\alpha_i \in \Sigma_m$. Indeed,

$$\begin{aligned} L_\alpha v_1(z) &\geq F^{1/m}(z+\tau, \mathtt{U}(z+\tau)) + g(|\tau|)L_\alpha(|z-z_0|^2) \\ &\geq F^{1/m}(z+\tau, \mathtt{U}(z+\tau)) + g(|\tau|) \\ &\geq F^{1/m}(z+\tau, \mathtt{U}(z+\tau)) + |F^{1/m}(z+\tau, \mathtt{U}(z+\tau)) - F^{1/m}(z, \mathtt{U}(z+\tau))| \\ &\geq F^{1/m}(z, \mathtt{U}(z+\tau)), \\ &\geq F^{1/m}(z, v_1(z)), \end{aligned}$$

for all $\alpha_i \in \Sigma_m, 1 \leq i \leq m-1$.
If $z \in \partial\Omega$, $z+\tau \notin \Omega$, then $V(z,\tau) = \mathtt{U}(z) = \varphi(z)$. On the other hand, $z \in \partial\Omega, z+\tau \in \Omega$, we get by (4.4.1) that $V(z,\tau) = \max\{\mathtt{U}(z), v_1(z)\} = \mathtt{U}(z) = \varphi(z)$. Then $V(z,\tau) = \varphi(z)$ on $\partial\Omega$, hence $V \in \mathcal{V}_m(\Omega, \varphi, F)$.
Consequently, $V(z,\tau) \leq \mathtt{U}(z); \forall z \in \bar{\Omega}$. This implies that if $z \in \Omega$, $z+\tau \in \bar{\Omega}$, we have

$$\mathtt{U}(z+\tau) + g(|\tau|)|z-z_0|^2 - d^2 g(|\tau|) - g(|\tau|) \leq \mathtt{U}(z).$$

Hence,

$$\mathtt{U}(z+\tau) - \mathtt{U}(z) \leq (d^2+1)g(|\tau|) - g(|\tau|).|z-z_0|^2 \leq (d^2+1)g(|\tau|).$$

Reversing the roles of $z+\tau$ and z, we get

$$|\mathtt{U}(z+\tau) - \mathtt{U}(z)| \leq (d^2+1)g(|\tau|); \forall z \in \bar{\Omega}.$$

Thus,

$$\omega_{\mathtt{U}}(t) \leq (d^2+1)g(t).$$

Finally, we have

$$\omega_{\mathtt{U}}(t) \leq \gamma(1 + \|F\|^{1/m}_{L^\infty(K)}) \max\{\omega_\varphi(t^{1/2}), \omega_{F^{1/m}}(t), t^{1/2}\},$$

where $\gamma := \eta(d^2+1)$, η is a positive constant depending on Ω and $K = \bar{\Omega} \times \{\sup_{\partial\Omega} |\varphi|\}$. □

Remark 4.4.4. When $m = n$ we can get by a slight modification that the proof is still true for a bounded strongly hyperconvex Lipschitz domain in $\mathbb{C}^n$ and this yields Theorem 2.1.3 in Chapter 2.

Theorem 4.1.1 has the following consequence.

Corollary 4.4.5. *Let Ω be a smoothly bounded strongly m-pseudoconvex domain in $\mathbb{C}^n$. Let $\varphi \in \mathcal{C}^{2\alpha}(\partial\Omega)$ and $0 \leq f^{1/m} \in \mathcal{C}^{\alpha}(\bar{\Omega})$, $0 < \alpha \leq 1/2$. Then the solution $\mathtt{U}$ of the Dirichlet problem (4.1.1) belongs to $\mathcal{C}^{\alpha}(\bar{\Omega})$.*

This result was proved by Nguyen in [N14] for the homogeneous case ($f \equiv 0$). H.C. Lu proved in [Lu12, Lu13b] the Hölder continuity of the solution $\mathtt{U}$ under the same assumption of Corollary 4.4.5 in a bounded strongly pseudoconvex domain. A similar result for $m = n$ goes back to Bedford and Taylor in [BT76] and the main idea of the proof depends on Walsh's method [Wal69].

We now give an example to point out that there is a loss in the regularity up to the boundary and show that our result is optimal.

Example 4.4.6. Let ψ be a concave modulus of continuity on $[0, 1]$ and

$$\varphi(z) = -\psi[\sqrt{(1 + Rez_1)/2}], \text{ for } z = (z_1, z_2, ..., z_n) \in \partial\mathbb{B} \subset \mathbb{C}^n.$$

We can show that $\varphi \in \mathcal{C}(\partial\mathbb{B})$ with modulus of continuity $\omega_\varphi(t) \leq C\psi(t)$, for some $C > 0$. We consider the following Dirichlet problem:

$$\begin{cases} u \in SH_m(\Omega) \cap \mathcal{C}(\bar{\Omega}), \\ (dd^c u)^m \wedge \beta^{n-m} = 0 & in\ \ \mathbb{B}, \\ u = \varphi & on\ \ \partial\mathbb{B}, \end{cases}$$

where $2 \leq m \leq n$ is an integer. Then by the comparison principle, $\mathtt{U}(z) := -\psi[\sqrt{(1 + Rez_1)/2}]$ is the unique solution to this problem. One can observe by a radial approach to the boundary point $(-1, 0, ..., 0)$ that

$$C_1\psi(t^{1/2}) \leq \omega_{\mathtt{U}}(t) \leq C_2\psi(t^{1/2}),$$

for some $C_1, C_2 > 0$.

4.5 Hölder continuous solutions for L^p-densities

4.5.1 Preliminaries and known results

The existence of a weak solution to the complex Hessian equation in some bounded domain in $\mathbb{C}^n$ was established in the work of Dinew and Kołodziej [DK14]. More precisely, let $\Omega \Subset \mathbb{C}^n$ be a smoothly $(m-1)$-pseudoconvex domain, $\varphi \in \mathcal{C}(\partial\Omega)$ and $0 \leq f \in L^p(\Omega)$ for some $p > n/m$. Then there exists $\mathtt{U} \in SH_m(\Omega) \cap \mathcal{C}(\bar{\Omega})$ such that $(dd^c\mathtt{U})^m \wedge \beta^{n-m} = f\beta^n$ in Ω and $\mathtt{U} = \varphi$ on $\partial\Omega$.

Recently, N.C. Nguyen in [N14] proved that the Hölder continuity of this solution under some technical conditions: the density $f \in L^p(\Omega)$, $p > n/m$ is bounded near the boundary $\partial\Omega$ or $f \leq C|\rho|^{-m\nu}$ there and the boundary data φ belongs to $\mathcal{C}^{1,1}(\partial\Omega)$.

Here we follow the approach proposed in [GKZ08] for the complex Monge-Ampère equation. A crucial role in this approach is played by an a priori weak stability estimate of the solution. This approach has been adapted to the complex Hessian equation in [N14] and [Lu12]. Here is the precise statement.

In order to simplify the notation, we set from now on for $r \geq 1$,

$$\gamma_r = \frac{r}{r + mq + \frac{pq(n-m)}{p - \frac{n}{m}}}, \tag{4.5.1}$$

where $p > n/m$, $1 \leq m \leq n$ and $1/p + 1/q = 1$.

Theorem 4.5.1. *Fix $0 \leq f \in L^p(\Omega)$ for $p > n/m$. Let $u, v \in SH_m(\Omega) \cap L^\infty(\bar{\Omega})$ be such that $(dd^c u)^m \wedge \beta^{n-m} = f\beta^n$ in Ω, and $\liminf_{z \to \partial\Omega}(u - v)(z) \geq 0$. Fix $r \geq 1$ and $0 < \gamma < \gamma_r$, where γ_r is as in (4.5.1). Then there exists a uniform positive constant $C = C(\gamma, \|f\|_{L^p(\Omega)})$ such that*

$$\sup_\Omega (v - u) \leq C \left[\|(v - u)_+\|_{L^r(\Omega)} \right]^\gamma,$$

where $(v - u)_+ := \max\{v - u, 0\}$.

The proof of this stability theorem is similar to the one for the complex Monge-Ampère equation (see Theorem 3.2.4).

The second result gives the Hölder continuity under some additional hypothesis.

Theorem 4.5.2. *([N14]). Let $0 \leq f \in L^p(\Omega)$ for $p > n/m$, and $\varphi \in \mathcal{C}(\partial\Omega)$. Let $\mathtt{U}$ be the continuous solution to (4.1.1). Suppose that there exists $v \in \mathcal{C}^{0,\nu}(\bar{\Omega})$ for $0 < \nu \leq 1$ such that $v \leq \mathtt{U}$ in Ω and $v = \varphi$ on $\partial\Omega$.*
1) If $\nabla \mathtt{U} \in L^2(\Omega)$ then $\mathtt{U} \in \mathcal{C}^{0,\alpha}(\bar{\Omega})$ for any $\alpha < \min\{\nu, \gamma_2\}$.
2) If the total mass of $\Delta \mathtt{U}$ is finite in Ω then $\mathtt{U} \in \mathcal{C}^{0,\alpha}(\bar{\Omega})$ for any $\alpha < \min\{\nu, 2\gamma_1\}$, where γ_r is defined by (4.5.1) for $r \geq 1$.

This result is analogue to that proved by Guedj, Kołodziej and Zeriahi [GKZ08].

4.5.2 Construction of Hölder barriers

The remaining problem is to construct a Hölder continuous barrier with the right exponent which guarantees one of the conditions in Theorem 4.5.2.

Using the interplay between the real and complex Monge-Ampère measures, we will construct Hölder continuous m-subharmonic barriers for the Dirichlet problem (4.1.1) when $f \in L^p(\Omega)$, $p \geq 2n/m$.

Proposition 4.5.3. *Let $0 \leq f \in L^p(\Omega)$, $p \geq 2n/m$ and let u be a locally convex function in Ω and continuous on $\bar{\Omega}$. If the real Monge-Ampère measure $Mu \geq f^{2n/m} dV_{2n}$ then the complex Hessian measure satisfies the inequality $(dd^c u)^m \wedge \beta^{n-m} \geq f dV_{2n}$ in the weak sense of measures on Ω.*

Proof. It stems from Proposition 3.5.3 that $(dd^c u)^n \geq f^{n/m} dV_{2n}$ in Ω. Set $v = |z|^2 \in PSH(\Omega)$. Since $(dd^c u)^n \geq f^{n/m} dV_{2n}$ and $(dd^c v)^n \geq dV_{2n}$, we get by Theorem 1.2.8 that

$$(dd^c u)^m \wedge \beta^{n-m} \geq f dV_{2n}.$$

□

The following result gives the existence of a 1/2-Hölder continuous m-subharmonic barrier for the problem (4.1.1) when $f \in L^p(\Omega)$, $p \geq 2n/m$.

Theorem 4.5.4. *Let $\varphi \in \mathcal{C}^{0,1}(\partial\Omega)$ and $f \in L^p(\Omega)$, $p \geq 2n/m$. Then there exists $v \in SH_m(\Omega) \cap \mathcal{C}^{0,1/2}(\bar{\Omega})$ such that $v = \varphi$ on $\partial\Omega$ and $(dd^c v)^m \wedge \beta^{n-m} \geq f\beta^n$ in the weak sense of currents. In particular, $v \leq \mathtt{U}$ in Ω.*

Proof. Let B be a large ball containing $\bar{\Omega}$ and let $\tilde{f}$ be the function defined by $\tilde{f} = f$ on Ω and $\tilde{f} = 0$ on $B \setminus \Omega$. Then $\tilde{f} \in L^p(B)$, $p \geq 2n/m$. Let us set $\mu := \tilde{f}^{2n/m}(n!)^{2n/m} dV_{2n}$ that is a nonnegative Borel measure on B with $\mu(B) < \infty$. Thanks to Theorem 3.5.2 there exists a unique convex function $u \in \mathcal{C}(\bar{B})$ such that $Mu = \mu$ in B and $u = 0$ on ∂B. Hence u is Lipschitz continuous on $\bar{\Omega}$. By Proposition 4.5.3, we have $(dd^c u)^m \wedge \beta^{n-m} \geq f\beta^n$ in Ω.
We will construct the required barrier as follows. Let $h_{\varphi-u}$ be the upper envelope of $\mathcal{V}_m(\Omega, \varphi - u, 0)$. Then, thanks to Theorem 4.1.1, $h_{\varphi-u}$ is Hölder continuous of exponent $1/2$ in $\bar{\Omega}$. Now it is easy to check that $v := u + h_{\varphi-u}$ is m-sh in Ω and satisfies $v = \varphi$ in $\partial\Omega$ and $(dd^c v)^m \wedge \beta^{n-m} \geq f\beta^n$ on Ω. Hence $v \leq \mathtt{U}$ in Ω by the comparison principle. □

The last theorem provides us with a Hölder continuous barrier for the Dirichlet problem (4.1.1) with better exponent.

However, when $f \in L^p(\Omega)$ for $p > n/m$, we can find a Hölder continuous barrier with exponent less than γ_1.

Proof of Theorem 4.1.2. We first prove that the total mass of $\Delta\mathtt{U}$ is finite in Ω. Let $\mathtt{U}_0$ be the solution to the Dirichlet problem (4.1.1) with zero boundary values and the density f. We first claim that the total mass of $\Delta\mathtt{U}_0$ is finite in Ω. Indeed, let ρ be the defining function of Ω. By Corollary 1.3.24 we obtain

$$(4.5.2)\quad \int_\Omega dd^c\mathtt{U}_0 \wedge (dd^c\rho)^{m-1} \wedge \beta^{n-m} \leq \left[\int_\Omega (dd^c\mathtt{U}_0)^m \wedge \beta^{n-m}\right]^{\frac{1}{m}} \left[\int_\Omega (dd^c\rho)^m \wedge \beta^{n-m}\right]^{\frac{m-1}{m}}.$$

Since Ω is a bounded strongly m-pseudoconvex domain, there exists a constant $c > 0$ such that $(dd^c\rho)^j \wedge \beta^{n-j} \geq c\beta^n$ in Ω for $1 \leq j \leq m$. We find $A \gg 1$ such that $A\rho - |z|^2$ is m-sh function. Now, it is easy to see that

$$\int_\Omega dd^c\mathtt{U}_0 \wedge \beta^{n-1} \leq \int_\Omega dd^c\mathtt{U}_0 \wedge (Add^c\rho)^{m-1} \wedge \beta^{n-m}.$$

Hence, the inequality (4.5.2) yields

$$\int_\Omega dd^c\mathtt{U}_0 \wedge \beta^{n-1} \leq A^{m-1} \left[\int_\Omega (dd^c\mathtt{U}_0)^m \wedge \beta^{n-m}\right]^{\frac{1}{m}} \left[\int_\Omega (dd^c\rho)^m \wedge \beta^{n-m}\right]^{\frac{m-1}{m}}.$$

Now, we note that the total mass of complex Hessian measures of ρ and $\mathtt{U}_0$ are finite in Ω. Therefore, the total mass of $\Delta\mathtt{U}_0$ is finite in Ω.
Let $\tilde{\varphi}$ be a $\mathcal{C}^{1,1}$-extension of φ to $\bar{\Omega}$ such that $\|\tilde{\varphi}\|_{\mathcal{C}^{1,1}(\bar{\Omega})} \leq C\|\varphi\|_{\mathcal{C}^{1,1}(\partial\Omega)}$, for some $C > 0$. Let $v = B\rho + \tilde{\varphi} + \mathtt{U}_0$, where $B \gg 1$ is so that $B\rho + \tilde{\varphi} \in SH_m(\Omega) \cap \mathcal{C}(\bar{\Omega})$. By the comparison principle, we see that $v \leq \mathtt{U}$ in Ω and $v = \mathtt{U} = \varphi$ on $\partial\Omega$. Since ρ is smooth in a neighborhood of $\bar{\Omega}$ and $\|\Delta\mathtt{U}_0\|_\Omega < +\infty$, we derive that $\|\Delta v\|_\Omega < +\infty$. Then, by Lemma 3.4.6, we have $\|\Delta\mathtt{U}\|_\Omega < +\infty$.

To apply Theorem 4.5.2 we construct a Hölder continuous function v such that $v \leq \mathtt{U}$ in Ω and $v = \varphi$ on $\partial\Omega$. We first assume that $f = 0$ near the boundary of Ω, that is there exists a compact $K \Subset \Omega$ such that $f = 0$ in $\Omega \setminus K$. We set $A > 0$ large enough so that $v := A\rho + \tilde{\varphi} \in SH_m(\Omega) \cap \mathcal{C}^{0,1}(\bar{\Omega})$ and $v \leq \mathtt{U}$ in a neighborhood of K. By the comparison principle, we can find that $v \leq \mathtt{U}$ in $\Omega \setminus K$ and hence $v \leq \mathtt{U}$ in Ω and $v|_{\partial\Omega} = \mathtt{U}|_{\partial\Omega} = \varphi$. Thus, Theorem 4.5.2 implies that the solution $\mathtt{U}$ is Hölder continuous in $\bar{\Omega}$ of exponent $\alpha_1 < 2\gamma_1$, where γ_1 is as in (4.5.1).

For the general case, when $0 \leq f \in L^p(\Omega)$, $p > n/m$. Let us fix a large ball $B \subset \mathbb{C}^n$ such that $\Omega \Subset B \subset \mathbb{C}^n$. We define $\tilde{f} = f$ in Ω and $\tilde{f} = 0$ in $B \setminus \Omega$. Let h_1 to the Dirichlet problem in B with the density $\tilde{f}$ and zero boundary values. Since $\tilde{f} \in L^p(\Omega)$ is bounded near ∂B, h_1 is Hölder continuous on $\bar{B}$ of exponent $\alpha_1 < 2\gamma_1$ by the previous case. Now let h_2 denote the solution to the Dirichlet problem in Ω with boundary values $\varphi - h_1$ and zero density. Thanks to Theorem 4.1.1, we infer that $h_2 \in \mathcal{C}^{0,\alpha_2}(\bar{\Omega})$, where $\alpha_2 = \alpha_1/2$. Therefore, the required barrier will be $v = h_1 + h_2$. It is clear that $v \in SH_m(\Omega) \cap \mathcal{C}(\bar{\Omega})$, $v|_{\partial\Omega} = \varphi$ and $(dd^c v)^m \wedge \beta^{n-m} \geq f\beta^n$ in the weak sense in Ω. Hence, by the comparison principle, we get that $v \leq \mathtt{U}$ in Ω and $v = \mathtt{U} = \varphi$ on $\partial\Omega$. Moreover, we have $v \in \mathcal{C}^{0,\alpha}(\bar{\Omega})$ for any $\alpha < \gamma_1$.
Hence, when $p > n/m$, we get by Theorem 4.5.2 that $\mathtt{U} \in \mathcal{C}^{0,\alpha}(\bar{\Omega})$ for any $\alpha < \gamma_1$.
Moreover, if $p \geq 2n/m$, Theorem 4.5.4 gives the existence of a 1/2-Hölder continuous barrier to the Dirichlet problem. Then, we obtain by Theorem 4.5.2 that $\mathtt{U} \in \mathcal{C}^{0,\alpha}(\bar{\Omega})$ for any $\alpha < \min\{1/2, 2\gamma_1\}$. □

We are able to find a better Hölder-exponent of the solution, when the density $f \in L^p(\Omega)$, $p > n/m$, satisfies the following condition near the boundary $\partial\Omega$,

$$f(z) \leq (h \circ \rho(z))^m \text{ near } \partial\Omega,$$

where $0 \leq h \in L^2([-A, 0[)$ is an increasing function and $A \geq \sup_\Omega |\rho|$.

Proof of Theorem 4.1.3. Let $\chi : [-A, 0] \to \mathbb{R}^-$ be the primitive of h such that $\chi(0) = 0$. It is clear that χ is a convex increasing function. By the Hölder inequality, we see that

$$|\chi(t_1) - \chi(t_2)| \leq \|h\|_{L^2} |t_1 - t_2|^{1/2},$$

for all $t_1, t_2 \in [-A, 0]$. From the hypothesis, there exists a compact $K \Subset \Omega$ such that

$$f(z) \leq (h \circ \rho(z))^m \text{ for } z \in \Omega \setminus K. \tag{4.5.3}$$

Then the function $v = \chi \circ \rho$ is m-subharmonic in Ω, continuous on $\bar{\Omega}$ and satisfies

$$dd^c \chi \circ \rho = \chi''(\rho) d\rho \wedge d^c\rho + \chi'(\rho) dd^c\rho \geq \chi'(\rho) dd^c\rho,$$

in the weak sense of currents in Ω.

By Definition 4.2.3, there is a constant $c > 0$ such that $(dd^c\rho)^m \wedge \beta^{n-m} \geq c\beta^n$. Hence the inequality (4.5.3) yields

$$(dd^c v)^m \wedge \beta^{n-m} \geq c(h \circ \rho)^m \beta^n \geq c.f\beta^n \text{ in } \Omega \setminus K. \tag{4.5.4}$$

Now consider a $\mathcal{C}^{1,1}$-extension $\tilde{\varphi}$ of φ to $\bar{\Omega}$ and choose $B \gg 1$ large enough so that $B\rho + \tilde{\varphi}$ is m-subharmonic in Ω and

$$\tilde{v} := B(v + \rho) + \tilde{\varphi} \leq \mathtt{U} \text{ in a neighborhood of } K.$$

Then $\tilde{v}$ is m-subharmonic in Ω and if $B \geq (1/c)^{1/m}$, then it follows from (4.5.4) that

$$(dd^c\tilde{v})^m \wedge \beta^{n-m} \geq f\beta^n \text{ in } \Omega \setminus K.$$

By the comparison principle, we have $\tilde{v} \leq \mathtt{U}$ on $\Omega \setminus K$. Consequently, $\tilde{v} \leq \mathtt{U}$ on $\bar{\Omega}$, $\tilde{v} = \varphi$ on $\partial\Omega$ and $\tilde{v} \in \mathcal{C}^{0,1/2}(\bar{\Omega})$.
As in the proof of Theorem 4.1.2 we have that the total mass of $\Delta\mathtt{U}$ is finite in Ω. Hence, Theorem 4.5.2 yields that the solution $\mathtt{U}$ belongs to $\mathcal{C}^{0,\alpha}(\bar{\Omega})$ for any $\alpha < \min\{1/2, 2\gamma_1\}$. □

As an example of application of the last result, fix $p > n/m$, take $h(t) := (-t)^{-\alpha}$ with $0 < \alpha < 1/(pm)$, $t < 0$ and define $f := (h \circ \rho)^m$.

4.5.3 Hölder continuity for radially symmetric solution

Here we consider the case when the right hand side and the boundary data are radial. In this case, Huang and Xu [HX10] gave an explicit formula for the radial solution of the Dirichlet problem (4.1.1) with $f \in \mathcal{C}(\bar{B})$ (see also [Mo86] for complex Monge-Ampère equations). Moreover, they studied higher regularity for radial solutions (see also [DD13]).

Here, we will extend this explicit formula to the case when $f \in L^p(\mathbb{B})$, for $p > n/m$, is a radial nonnegative function and $\varphi \equiv 0$ on $\partial\mathbb{B}$. Then, we prove Hölder continuity of the radially symmetric solution.

Proof of Theorem 4.1.4. Let $f_k \in \mathcal{C}(\bar{\mathbb{B}})$ be a positive radial symmetric function such that $\{f_k\}$ converges to f in $L^p(\mathbb{B})$. Then there exists, by [HX10], a unique solution $\mathtt{U}_k \in \mathcal{C}(\bar{\mathbb{B}})$ to (4.1.1) with zero boundary values and the density f_k, given by the following formula:

$$\mathtt{U}_k(r) = -B\int_r^1 \frac{1}{t^{2n/m-1}}\left(\int_0^t \rho^{2n-1} f_k(\rho)d\rho\right)^{1/m} dt.$$

It is clear that $\mathtt{U}_k$ converges in $L^1(\mathbb{B})$ to the function $\tilde{u}$ given by the same formula i.e.

$$\tilde{u}(r) = -B\int_r^1 \frac{1}{t^{2n/m-1}}\left(\int_0^t \rho^{2n-1} f(\rho)d\rho\right)^{1/m} dt.$$

We claim that the sequence $\{\mathtt{U}_k\}$ is uniformly bounded and equicontinuous in $\bar{\mathbb{B}}$. Indeed, let $0 < r < r_1 \leq 1$, we have

$$\begin{aligned}
|\mathtt{U}_k(r_1) - \mathtt{U}_k(r)| &= B\int_r^{r_1} \frac{1}{t^{2n/m-1}}\left(\int_0^t \rho^{2n-1} f_k(\rho)d\rho\right)^{1/m} dt \\
&\leq B\int_r^{r_1} \frac{1}{t^{2n/m-1}}\left(\int_0^t \rho^{(2n-1)/q}\rho^{(2n-1)/p} f_k(\rho)d\rho\right)^{1/m} dt \\
&\leq C\|f_k\|_{L^p(\mathbb{B})}^{1/m}\int_r^{r_1} \frac{1}{t^{2n/m-1}}\left(\int_0^t \rho^{2n-1}d\rho\right)^{1/mq} dt \\
&\leq C\|f_k\|_{L^p(\mathbb{B})}^{1/m}(r_1^{2-\frac{2n}{mp}} - r^{2-\frac{2n}{mp}}).
\end{aligned}$$

Since f_k converges to f in $L^p(\mathbb{B})$, we get $\|f_k\|_{L^p(\mathbb{B})} \leq C_1$, where $C_1 > 0$ does not depend on k. Hence $\mathtt{U}_k$ is equicontinuous on $\bar{\mathbb{B}}$. By Arzelà-Ascoli theorem, there exists a subsequence

$\mathtt{U}_{k_j}$ converges uniformly to $\tilde{u}$.
Consequently, $\tilde{u} \in SH_m(\mathbb{B}) \cap \mathcal{C}(\bar{\mathbb{B}})$ and thanks to the convergence theorem for the complex Hessian operator (see [SA12]) we can see that $(dd^c\tilde{u})^m \wedge \beta^{n-m} = f\beta^n$ in $\mathbb{B}$.
Moreover, we have

$$|\tilde{u}(r_1) - \tilde{u}(r)| \leq C\|f\|^{1/m}_{L^p(\mathbb{B})}(r_1^{2-\frac{2n}{mp}} - r^{2-\frac{2n}{mp}}).$$

Hence, for $p \geq 2n/m$ we get $\tilde{u} \in Lip(\bar{\mathbb{B}})$, and for $n/m < p < 2n/m$ we have $\tilde{u} \in \mathcal{C}^{0,2-\frac{2n}{mp}}(\bar{\mathbb{B}})$. □

We give an example which illustrates that the Hölder exponent $2 - \frac{2n}{mp}$ given by Theorem 4.1.4 is optimal.

Example 4.5.5. Let $p \geq 1$ be a fixed exponent. Take $f_\alpha(z) = |z|^{-\alpha}$, with $0 < \alpha < 2n/p$. Then it is clear that $f_\alpha \in L^p(\mathbb{B})$. The unique radial solution to the Dirichlet problem (4.1.1) with right hand side f_α and zero boundary values is given by

$$\mathtt{U}_\alpha(z) = c(r^{2-\alpha/m} - 1); \;\; r := |z| \leq 1,$$

where $c = \left(\frac{C_n^m}{2^{m+1}n}\right)^{-1/m} (\frac{1}{2n-\alpha})^{1/m} \frac{m}{2m-\alpha}$. Then we have

1. If $p > n/m$ then $0 < \alpha < 2m$ and the solution $\mathtt{U}_\alpha$ is $(2 - \frac{2n}{mp} + \delta)$−Hölder with $\delta = (2n/p - \alpha)/m$. Since α can be chosen arbitrary close to $2n/p$, this implies that the optimal Hölder exponent is $2 - \frac{2n}{mp}$.

2. Observe that when $1 \leq p < n/m$ and $2m < \alpha < 2n$, then the solution $\mathtt{U}_\alpha$ is unbounded.

The next example shows that in Theorem 4.1.4, n/m is the critical exponent in order to have a continuous solution.

Example 4.5.6. Consider the density f given by the formula

$$f(z) := \frac{1}{|z|^{2m}(1 - log|z|)^\gamma},$$

where $\gamma > m/n$ is fixed.

It is clear that $f \in L^{n/m}(\mathbb{B}) \setminus L^{n/m+\delta}(\mathbb{B})$ for any $\delta > 0$. An elementary computation shows that the corresponding solution $\mathtt{U}$ given by the explicit formula (4.1.3) can be estimated by

$$\mathtt{U}(z) \leq C(1 - (1 - log|z|)^{1-\gamma/m}),$$

where $C > 0$ depends only on n, m and γ. Hence we see that if $m/n < \gamma < m$ then $\mathtt{U}$ goes to $-\infty$ when z goes to 0. In this case the solution $\mathtt{U}$ is unbounded.

4.6 Open questions

- Let Ω be a smooth bounded strongly m-pseudoconvex domain in $\mathbb{C}^n$, $\varphi \in \mathcal{C}(\partial\Omega)$. Let also μ be a Hausdorff-Riesz measure on Ω and $0 \leq f \in L^p(\Omega, \mu)$ for some $p > n/m$. Does there exist a continuous solution to (4.1.1)?
  Moreover, if φ is Hölder continuous, can we say that $\mathtt{U}$ is Hölder continuous in $\bar{\Omega}$?

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Résumé

Cette thèse est consacrée à l'étude de la régularité des solutions des équations de Monge-Ampère complexes ainsi que des équations hessiennes complexes dans un domaine borné de $\mathbb{C}^n$.
Dans le premier chapitre, on donne des rappels sur la théorie du pluripotentiel.
Dans le deuxième chapitre, on étudie le module de continuité des solutions du problème de Dirichlet pour les équations de Monge-Ampère lorsque le second membre est une mesure à densité continue par rapport à la mesure de Lebesgue dans un domaine strictement hyperconvexe lipschitzien.
Dans le troisième chapitre, on prouve la continuité hölderienne des solutions de ce problème pour certaines mesures générales.
Dans le quatrième chapitre, on considère le problème de Dirichlet pour les équations hessiennes complexes plus générales où le second membre dépend de la fonction inconnue. On donne une estimation précise du module de continuité de la solution lorsque la densité est continue. De plus, si la densité est dans L^p, on démontre que la solution est Hölder-continue jusqu'au bord.

Mots-clés

Problème de Dirichlet, Opérateur de Monge-Ampère, Mesure de Hausdorff-Riesz, Fonction m-sousharmonique, Opérateur hessien, Capacité, Module de continuité, Principe de comparaison, Théorème de stabilité, Domaine strictement hyperconvexe lipschitzien, Domaine strictement m-pseudoconvexe.