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Contributions à l'étude de l'instant de défaut d'un processus de Lévy en observation complète et incomplète

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Résumé

Dans nos travaux, nous avons considéré un processus de Lévy X avec une composante brownienne non nulle et dont la partie à sauts est un processus de Poisson composé. Nous avons supposé que la valeur d'une entreprise est modélisée par un processus stochastique de la forme $V = V_0 \exp -X$ et que cette entreprise est mise à défaut dès lors que sa valeur passe sous un certain seuil b déterminé de façon exogène et qui donc, est une donnée du problème. L'instant de défaut τ est alors de la forme τ_x pour $x = \ln(V_0) - \ln((b)$ où $x > 0$, $\tau_x = \inf\{t \geq 0 : X_t \geq x\}$.
Dans un premier temps, nous supposons que des agents observant la valeur V des actifs de la firme souhaitent connaître le comportement de l'instant de défaut. Dans ce modèle, au chapitre 2, nous avons étudié d'une part la régularité de la densité de la loi de l'instant de défaut. D'autre part, nous avons étudié la loi conjointe de l'instant de défaut, de l'overshoot et de l'undershoot. Au chapitre 3, nous avons obtenu une équation à valeurs mesures dont le quadriplet formé par la variable aléatoire X_t, le supremum du processus X à l'instant t, le supremum du processus X au dernier instant de saut avant l'instant t et le dernier instant de saut à l'instant t est solution au sens faible, puis une équation dont ce quadriplet est une solution forte. Dans un second temps, au chapitre 4, nous avons supposé que des investisseurs souhaitant détenir une part de cette entreprise ne disposent pas de l'information complète. Ils n'observent pas la valeur des actifs de la firme V, mais sa valeur bruitée. Leur information est modélisée par la filtration $\mathcal{G} = (\mathcal{G}_t, t \geq 0)$ engendrée par cette observation. Dans ce modèle, nous avons montré que la loi conditionnelle de l'instant de défaut sachant la tribu $\mathcal{G}_t$ admet une densité par rapport à la mesure de Lebesgue et obtenu une équation de Volterra dont cette densité est solution. Cette connaissance permet aux investisseurs de prévoir au vu de leur information, quand est-ce que l'instant de défaut va intervenir après l'instant t. Nous avons complété ce travail par des simulations numériques.

Mots clés : Processus de Lévy, Instant de défaut, Equations aux dérivées partielles, Théorie du filtrage, Observation complète, Observation incomplète.

Abstract

In this Ph.D thesis, we consider a jump-diffusion process which the diffusion part is a drifted Brownian motion and the jump part is a compound Poisson process. We assume that a firm value is modelling by a stochastic process $V = V_0 \exp -X$. This firm goes to default whenever its value is below a specified threshold b which is exogenously determined. For $x = \ln(V_0) - \ln(b) > 0$, the default time is of the form $\tau_x = \inf\{t \geq 0 : X_t \geq x\}$.
First, we suppose that agents observe perfectly the firm value. In this model, we showed in chapter 2 that the density of the default time is continuous, then study the joint law of the default time, overshoot an undershoot. We obtained in chapter 3 a valued measure differential equation which the solution is the quadruplet formed by the random variable X_t, the running supremum X_t^* of X at time t, the supremum of X at the last jump time before t and the last jump time before t.
Secondly, we assume that investors wishing detain a part of the firm can not observe the firm value. They observe a noisy value of the firm and their information is modelling by the filtration $\mathcal{G} = (\mathcal{G}_t, t \geq 0)$ generated by their observation. In this model, we have shown that the conditional density of τ_x with respect to $\mathcal{G}$ has a density which is solution of one stochastic integral-differential equation The knowledge of this density allows investors to predict the default time after time t. This second part is the chapter 4.

Keywords : Lévy processes, default time, Partial differential equation, Filtering theory, Complete observation, incomplete observation.

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Chapitre 1

Introduction générale

Si on s'intéresse à la réalisation d'un certain phénomène : par exemple un tremblement de terre avec une intensité supérieure à certain niveau ou un nombre de clients dépassant la sécurité exigée par un établissement ou encore un scénario défavorable conduisant à ne pas respecter ses engagements...etc, on est obligé d'accorder une attention particulière à l'instant aléatoire τ où le phénomène se manifeste pour la première fois. Sur un marché financier, un tel phénomène est la faillite d'entreprise et des agents disposant d'informations différentes essaient de prévoir l'instant où il se produit, appelé instant de défaut. Dans ce chapitre introductif, nous faisons une description générale de l'instant de défaut, présentons l'état de l'art, ensuite les contributions de cette thèse.

1.1 Présentation générale

L'instant de défaut est le thème principal de cette thèse. Il est alors tout à fait naturel d'en faire une description générale. Il s'agit ici d'un moment où survient un scénario défavorable conduisant une entreprise à l'impossibilité de faire face à ses engagements. Les modèles utilisés dans ce type d'étude se classifient selon deux approches.
L'approche réduite : le mécanisme liant le défaut à la valeur de la firme n'est pas explicite. On modélise directement le processus de défaut en se basant sur certains facteurs économiques ; Dans cette approche, qui n'est pas l'objet de notre étude, le temps de défaut est défini de manière exogène.
L'approche structurelle : on modélise le temps de défaut pour une firme donnée par une fonction de sa valeur. Par exemple, le défaut survient si la valeur de la firme atteint un certain seuil. L' approche structurelle dont le fondateur est Merton en 1974, [Mer74] a l'avantage de relier le défaut aux fondements économiques qui régissent la firme. Le modèle de Merton est le pionnier des modèles de l'approche structurelle, point de départ de nombreux développements et il est utilisé en pratique dans les sociétés d'assurance. Cependant, il manque de réalisme à plusieurs niveaux. L'une de ses faiblesses est que le

défaut ne peut survenir qu'à l'échéance de la dette. Les modèles de type barrière visent à corriger ce problème. L'idée est de modéliser le temps de défaut comme l'instant du premier passage du processus valeur de la firme sous une certaine barrière. On note dans ce type de modèle une certaine flexibilité : pour un même processus modélisant la valeur de la firme, le fait de varier le choix de la barrière conduit à différents modèles. Les éléments décrivant ce modèle étant aléatoires, une étude mathématique s'impose. Précisément, il est très utile d'étudier la distribution du premier temps de passage d'un processus sous un seuil. Malheureusement, mis à part le cas du mouvement brownien, qui peut-être est une exception, il est très difficile, voir presque impossible d'obtenir des résultats analytiques.

Du côté des mathématiques financières, ce type d'étude a été initié par Pye [Pye74] et Litterman-Iben [LI91], ensuite formalisé indépendamment par Jarrow et Turnbull [JT95], Lando [Lan94], Madan et Unal [MU98]. Puis elle a été développée par Hull et White [HW95] et Lotz [[Lot99], [L$^+$98]].

Les outils mathématiques qui sont à la base de telles études ont été introduits par Dellachérie [Del70], Chou et Meyer [CM75], Dellacherie et Meyer [DM80], Jeulin et Yor [JY78] et plus tard développés par Jeanblanc et Rutskowski [JR00], Bielecki et Rutskowski [BR02].

Dans la littérature, l'etude du temps d'atteinte des processus a passionné beaucoup d'auteurs. Dans le cas d'une diffusion, on peut citer, entre autres, Borodin et Salminen [BS02], Delong [DL81], Kent [Ken78] , Pitman et Yor [PY81] ou encore Jacod et Shiryaev [JS13].

Les dirigeants de l'entreprise, les actionnaires ou les investisseurs essaient de "prévoir" la faillite de la firme en fonction de l'information dont ils disposent. Ces agents peuvent observer de façon parfaite ou imparfaite, de manière continue ou à des instants discrets la valeur des actifs de la firme. Selon le cas, on parle d'observation complète à temps continu ou à des instants discrets, ou bien d'observation incomplète à temps continu ou à des instants discrets.

Dans ce manuscrit, nous avons considéré le cas d'information complète et le cas d'information incomplète à temps continu, le cas discret restant un problème ouvert.

1.1.1 Définitions et notations

Dans ce chapitre, on se place sur un espace de probabilité filtré $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), \mathbb{P})$ satisfaisant les conditions habituelles. Nous donnons la définition d'un processus de Lévy conformémemnt à [CT04].

Définition 1.1.1 *Un processus stochastique X continu à gauche, limité à droite (cadlag) à valeurs dans $\mathbb{R}^d$ tel que $X_0 = 0$ est un processus de Lévy s'il vérifie les propriétés suivantes :*

1. *L'indépendance des accroissements : pour toute suite croissante de temps* $(t_0, \cdots, t_n)$, *les variables aléatoires* $X_{t_0}, X_{t_1} - X_{t_0}, \cdots, X_{t_n} - X_{t_{n-1}}$ *sont indépendantes.*

2. *La stationnarité des accroissements : pour tout* t *et* h *positifs, la loi de* $X_{t+h} - X_t$ *ne dépend pas de* t.

3. *La continuité stochastique : pour tout* $\varepsilon > 0$, $\lim_{h \to 0} \mathbb{P}(|X_{t+h} - X_t| > \varepsilon) = 0$.

D'après le théorème de "Représentation de Lévy-Khinchine" (cf. Theorem 3.1, [CT04]), la fonction caractéristique de la loi de X_t est donnée par

$$\mathbb{E}(e^{izX_t}) = e^{t\Psi(z)}, \quad z \in \mathbb{R}.$$

Dans le cas d'un processus de Lévy à valeurs réelles, la fonction Ψ (caractéristique exponentielle de Lévy) est de la forme :

$$\Psi(z) = -\frac{1}{2}\sigma z^2 + i\ mz + \int_{-\infty}^{+\infty} (e^{izx} - 1 - izx\mathbf{1}_{\{|x| \le 1\}})\nu(dx).$$

La mesure ν est dite mesure de Lévy du processus X. Nous supposons dans tout ce document que $\lambda = \nu(\mathbb{R}) < +\infty$ et dans ce cas, le processus admet une décomposition de la forme :

$$X_t = mt + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \quad \forall \ \ t \ge 0. \tag{1.1.2}$$

Le processus $(W_t, t \ge 0)$ est un mouvement brownien standard, $m \in \mathbb{R}$, $\sigma > 0$, N_t est un processus de Poisson d'intensité constante positive λ et $(Y_i, i \in \mathbb{N}^*)$ une suite de variables aléatoires indépendantes et identiquement distribuées dont la loi $\nu(\mathbb{R})^{-1}\nu$ admet la fonction de répartition F_Y. Sans perte de généralité, nous supposons que $\sigma = 1$ et que $(W_t, t \ge 0)$, $(N_t, t \ge 0)$, et $(Y_i, i \in \mathbb{N}^*)$ sont indépendants. Sauf mention contraire, X modélise la valeur des actifs d'une firme et l'instant de défaut est modélisé par le temps d'arrêt

$$\tau_x = \inf\{t \ge 0 : X_t \ge x\} \tag{1.1.3}$$

où $x > 0$ est la barrière. Le mouvement brownien avec dérive $(mt + W_t, \ \ t \ge 0)$ sera noté $\tilde{X}_t$ et l'instant de défaut associé $\tilde{\tau_x}$.

1.1.2 Observation Complète

Il s'agit ici des modèles purement structurels correspondant au cas où l'information est complète. Les agents qui veulent "prévoir" la faillite observent parfaitement la valeur X de la firme. l'information de l'agent au temps t est modélisée par la tribu $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t\}$. Dans ce cas, les agents peuvent directement savoir si le défaut a eu lieu ou non à partir de leur observation sur le marché. En d'autres termes l'instant de défaut est un $(\mathcal{F}_t)-$ temps d'arrêt et on obtient

$$\mathbb{P}(\tau > t|\mathcal{F}_t) = \mathbf{1}_{\{\tau>t\}}, \quad \forall \quad t > 0.$$

Pour un agent qui observe parfaitement le processus valeur X de la firme étudier le comportement de l'instant de défaut revient à étudier le comportement du supremum X^* de X et de sa valeur terminale grâce à la relation :

$$\mathbb{P}(X_t \geq a, X_t^* \geq b) = \mathbb{P}(X_t \geq a, \tau_b \leq t)$$

pour tous les nombres réels a et b tels $b \geq a$ et $b > 0$. On trouve beaucoup de résultats dans la littérature suivant le processus utilisé.

Le cas du mouvement brownien

Si W est le mouvement brownien standard et $\tau_x = \inf\{t \geq 0 : W_t = x\}$ pour $x > 0$ fixé, on trouve dans [KS12] la densité de τ_x définie pour tout $t > 0$, par

$$f_0(t,x) = \frac{|x|}{\sqrt{2\pi t^3}} e^{-\frac{x^2}{2t}}.$$

Si W^* est son supremum, la densité de la loi du couple (W^*, W) est donnée par

Proposition 1.1.4 *Pour tout* $t > 0$, *la densité de la loi du couple* (W_t^*, W_t) *est*

$$p^0(b,a,t) = \frac{2(2b-a)}{\sqrt{2\pi t^3}} \exp -\frac{(2b-a)^2}{2t} \mathbf{1}_{b>a\vee 0}.$$

En appliquant la formule de Girsanov, on obtient les résultats analogues suivants pour le mouvement brownien avec dérive $\tilde{X}$.

Proposition 1.1.5 *Pour tout* $t > 0$, *la loi du couple* $(\tilde{X}_t^*, \tilde{X}_t)$ *admet une densité par rapport à la densité de Lebesgue donnée par*

$$\tilde{p}(b,a,t) = \frac{2(2b-a)}{\sqrt{2\pi t^3}} \exp(-\frac{(2b-a)^2}{2t} + ma - \frac{m^2}{2}t)\mathbf{1}_{b>a\vee 0}. \tag{1.1.6}$$

De plus, la loi de $\tilde{\tau}_x$ *admet la densité*

$$\tilde{f}(u,x)du + \mathbb{P}(\tilde{\tau}_x = \infty)\delta_\infty(du)$$

où

$$\tilde{f}(u,x) = \frac{|\ x\ |}{\sqrt{2\pi u^3}} \exp[-\frac{1}{2u}(x-mu)^2] 1_{]0,+\infty[}(u) \ \textit{et}\ \mathbb{P}(\tilde{\tau}_x = \infty) = 1 - e^{mx-|mx|}. \tag{1.1.7}$$

Lorsque la barrière x est une fonction continue du temps, un autre résultat a été récemment obtenu par Herrmann et Tanré dans [HT16] sous les hypothèses :

— $x(0) > 0$ et $\overline{\lim}_{t\to+\infty} \frac{x(t)}{\sqrt{2t\log\log t}} < 1$

— x est non décroissante et de classe $\mathcal{C}^1$

— $2x'(t)\sqrt{1+T} < 1, \quad \forall t \geq 0.$

Ils donnent un nouvel algorithme permettant d'approcher τ_x.

Le cas du processus d'Ornstein-Uhlenbeck

Dans [Pat04], Patie P. a donné différentes représentations et illustrations numériques de la densité de la loi du premier temps de passage d'un processus d'Ornstein-Uhlenbeck au niveau d'une barrière $x \in \mathbb{R}_+^*$. Soit W un mouvement brownien standard. Nous rappelons que le processus d'Ornstein-Uhlenbeck est l'unique solution de l'équation

$$dX_t = dW_t - \beta X_t dt, \quad X_0, \quad \beta \quad \in \mathbb{R}.$$

Si $p^{(\beta)}_{X_0\to x}(.)$ est la densité de la loi de $\tau_x^{o.u} = \inf\{t \geq 0; X_t = x\}$, alors, on a le théorème suivant qui donne sa représentation intégrale.

Théorème 1.1.8 *Soient $X_0 < x$ fixés, alors la densité de $\tau_x^{o.u}$ est données par*

$$p^{(\beta)}_{X_0\to x}(t) = \int_0^{+\infty} \cos(\alpha\beta t)\hat{H}_{-\alpha}(-(x-X_0)\sqrt{\beta})d\alpha$$

où

$$\hat{H}_\alpha(x, X_0) = \frac{H_{r\alpha}(x)H_{r\alpha}(X_0) + H_{i\alpha}(X_0)H_{i\alpha}(x)}{H^2_{r\alpha}(\alpha) + H^2_{i\alpha}(x)}$$

et

$$H_{r\alpha}(z) = \int_0^{+\infty} e^{-u^2} \cos\left(\frac{\alpha}{2}\log(1+(\frac{z}{u})^2)\right) du$$
$$H_{i\alpha}(z) = \int_0^{+\infty} e^{-u^2} \sin\left(\frac{\alpha}{2}\log(1+(\frac{z}{u})^2)\right) du.$$

Le cas des processus de Bessel

Le processus de Bessel de dimension δ issu de x_0 est la solution de

$$X_t = x_0 + \frac{\delta-1}{2}\int_0^t \frac{1}{X_s}ds + B_t, \quad t \geq 0.$$

Si la dimension δ est entière la transformée de Laplace est donnée par

$$\mathbb{E}\left(\exp -\lambda\tau_x\right) = \frac{x_0^{-\nu}}{x^{-\nu}}\frac{I_\nu(x_0\sqrt{2\lambda})}{I_\nu(x\sqrt{2\lambda})}$$

$\lambda > 0$, I_ν est la fonction de Bessel modifié et $\nu = \frac{\delta-1}{2}$. Dans le cas des dimensions non entières, Hamana et Matsumoto [HM13] donnent l'expression de la fonction de répartition de τ_x en fonction des zéros des fonctions de Bessel.
Pour compléter ce paragraphe, Deaconu et Herrmann [DH$^+$13] et [DH14] proposent des algorithmes de simulation du temps d'atteinte.

Le cas des martingales

Rogers caractérise dans [Rog93] toutes les lois jointes d'une martingale (martingale locale continue issue de zéro) et de son supremum. Il introduit la notation $\bar{M}_\infty = \sup_{t\geq 0} M_t$ pour une martingale M et définit pour toute mesure de probabilité ν sur $\mathbb{R}_+^2$, la fonction

$$c(s) := \begin{cases} \int_{[s,\infty[\times\mathbb{R}_+} \frac{(x-y)\nu(dx,dy)}{\nu([s,\infty[\times\mathbb{R}_+)} & \text{si } \nu([s,\infty[\times\mathbb{R}_+) > 0 \\ s & \text{sinon} \end{cases}$$

Il obtient dans le cas des martingales uniformément intégrables le résultat suivant :

Théorème 1.1.9 *Pour qu'une mesure de probabilité ν définie sur $\mathbb{R}_+ \times \mathbb{R}_+$ soit la loi du couple $(\bar{M}_\infty, \bar{M}_\infty - M_\infty)$ pour toute martingale M uniformément intégrable il faut et il suffit que*

$$\int\int |x-y|\nu(dx,dy) < \infty,$$
$$c(.) \textit{ soit croissante },$$
$$c(s) \geq s \textit{ pour tout } s.$$

Nous rappelons l'ordre entre deux mesures sur $\mathbb{R}$: Deux mesure μ et μ vérifient $\nu \leq \mu$ si et seulement si pour tout $t > 0$, $\nu([t,+\infty[) \leq \mu([t,+\infty[)$. Dans le cas des martingales continues, il utilise cet ordre et obtient

Théorème 1.1.10 *La mesure de probabilité ν sur $\mathbb{R}_+ \times \mathbb{R}_+$ est la loi jointe du couple $(\bar{M}_\infty, \bar{M}_\infty - M_\infty)$ pour toute martingale locale continue M presque sûrement convergente telle que $M_0 = 0$ si et seulement si*

$$\left(\int\int_{]t,\infty[\times\mathbb{R}_+} \nu(ds,dy)\right) dt \geq \int_{]0,\infty[} y\nu(dt,dy).$$

Si de plus M est uniformément intégrable, on obtient une égalité.

Il énonce et démontre à nouveau le résultat suivant dû à Vallois dans [Val94].

Théorème 1.1.11 *Soit F une mesure de probabilité sur $\mathbb{R}_+$ telle que $F(dt) = \rho(t)dt + \alpha(dt)$ où α est singulière par rapport à la mesure de Lebesgue. Alors F est la loi du maximum d'une martingale continue uniformément intégrable si et seulement si*

$$\rho(t) > 0 \text{ pour tout } t < a := \sup\{u : F(u) < 1\},$$
$$\lim_{t \to +\infty} t\bar{F}(t) = 0 \text{ où } \bar{F}(t) = 1 - F(t),$$
$$\int_0^{+\infty} t\alpha(dt) + \int_0^{+\infty} |t\rho(t) - \bar{F}(t)|dt < +\infty.$$

Définition 1.1.12 *Soit X une sous martingale positive de décomposition*

$$X_t = N_t + A_t \quad \forall \;\; t \geq 0. \tag{1.1.13}$$

On dit que X est de classe (Σ) si

1. *N est une martingale locale issue de* 0.
2. *A est un processus continu croissant issu de* 0.
3. *La mesure dA est prise sur l'ensemble $\{t : X_t = 0\}$.*

Si de plus X vérifie $\{X_\tau, \;\; \tau < \infty, \;\; \tau$ un temps d'arrêt $\}$ est uniformément intégrable, on dit que X est de classe (ΣD).

Le résultat suivant est une caractérisation "martingale" des processus de classe (Σ).

Théorème 1.1.14 *Il y a équivalence entre*

1. *La sous martingale X est de classe* (Σ).
2. *Il existe un processus (C_t) croissant, adapté et continu tel pour toute fonction f borélienne bornée et pour F définie par $F(x) := \int_0^x f(z)dz$, le processus*

   $$F(C_t) - f(C_t)X_t$$

   est une martingale locale. De plus, $C_t = A_t$.

Pour une sous martingale admettant la décomposition (1.1.13), nous caractérisons le premier temps de passage à un niveau u du processus A.

Théorème 1.1.15 *Soit X une sous martingale locale de classe (Σ), de décomposition (1.1.13) et à sauts uniquement négatifs telle que $A_\infty = \infty$. Considérons le premier temps de passage de X à un niveau u*

$$\tau_u := \inf\{t : X_t > u\}$$

Soit $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ une fonction borélienne. Alors

$$\mathbb{P}\left(\exists\ \ t \geq 0, X_t > \varphi(A_t)\right) = 1 - \exp\left(-\int_0^\infty \frac{dx}{\varphi(x)}\right)$$

et

$$\mathbb{P}\left(\exists\ \ t < \tau_u, X_t > \varphi(A_t)\right) = 1 - \exp\left(-\int_0^u \frac{dx}{\varphi(x)}\right).$$

Les deux théorèmes précédents permettent de résoudre le problème d'arrêt de Skorokhod pour une mesure de probabilité sur $\mathbb{R}_+$ non atomique. La littérature étant vaste sur ce sujet, nous donnons ici une méthode générale permettant de traiter une grande variété de processus stochastiques. Plus précisément, nous considérons d'abord une mesure de probabilité sans atome ϑ sur $\mathbb{R}_+$ et une sous martingale locale X de classe (Σ) n'ayant que des sauts négatifs et de décomposition (1.1.13) tel que $\lim_{t\to+\infty} A_t = +\infty$. Le résultat suivant permet de trouver un temps d'arrêt T_ϑ tel que X_{T_ϑ} suit la loi de ϑ. Pour cela, on introduit la queue de distribution $\bar{\vartheta}$ de ϑ : $\bar{\vartheta}(x) := \vartheta([x, +\infty[)$ et

$$a_\vartheta = \sup\{x \geq 0 : \bar{\vartheta}(x) = 1\}$$
$$b_\vartheta = \inf\{x \geq : \bar{\vartheta}(x) = 0\}, \quad -\infty \leq a_\vartheta \leq b_\vartheta \leq +\infty$$

respectivement le supremum et l'infimum du support de ϑ. Nous introduisons ensuite la fonction duale de Hardy-Littlewood $\Psi_\vartheta : \mathbb{R}_+ \mapsto \mathbb{R}_+$ en posant

$$\Psi_\vartheta(x) = \int_0^x \frac{z}{\bar{\vartheta}(z)} d\vartheta(z), \text{ si } a_\vartheta \leq x < b_\vartheta,$$
$$\Psi_\vartheta(x) = 0 \text{ si } 0 \leq x < a_\vartheta,$$
$$\Psi_\vartheta(x) = \infty \text{ si } x \leq b_\vartheta.$$

La fonction Ψ_ϑ étant continue et strictement croissante, nous définissons son inverse (continu à droite) par

$$\varphi_\vartheta(z) = \inf\{x \geq 0 : \Psi_\vartheta(x) > z\}.$$

Cette fonction aussi est strictement croissante.

Théorème 1.1.16 *Soit X une sous martingale locale de classe (Σ) et de décomposition (1.1.13) avec uniquement des sauts négatifs telle que $A_\infty = \infty$. Le temps d'arrêt*

$$T_v = \inf\{x \geq 0 : X_t \geq \varphi_\vartheta(A_t)\}$$

est presque sûrement fini et il est solution du problème de Skorokhod pour le processus X : X_{T_ϑ} suit la loi de ϑ.

On peut voir en détail les théorèmes 1.1.14, 1.1.15 et 1.1.16 dans [Nik06].

Le cas d'un processus de diffusion

Dans le cas d'une diffusion, Jeanblanc et al. [JYC09] ont donné dans la sous-section 5.4.3 une méthode d'étude basée sur l'équation de Fokker-Planck par la proposition suivante.

Proposition 1.1.17 *Soit*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t.$$

Supposons h une fonction déterministe telle que $X_0 > h(0)$ et définissons

$$\tau := \inf\{t \geq 0 : X_t \leq h(t)\} \text{ et } g(t,x)dx := \mathbb{P}(X_t \in dx \cap \tau > t).$$

La fonction $g(t,x)$ satisfait à l'équation de Fokker-Planck

$$\frac{\partial g(t,x)}{\partial t} = -\frac{\partial (b(t,x)g(t,x))}{\partial x} + \frac{1}{2}\frac{\partial^2(\sigma^2(t,x)g(t,x))}{\partial x}, \quad x > h(t)$$

avec les conditions aux limites

$$\lim_{t\to 0} g(t,x)dx = \delta(x - X_0)$$
$$g(t,x)_{|x=h(t)} = 0.$$

Le cas du processus de Lévy

Pour un processus diffusion-saut dont les sauts suivent une loi double exponentielle, Kou et Wang ont obtenu dans [KW03] des expressions explicites pour la transformée de Laplace de τ_x et celle du couple (X^*, X).
Lorsque X est un processus de Lévy quelconque de mesure de Lévy ν, Volpi et al. considèrent dans [RVV08] l'instant de défaut τ_x, le déficit aussitôt après la ruine $K_x := X_{\tau_x} - x$, appelé 'overshoot'ou 'sévérité de la ruine' et la fortune immédiatement avant la ruine $L_x := x - X_{\tau_x^-}$. Dans un second temps, les auteurs montrent que la transformée de Laplace du vecteur (τ_x, K_x, L_x) est solution d'une certaine équation intégro-différentielle. Dans un second temps, supposant que ν admet des moments finis, ils montrent que le triplet $(\bar{\tau}_x, K_x, L_x)$, où $\bar{\tau}_x$ est une normalisation convenable de τ_x, converge en distribution lorsque x tend vers $+\infty$.
Doney et Kyprianou ont donné dans [DK06] la loi du vecteur

$$(G^*_{\tau_x-}, \tau_x - G^*_{\tau_x-}, X_{\tau_x} - x, x - X_{\tau_x-}, x - X^*_{\tau_x-}) \text{ où } G^*_t = \sup\{s < t : X^*_s = X_s\}$$

en utilisant la fonction d'échelle associée à X.
Cependant, dans la pratique, il est souvent difficile pour les investisseurs d'observer la valeur des actifs de la firme. En d'autres termes, l'information disponible de leur

point de vue est véritablement incomplète. Ainsi notre objectif est de tenir compte de cet état de fait en trouvant un moyen de modéliser l'information disponible pour les investisseurs. Ceci permettra à ces derniers, au vu de l'information dont ils disposent, de décrire le comportement de l'instant de défaut. C'est l'objet du paragraphe suivant.

1.1.3 Observation Incomplète

Les modèles utilisés pour ce type d'étude sont des modèles hybrides (mélange de modèles structurel et réduit). Nous définissons explicitement le mécanisme qui provoque le défaut (structurel) mais nous supposons que les agents n'ont pas une connaissance parfaite de ce mécanisme. Plusieurs modèles sont candidats pour ce type d'étude. Pour ce paragraphe, nous notons $\mathcal{I}$ l'information sur la firme, $\mathcal{D}$ l'information sur le temps de défaut et $\mathcal{G} = \mathcal{I} \vee \mathcal{D}$ l'information disponible pour les agents. Duffie et Lando [DL01] étudient le cas où les investisseurs reçoivent à certaines dates prédéterminées des rapports imparfaits sur la valeur des actifs de la firme. Dans leur modèle, X est un mouvement brownien géométrique :

$$X_t = e^{Z_t} \text{ où } Z_t = Z_0 + mt + \sigma W_t, \quad \forall \quad t \geq 0$$

avec W un mouvement brownien standard, $\sigma > 0$ un paramètre de volatilité et $m \in \mathbb{R}$ un paramètre tel que $\frac{1}{t}\log(\frac{X_t}{X_0}) = m + \frac{\sigma^2}{2}$. Ils notent Q le processus modélisant l'information des investisseurs, ils supposent que $Q_t = Z_t + B_t$ où B est un mouvement brownien indépendant de Z. Ils considèrent un nombre arbitraire de dates (t_i) vérifiant $t_1 < t_2 <, \cdots, < t_n \leq t$. Ils définissent la filtration $(\mathcal{I}_t, t \geq 0)$ comme suit :

$$\mathcal{I}_t = \sigma(Q_{t_1}, Q_{t_2}, \cdots, Q_{t_n}),$$

l'information globale dont disposent les investisseurs à l'instant t étant

$$\mathcal{G}_t = \mathcal{I}_t \vee \mathcal{D}_t.$$

où $\mathcal{D}_t = \sigma(\mathbf{1}_{\tau \leq s, s \leq t})$. Leur objectif étant d'étudier la loi conditionnelle de X sachant $\mathcal{G}$, ils commencent par le cas le plus simple, c'est-à-dire $t = t_1 > 0$. Ceci leur permet de déterminer explicitement les probabilités conditionnelles de survie à la date du rapport et après cette date. Après avoir donné des exemples numériques et quelques applications en finance, ils discutent d'une technique de type itératif qui permet de traiter le cas où plus d'un rapport est dévoilé au cours du temps.
A la différence de ce qui précède, Robert A. Jarrow et al. [GJZ09] donnent une autre définition de la filtration modélisant l'information dont disposent les investisseurs.
Supposant que les agents observent la firme de manière continue, ils donnent la définition suivante :

Définition 1.1.18 *Soit $(\mathcal{H}_t, t \geq t)$ une filtration continue à droite et contenant les ensembles $\mathbb{P}$-négligeables. Soit $(\alpha_t)_{t\geq 0}$ un processus croissant, continu à droite tel que $\alpha_0 = 0$ et α_t est un $\mathcal{H}$-temps d'arrêt pour tout $t \geq 0$ et pour tout ω, $\alpha_t(\omega) \leq t$. La filtration $\mathcal{I}$ est définie par*

$$\mathcal{I}_t = \mathcal{H}_{\alpha_t} \text{ pour tout } t \geq 0.$$

Supposant que les agents observent la firme à des instants de temps discrets, ils donnent la définition suivante :

Définition 1.1.19 *Soit $(\mathcal{H}_t, t \geq t)$ une filtration continue à droite et contenant les ensembles $\mathbb{P}$-négligeables. Soit K suites strictement croissantes de $\mathcal{H}$-temps d'arrêt $(T_n^k)_{n\geq 0, 1\leq k\leq K}$. Soit $(\mathcal{A}_{i_1,\cdots,i_k})_{i_1,\cdots,i_k\in\mathbb{N}}$ une famille de sous-tribus de $\mathcal{H}_\infty = \vee_{t\geq 0}\mathcal{H}_t$ telle que*

$$\begin{aligned}
&1) \quad \mathcal{A}_{i_1,\cdots,i_k} \subset \mathcal{A}_{j_1,\cdots,j_k} \text{ si } i_1 < j_1 <, \cdots, i_k < j_k \\
&2) \quad \mathcal{A}_{i_1,\cdots,i_k} \subset \mathcal{H}_{T_1^1 \vee \cdot \vee T_1^k}, \text{ où } T_1^1 \vee \cdot \vee T_1^k = \max_{i=1,..,k} \{T_1^i\}, \\
&3) \quad \text{tout } T_n^k \text{ est } \mathcal{A}_{i_1,\cdots,i_k} \text{ mesurable dès que } n \leq i_k.
\end{aligned}$$

La filtration modélisant l'information est définie par

$$\mathcal{I}_t^0 = \cup_{i_1,\cdots,i_k} \left(\mathcal{A}_{i_1,\cdots,i_k} \cap \{T_{i_k}^k \leq t \leq T_{i_{k+1}}^k, 1 \leq k \leq K\} \right)$$

et

$$\mathcal{I}_t = \cup_{i_1,\cdots,i_k} \left((\mathcal{A}_{i_1,\cdots,i_k} \vee \sigma(\mathcal{N})) \cap \{T_{i_k}^k \leq t \leq T_{i_{k+1}}^k, 1 \leq k \leq K\} \right)$$

où $\mathcal{N}$ est l'ensemble des $\mathbb{P}$-négligeables.

Les auteurs appellent ces filtrations “filtrations retardées” et illustrent à travers un exemple l'importance de ces définitions. Ils utilisent les mêmes techniques de calcul que dans Duffie et Lando [DL01] pour calculer le processus intensité dont la définition est la suivante :

Définition 1.1.20 *Le processus intensité $(\lambda_t, t \geq 0)$ de l'instant de défaut τ, associé à une certaine filtration $(\mathcal{G}_t, t \geq 0)$, lorsqu'il existe, est une fonction mesurable, positive, non identiquement nulle telle que le processus*

$$t \longrightarrow D_t = \mathbf{1}_{\{\tau\leq t\}} - \int_0^t bf1_{\{\tau>u\}}\lambda_u du$$

est une $(\mathcal{G}_t, t \geq 0)-$ martingale.

Dans la deuxième partie de sa thèse [Dor07], D. Dorobantu a modélisé la valeur de la firme par un processus stochastique diffusion-saut

$$V_t = xe^{Z_t} \text{ où } Z_t = mt + \sigma W_t + \sum_{i=1}^{N_t} Y_i$$

et l'instant de défaut par un temps d'arrêt

$$\tau = \inf\{t \geq 0 : V_t \leq b\} \text{ avec b une constante.}$$

Au temps d'arrêt τ, elle associe le processus croissant $H^\tau : t \longrightarrow \mathbf{1}_{\{\tau \leq t\}}$ et la filtration càd $\mathcal{F}^{H^\tau}$ engendrée par ce processus . Elle montre que

Proposition 1.1.21 *Soit S un temps d'arrêt tel que $\mathcal{P}(\tau = 0) = 0$ et pour tout $t > 0$, $\mathcal{P}(\tau > t) > 0$. Le temps d'arrêt τ admet une $\mathcal{F}^{H^\tau}-$ intensité si et seulement si la loi de τ admet une densité par rapport à la mesure de Lebesgue, notée f. Dans ce cas, la $\mathcal{F}^{H^\tau}-$ intensité est unique et elle est égale à*

$$\lambda(t) = \frac{f(t)}{1 - F_\tau(t)}, \quad t \geq 0 \text{ avec } F_\tau(t) = \mathbb{P}(\tau \leq t).$$

Elle introduit une décomposition de $\tau : \tau = \tau_1 \vee \tau_2$, où τ_1 est un $\mathcal{F}^V-$ temps d'arrêt prévisible et τ_2 un $\mathcal{F}^V-$ temps d'arrêt totalement inaccessible. En remarquant que τ_2 coïncide avec un instant de saut T_n du processus de Poisson où $n \in \mathbb{N}^*$ est tel que $\tau = T_n$ et $Y_n \neq 0$, elle prouve :

Proposition 1.1.22 *La $\mathcal{F}^V-$ projection duale prévisible du processus $(H_t^\tau = \mathbf{1}_{\{\tau \leq t\}}, t \geq 0)$ est égale à*

$$\mathbf{1}_{\{\tau_1 \leq t\}} + \int_0^t \mathbf{1}_{\{\tau_2 > u\}} \lambda F_Y(\frac{\ln b}{V_u}) du, \quad t \geq 0$$

où λ est le paramètre du processus de Poisson et F_Y est la fonction de répartition de la loi de Y.
De plus, τ n'admet pas de $\mathcal{F}^V-$ intensité.

L'information apporté par la connaissance de l'intensité $\lambda(t)$ est faible. En effet, pour un intervalle de temps assez petit Δh, la probabilité conditionnelle à l'instant t, que le défaut se produise entre t et $t + \Delta h$ $(\mathbb{P}(\tau \leq t + \Delta h | \tau > t))$ est de l'ordre de $\lambda(t)\Delta h$. Ainsi, en 2010, M. Jeanblanc, N. Elkaroui et Ying Jiao [EKJJ10] utilisent un modèle réduit pour mettre en évidence les limites du processus intensité et pour montrer que le processus densité caractérise entièrement les liens entre l'instant de défaut et la filtration de référence. En 2014, N. Elkaroui et al. [EKJJZ14] ont construit un des modèles explicites de densité conditionnelle d'un (ou de plusieurs) instants de défaut sachant une filtration de référence. Pour ce faire, les auteurs ont eu recours aux méthodes de changement de temps, de changement de mesure de probabilité et de filtrations et des méthodes de copules dynamiques. Plus récemment, en 2015, dans [KJJ15], elles appliquent cette approche à l'étude d'une famille $\tau = (\tau_1, \cdots, \tau_n)$ de temps d'arrêts représentant les instants de défaut de n firmes. Elles concluent dans ce papier que l'approche de la densité offre un choix flexible par rapport à celle de l'intensité. Nous utilisons dans cette thèse un modèle structurel comme dans [DL01], un processus diffusion-saut comme dans [Dor07] et une approche densité comme dans [KJJ15].

1.2 Contribution de la thèse

1.2.1 Observation complète

Dans le chapitre 2, nous avons dans un premier temps montré la régularité de la densité (de l'instant de défaut τ_x) obtenue par Coutin et Dorobantu dans [CD$^+$11]. Dans un second temps, nous avons obtenu une expression explicite qui caractérise la loi du triplet (instant de défaut, overshoot, undershoot) pour un processus mixte diffusion-saut. Ce chapitre a fait l'objet d'un article soumis au journal ESAIM : Probabability and Statistics, [CN16].

Afin d'obtenir la régularité en fonction du seuil, le chapitre 3 est motivé par l'existence d'une équation au sens faible dont est solution le quadruplet formé par la variable aléatoire X_t, le supremum du processus X à l'instant t, le supremum au dernier instant de saut avant t et le dernier instant de saut avant l'instant t.

Chapitre 2 : Loi jointe de l'instant de défaut, de l'overshoot et de l'undershoot d'un processus de Lévy

Nous nous inspirons ici de [Vol03] : une compagnie d'assurance, ayant un capital initial $x > 0$ à l'instant 0 peut modéliser son capital à l'instant t par un processus stochastique :

$$Z_t^x = x - X_t \tag{1.2.1}$$

où

$$X_t = mt + W_t + \sum_{i=1}^{N_t} Y_i. \tag{1.2.2}$$

Ici, $(W_t, t \geq 0)$ représente les "petites fluctuations" régulières de la fortune et c'est un mouvement brownien standard ; m est un réel, mt représente les rentrées non aléatoires

et $\sum_{i=1}^{N_t} Y_i$ est un processus de Poisson composé représentant les fluctuations discontinues. Tous ces éléments sont supposés indépendants.
L'instant de défaut, encore appelé instant de faillite, est le premier temps de passage au niveau 0 du processus $(Z_t^x, t \geq t)$. Il est noté τ_x et il est défini par :

$$\tau_x = \inf\{t \geq 0 : Z_t^x \leq 0\}. \tag{1.2.3}$$

Il s'agit d'un temps d'arrêt qui, de manière équivalente, est le premier temps de dépassement du niveau x par le processus de Lévy $(X_t, t \geq 0)$:

$$\tau_x = \inf\{t \geq 0 : X_t \geq x\}. \tag{1.2.4}$$

Nous utiliserons la notation (1.2.4) dans toute la suite.
Coutin et Dorobantu ont montré dans [CD$^+$11] que la loi de τ_x admet une densité f par rapport à la mesure de Lebesgue définie par

$$f(t,x) = \begin{cases} \lambda\mathbb{E}(1_{\tau_x>t}(1-F_Y)(x-X_t)) + \mathbb{E}(1_{\tau_x>T_{N_t}}\tilde{f}(t-T_{N_t}, x-X_{T_{N_t}})), & \forall t>0 \\ \frac{\lambda}{2}(2-F_Y(x)-F_Y(x_-)) + \frac{\lambda}{4}(F_Y(x)-F_Y(x_-)) & \text{if } t=0 \end{cases} \tag{1.2.5}$$

et $\mathbb{P}(\tau_x = \infty) = 0$ si et seulement si $m + \lambda\mathbb{E}(Y_1) \geq 0$.
La fonction $\tilde{f}$ est la densité de la loi du temps d'atteinte d'un mouvement brownien avec dérive m obtenue par I. Karatzas et S. E. Shreve dans [KS12] et définie par

$$\tilde{f}(u,x)du + \mathbb{P}(\tilde{\tau}_x = \infty)\delta_\infty(du)$$

où

$$\tilde{f}(u,x) = \frac{|\, x \,|}{\sqrt{2\pi u^3}} \exp[-\frac{1}{2u}(x-mu)^2] 1_{]0,+\infty[}(u) \text{ et } \mathbb{P}(\tilde{\tau}_x = \infty) = 1 - e^{mx-|mx|} \tag{1.2.6}$$

avec $\tilde{f}$ est la densité de la loi de $\tilde{\tau}_x = \inf\{t \geq 0 : \tilde{X}_t = x\}$; $\tilde{X}_t = mt + W_t$.
En guise de contribution, nous avons obtenu tout d'abord un résultat de régularité de la densité f. Nous avons dans un premier temps montré un résultat de continuité en temps et en espace dont l'énoncé est le suivant

Proposition 1.2.7 *L'application définie sur* $]0,+\infty[\times]0,+\infty[$ *par*

$$(t,x) \longrightarrow f_{\tau_x}(t,x) = \lambda\mathbb{E}\left(\mathbf{1}_{\{\tau_x>t\}}(1-F_Y)(x-X_t)\right) + \mathbb{E}\left(1_{\tau_x>T_{N_t}}\tilde{f}(t-T_{N_t}, x-X_{T_{N_t}})\right)$$

est continue. De plus, Si $x > 0$ fixé, nous avons

$$\lim_{t\mapsto 0}\mathbb{E}\left(\mathbf{1}_{\tau_x>t}[1-F_Y](x-X_t)\right) = \frac{1}{2}\left(2-F_Y(x)-F_Y(x_-)\right)$$

et

$$\lim_{t\to 0}\mathbb{E}\left(\mathbf{1}_{\{\tau_x>T_{N_t}\}}\tilde{f}(t-T_{N_t}, x-X_{T_{N_t}})\right) = \frac{\lambda}{4}\left(F_Y(x)-F_Y(x_-)\right).$$

Remarque 1.2.8 *Pour $t = 0$, on ne peut rien dire sur la continuité en x de $x \mapsto f_{\tau_x}(0, x)$ car F_Y n'est pas forcément continue.*

Si $K_x := X_{\tau_x} - x$ designe le déficit aussitôt après la ruine, appelé 'overshoot' ou 'sévérité de la ruine', et si $L_x := x - X_{\tau_x^-}$ désigne la fortune immédiatement avant la ruine, Kyprianou a étudié [Kyp14] la loi conjointe de (K_x, L_x) lorsque le processus X est un subordinateur (processus de Lévy dont les trajectoires sont non décroissantes). A. Volpi et al. [RVV08] se sont intéressés à l'étude asymptotique quand x tend vers $+\infty$, dans le cas du processus (1.1.2), de la loi conjointe du triplet (τ_x, K_x, L_x). Ils établissent que la transformée de Laplace de ce triplet est l'unique solution d'une certaine équation intégro-différentielle. Puis ils montrent après une renormalisation convenable de τ_x que sous une certaine hypothèse sur la mesure de Lévy, ce triplet converge en distribution lorsque x tend vers $+\infty$. A ce sujet, notre contribution consiste à donner la première expression explicite caractérisant la loi du triplet (τ_x, K_x, L_x) par le théorème suivant.

Théorème 1.2.9 *(Théorème 4.3.4) La loi jointe du triplet (τ_x, K_x, L_x), sachant que $\{\tau_x < \infty\}$, est donnée sur $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ par $p(.,.,.,x)$ telle que :*

$$\begin{aligned} p(0, dk, dl, x) &= \frac{\lambda}{4}[F_Y(x) - F_Y(x_-)]\delta_{\{0,0,0\}}(dt, dk, dl) + \lambda F_l(dk)\delta_{\{0,x\}}(dt, dl) \\ &+ \frac{\lambda}{2}\Delta F_Y(x)\delta_{\{0,0,x\}}(dt, dk, dl) \end{aligned}$$

et pour tout $t > 0$,

$$\begin{aligned} p(dt, dk, dl, x) = \mathbb{E}[&\mathbf{1}_{\{\tau_x > T_{N_t}\}}\tilde{f}(t - T_{N_t}, x - X_{T_{N_t}})]\delta_{\{0,0\}}(dk, dl)dt \\ &+ \lambda\mathbb{E}\left[\mathbf{1}_{\{k\geq 0, l\geq 0\}}\mathbf{1}_{\{\tau_x > T_{N_t}\}}f_0(x - X_{T_{N_t}} - l)\right] F_l(dk)dldt \\ &- \lambda\mathbb{E}\left[\mathbf{1}_{\{k\geq 0, l\geq 0\}}\mathbf{1}_{\{\tau_x > T_{N_t}\}}f_0(X_{T_{N_t}} - x - l)\exp(2m(x - X_{T_{N_t}}))\right] F_l(dk)dldt \end{aligned}$$

où δ est la masse de Dirac et f_0 est la densité de la loi normale d'espérance $\mu = m(t - T_{N_t})$ et de variance $\sigma^2 = t - T_{N_t}$, $\tilde{f}$ est définie par (1.1.7), $F_l(dk)$ est l'image de $F_Y(dk)$ par l'application $y \mapsto y - l$ et $\Delta F_Y(x) = F_Y(x) - F_Y(x^-)$.

Remarque 1.2.10 *Se référant à [RVV08], pour tout $x > 0$, le premier instant de passage τ_x est fini presque sûrement si et seulement si $m + \mathbb{E}(Y_1) \geq 0$.*

La loi du triplet (τ_x, K_x, L_x) en observation incomplète reste aussi un problème ouvert.

Chapitre 3 : Equation à valeur mesure pour la loi du couple formé par un processus de Lévy et son supremum

En observation complète, la liste de travaux donnée dans le paragraphe 1.1.2 est loin d'être exhaustive, mais nous y joignons notre contribution. Dans le chapitre 3, nous avons obtenu une équation aux dérivées partielles permettant de caractériser au sens faible le quadruplet

$$U_t := (X_t^*, X_t, X_{T_{N_t}}^*, T_{N_t}) \tag{1.2.11}$$

où $X_t = mt + W_t + \sum_{i=1}^{N_t} Y_i$. Nous introduisons les notations respectives du mouvement brownien avec dérive m et de son supremum :

$$\tilde{X}_t = mt + W_t$$
$$\tilde{X}_t^* = \sup_{u \leq t} \tilde{X}_u.$$

Pour tout $t > 0$, on introduit aussi la densité de la loi du couple $(\tilde{X}_t^*, \tilde{X}_t)$:

$$\tilde{p}(b, a, t) = \frac{2(2b-a)}{\sqrt{2\pi t^3}} \exp(-\frac{(2b-a)^2}{2t} + ma - \frac{m^2}{2}t). \tag{1.2.12}$$

Théorème 1.2.13 *Soit* $\varphi\ : \mathbb{R}^4 \to \mathbb{R}$ *une fonction* C^3- *bornée.*
Pour tout $t > 0$,

$$\begin{aligned}
&\mathbb{E}\left(\varphi(U_t)\right) = \varphi(0,0,0,0) + \int_0^t \mathbb{E}\left[m\partial_2\varphi(U_s) + \frac{1}{2}\partial_{22}^2\varphi(U_s)\right] ds \\
&+ \int_0^t \frac{1}{2} E\left[\mathbf{1}_{\{X_s^* > X_{T_{N_s}}^*\}} \partial_1\varphi(X_s^*, X_s^*, X_{T_{N_s}}^*, T_{N_s}) \frac{\tilde{p}(X_s^* - X_{T_{N_s}}^*, X_s^* - X_{T_{N_s}}^*, s - T_{N_s})}{\tilde{p}^*(X_s^* - X_{T_{N_s}}^*, s - T_{N_s})}\right] ds \\
&+ \lambda \int_0^t \mathbb{E}\left(\int_{\mathbb{R}} \left[\varphi(U_s(y)) - \varphi(U_s)\right] d\mu_Y(y)\right) ds.
\end{aligned}$$

où

$$U_s(y) = (\max(X_s^*, X_s + y), X_s + y, max(X_s^*, X_s + y), s), \quad s \geq 0. \tag{1.2.14}$$

Nous donnons les grandes lignes de la preuve de ce théorème. Il s'agit de calculer

$$\lim_{h \to 0} \frac{1}{h} A(t, h) \text{ où } A(t, h) := \mathbb{E}\left[\varphi(U_{t+h}) - \varphi(U_t)\right]$$

puis montrer que cette limite est bornée. Cela suffit à écrire :

$$\mathbb{E}\left(\varphi(U_t)\right) = \int_0^t a(s) ds + \varphi(0,0,0,0).$$

L'idée consiste à décomposer $A(t,h)$ suivant les valeurs prises par $N_{t+h} - N_t$:

$$A(t,h) = \sum_{i=0}^{2} A_i(t,h) \tag{1.2.15}$$

avec

$$\begin{aligned} A_i(t,h) &:= \mathbb{E}\left([\varphi(U_{t+h}) - \varphi(U_t)]\, \mathbf{1}_{\{N_{t+h}-N_t=i\}}\right), \quad i = 0,1 \\ A_2(t,h) &:= \mathbb{E}\left([\varphi(U_{t+h}) - \varphi(U_t))]\, \mathbf{1}_{\{N_{t+h}-N_t\geq 2\}}\right). \end{aligned} \tag{1.2.16}$$

Ces trois termes sont traités chacun par une proposition ou par un lemme.
Pour $A_2(t,h)$, nous avons

Lemme 1.2.17 *Pour toute fonction $\varphi : \mathbb{R}^4 \mapsto \mathbb{R}$ de classee $\mathcal{C}^3$, bornée,*

$$\lim_{h\to 0} h^{-1} A_2(t,h) = 0. \tag{1.2.18}$$

Pour le terme $A_1(t,h)$, nous prouvons :

Proposition 1.2.19 *Pour toute fonction $\varphi : \mathbb{R}^4 \mapsto \mathbb{R}$ de classee $\mathcal{C}^3$, bornée,*

$$\lim_{h\to 0} h^{-1} A_1(t,h) = \lambda \mathbb{E} \int_{\mathbb{R}} [\varphi(U_t(y)) - \varphi(U_t)] F_Y(dy).$$

où U_t est défini par (1.2.11) et $U_t(y)$ par (1.2.14).

Etudions maintenat le terme $h^{-1}A_0(t,h)$ quand h tend vers 0. Sur l'évèment $\{N_{t+h} - N_t = 0\}$, $T_{N_t} = T_{N_{t+h}}$, ainsi $X^*_{T_{N_{t+h}}} = X^*_{T_{N_t}}$ and $X_{T_{N_{t+h}}} = X_{T_{N_t}}$.

$$\begin{aligned} X_{t+h} &= X_t + \tilde{X}_h \circ \theta_t, \\ X^*_{t+h} &= \max(X^*_t, X_t + \tilde{X}^*_h \circ \theta_t). \end{aligned}$$

La propriétde Markov en t et le fait N soit indépendant de $\tilde{X}$, entrainent que

$$A_0(t,h) = e^{-\lambda h} \mathbb{E}\left(\mathbb{E}\left(\varphi(\max(x^*, x + \tilde{X}^*_h), x + \tilde{X}_h, y, u) - \varphi(x^*, x, y, u)\right)_{|x^* = X^*_t, x = X_t, y = X^*_{T_{N_t}}, T_{N_t} = u} \right).$$

Introduisons

$$a_0(h, x^*, x, y, u) := \mathbb{E}\left(\varphi(\max(x^*, x + \tilde{X}^*_h), x + \tilde{X}_h, y, u) - \varphi(x^*, x, y, u)\right).$$

Pour étudier le terme $a_0(h, x^*, x, y, u)$, nous faisons un développement de Taylor au voisinage de (x^*, x) (y, u sont vus comme des constantes)

$$\begin{aligned} a_0(h, x^*, x, y, u) : =\ & \partial_1 \varphi(x^*, x, y, u) \mathbb{E}\left(\left[\max(x^*, x + \tilde{X}^*_h) - x^*\right]\right) \\ & + \partial^2_{1,2} \varphi(x^*, x, y, u) \mathbb{E}\left(\left[\max(x^*, x + \tilde{X}^*_h) - x^*\right] \tilde{X}_h\right) \\ & + \frac{1}{2} \partial^2_{1,1} \varphi(x^*, x, y, u) \mathbb{E}\left(\left[\max(x^*, x + \tilde{X}^*_h) - x^*\right]^2\right) \\ & + \partial_2 \varphi(x^*, x, y, u) m h + \frac{1}{2} \partial^2_{22} \varphi(x^*, x, y, u)[m^2 h^2 + h] + R_0(t, h, x^*, x, y, u), \end{aligned}$$

où, pour tout y et tout u,

$$|R_0(h,x^*,x,y,u)| \leq 4\|\nabla^3\varphi\|_\infty \left[\mathbb{E}\left(\left|\max(x^*,x+\tilde{X}_h^*)-x^*\right|^3\right)+\mathbb{E}\left(\left|\tilde{X}_h\right|^3\right)\right].$$

Ce qui permet d'écrire :

$$A_0(t,h)=\sum_{i=1}^{3} A_{0,i}(t,h) \tag{1.2.20}$$

avec $A_{0,i}(t,h) := \mathbb{E}\left(a_{0,i}(h,,x^*,x,y,u)_{|x^*=X_t^*,x=X_t,y=X^*_{T_{N_t}},T_{N_t}=u}\right)$ où

$$\begin{aligned}
a_{0,1}(h,x^*,x,y,u): &= \partial_2\varphi(x^*,x,y,u)mh+\frac{1}{2}\partial^2_{22}\varphi(x^*,x,y,u)[m^2h^2+h]\\
a_{0,2}(h,x^*,x,y,u): &= \partial^2_{1,2}\varphi(x^*,x,y,u)\mathbb{E}\left(\left[\max(x^*,x+\tilde{X}_h^*)-x^*\right]\tilde{X}_h\right)\\
&\quad +\frac{1}{2}\partial^2_{2,2}\varphi(x^*,x,y,u)\mathbb{E}\left(\left[\max(x^*,x+\tilde{X}_h^*)-x^*\right]^2\right)+R_0(h,x^*,x,y,u),\\
a_{0,3}(h,x^*,x,y,u): &= \partial_1\varphi(x^*,x,y,u)\mathbb{E}\left(\left[\max(x^*,x+\tilde{X}_h^*)-x^*\right]\right).
\end{aligned}$$

Proposition 1.2.21 *Sous les hypothèses du théorème 1.2.13,*

$$\lim_{h\to 0} h^{-1}(A_{0,1}+A_{0,2})(t,h)=\mathbb{E}\left(\partial_2\varphi(U_t)m+\frac{1}{2}\partial^2_{22}\varphi(U_t)\right). \tag{1.2.22}$$

Proposition 1.2.23 *Pour tout $t>0$, la loi du vecteur (X_t^*,X_t) admet une densité par rapport à la mesure de Lebesgue. Cette densité est donnée par*

$$p(b,a,t)=\mathbb{E}\left(\sum_{k=0}^{N_t}\tilde{p}\left(b-X_{T_k},a-X_{T_k}-Y_{k+1}\mathbf{1}_{k<N_t}-(X_t-X_{T_{k+1}\wedge t}),t\wedge T_{k+1}-T_k\right)\mathbf{1}_{\Delta'_{k,t}}(b,a)\right)$$

où

$$\Delta'_{k,t}=\left\{(b,a),\mid b>\max\left(X^*_{T_k},[a+\sup_{u\in[T_{k+1}\wedge t,t]}(X_u-X_t)]\mathbf{1}_{\{T_{k+1}<t\}}\right)\right\} \tag{1.2.24}$$

et $\tilde{p}$ est introduit par (1.2.12).

Proposition 1.2.25 *Sous les hypothèses du théorème 1.2.13,*

$$\lim_{h\to 0}\frac{1}{2h}\mathbb{E}\left(\partial_1\varphi(U_t)\mathbb{E}\left(\max(x^*,x+\tilde{X}_h^*)-x^*\right)_{x^*=X_t^*,x=X_t}\right)=$$
$$+\frac{1}{4}\mathbb{E}\left[\mathbf{1}_{X_t^*>X^*_{T_{N_t}}}\partial_1\varphi(X_t^*,X_t^*,X^*_{T_{N_t}},T_{N_t})\frac{\tilde{p}(X_t^*-X^*_{T_{N_t}},X_t^*-X^*_{T_{N_t}},t-T_{N_t})}{\tilde{p}^*(X_t^*-X^*_{T_{N_t}},t-T_{N_t})}\right]. \tag{1.2.26}$$

De manière équivalente,

$$\lim_{h\to 0} A_{0,3}(t,h) = 1/2\mathbb{E}\left[\mathbf{1}_{\{X_s^*>X_{T_{N_s}}^*\}}\partial\varphi(X_t^*,X_t^*,X_{T_{N_t}}^*,T_{N_t})\frac{\tilde{p}(X_t^*-X_{T_{N_t}}^*,X_t^*-X_{T_{N_t}}^*,t-T_{N_t})}{\tilde{p}^*(X_t^*-X_{T_{N_t}}^*,t-T_{N_t})}\right].$$

Ce chapitre ouvre des perspectives que l'on présente dans le paragraphe suivant.

Perspectives

Parmi les problèmes restant ouverts et sur lesquels débouchent les travaux précédents nous pouvons citer

— la régularité de la probabilité de survie $(x,t) \mapsto G(t,x) := \mathbb{P}(\tau_x > t)$, c'est à dire $(x,t) \mapsto G(t,x)$ est $\mathcal{C}_b^1$.

— La représentation intégrale de $\mathbf{1}_{\tau_x>t}$ sous la forme

$$\mathbf{1}_{\tau_x>t} = \mathbb{P}(\tau_x > t) + \int_0^t H_s^t dW_s + \int_0^t \int_{\mathbb{R}} H^t(s,z)dM(s,z)$$

où H^t et $H^t(.,.)$ sont des processu prévisibles et M est la mesure aléatoire compensée associée au processus $t \mapsto \sum_{i=1}^{N_t} Y_i$.

— La caractérisation de la loi de $(X_t^*, X_t, X_{T_{N_t}}^*, T_{N_t})$ via une équation différentielle à valeur mesure au sens faible.

1.2.2 Observation incomplète

L'information apportée par la connaissance de l'intensité est faible. Dans un premier temps, le chapitre 4 montre dans le modèle étudié par D. Dorobantu que la loi conditionnelle de l'instant de défaut sachant la tribu $\mathcal{G}_t$ admet une densité par rapport à la mesure de Lebesgue et obtient une équation de Volterra dont est solution cette densité. Cette connaissance permet aux investisseurs de prévoir l'instant de défaut après l'instant t, au vu de leur information. Ce travail est complété par des simulations numériques et a fait l'objet d'un article publié dans le journal Journal of Mathematical Finance, voir [Ngo15].

Chapitre 4 : Loi conditionnelle de l'instant de défaut d'un processus de Lévy en observation incomplète

L'objectif du chapitre 4 est dans un premier temps de prouver l'existence de la densité conditionnelle de la loi de l'instant de défaut sachant une information bruitée, puis d'établir une équation intégro-différentielle dont cette densité est solution, ensuite d'assurer l'unicité de cette solution et enfin de donner quelques schémas numériques.

Nous nous situons sur un espace de probabilité $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), \mathbb{P}^0)$ filtré satisfaisant les conditions habituelles sur lequel sont définis :

$$X_t = mt + W_t + \sum_{i=1}^{N_t} Y_i \text{ modélise la valeur de la firme au temps } t,$$

$$\tau_x = \inf\{t \geq 0 : X_t > x\} \text{ représente l'instant de défaut,}$$

$$Q_t = \int_0^t h(X_s)ds + B_t, \quad t \in \mathbb{R}_+, \text{ est le processus d'observation.}$$

Les processus Q, $,W$, N et Y sont supposés indépendants, le couple (W, Q) est un $(\mathbb{P}^0,\ (\mathcal{F}_t, t \geq\ 0))-$ mouvement brownien et la fonction h est borélienne bornée. Nous avons utilisé la théorie du filtrage dont le cadre est le suivant.

Définition 1.2.27 *Le processus X est appelé le signal et le processus Q l'observation.*

La condition de Novikov : $\forall T > 0 \quad \mathbb{E}^0\left(e^{\frac{1}{2}\int_0^T h^2(X_s)ds}\right) < \infty$, est satisfaite et nous définissons la $(\mathcal{F}_t)-$martingale

$$L_t = \exp\left(\int_0^t h(X_s)dQ_s - \frac{1}{2}\int_0^t h^2(X_s)ds\right), \quad t \in \mathbb{R}_+.$$

Pour une maturité $T > 0$ fixée, ceci, à l'aide de la formule de Girsanov, permet de définir une nouvelle mesure de probabilité $\mathbb{P}$ sur toute tribu $\mathcal{F}_t$, $\ t < T$.

Définition 1.2.28 *Pour tout $t > 0$, nous définissons une mesure de probabilité $\mathbb{P}$ telle que*

$$\mathbb{P}_{|\mathcal{F}_t} := L_t \mathbb{P}^0_{|\mathcal{F}_t}$$

En plus de l'observation, les investisseurs obtiennent une information sur le défaut donnée par

$$\mathcal{D}_t = \sigma(\mathbf{1}_{\tau_x \leq u}, u \leq t).$$

Nous notons $\mathcal{F}^Q$ la filtration modélisant l'information obtenue par l'observation, alors l'information globale des investisseurs est

$$\mathcal{G} := (\ \mathcal{G}_t = \mathcal{F}_t^Q \vee \mathcal{D}_t, \quad t \geq 0).$$

Notre résultat d'existence est le suivant :

Proposition 1.2.29 *(Proposition 4.3.1) Pour tout $t > 0$, sur l'événement $\{\tau_x > t\}$, la loi conditionnelle de τ_x sachant la filtration $(\mathcal{G}_t)$ admet une densité qui est de la forme*

$$\begin{aligned} &\bar{f}(r,t,x)dr + \mathbb{P}(\tau_x = \infty|\mathcal{G}_t)\delta_\infty(dr) \\ &\textit{et } \mathbb{P}(\tau_x = \infty|\mathcal{G}_t) = \mathbf{1}_{\tau_x > t}\mathbb{E}(G(\infty, x - X_t)|\mathcal{G}_t), \end{aligned} \tag{1.2.30}$$

où

$$\bar{f}(r,t,x) := \mathbb{E}[f(r-t, x-X_t)|\mathcal{G}_t].$$

et

$$G(t,x) := \mathbb{P}(\tau_x > t) = \mathbb{P}^0(\tau_x > t) = \int_t^\infty f(u,x)du.$$

Remarque 1.2.31 *Se reférant à [RVV08], pour tout $x > 0$, le premier temps de passage τ_x est fini presque sûrement si et seulement si $m + \mathbb{E}(Y_1) \geq 0$.*

A ce résultat est associé le théorème suivant :

Théorème 1.2.32 *(Théorème 4.3.4) Soit $t > 0$ un nombre réel. Pour tout $r > t$, sur l'évènement $\{\tau_x > t\}$, la densité conditionnelle de τ_x sachant $\mathcal{G}_t$ satisfait l'équation intégrodifférentielle :*

$$\begin{aligned}\bar{f}(r,t,x) = {} & \frac{f(r,x)}{\mathbb{P}(\tau_x > t)} + \int_0^t \Pi^1(h)(r,t,u)dQ_u \\ & - \int_0^t \frac{\bar{f}(r,u,x)}{\mathbb{E}(\mathbf{1}_{\tau_x>u}G(t-u,x-X_u)|\mathcal{G}_u)}\Pi(h)(t,u)dQ_u \\ & + \int_0^t \frac{\bar{f}(r,u,x)}{\mathbb{E}(\mathbf{1}_{\tau_x>u}G(t-u,x-X_u)|\mathcal{G}_u)}[\Pi(h)(t,u)]^2 du \\ & - \int_0^t \Pi^1(h)(r,t,u)\Pi(h)(t,u)du.\end{aligned} \tag{1.2.33}$$

où

$$\Pi^1(\Phi)(r,t,u) = \frac{\mathbb{E}(\mathbf{1}_{\tau_x>u}\Phi(X_u)f(r-u,x-X_u)|\mathcal{G}_u)}{\mathbb{E}(\mathbf{1}_{\tau_x>u}G(t-u,x-X_u)|\mathcal{G}_u)},$$
$$\Pi(\Phi)(t,u) = \frac{\mathbb{E}(\mathbf{1}_{\tau_x>u}\Phi(X_u)G(t-u,x-X_u)|\mathcal{G}_u)}{\mathbb{E}(\mathbf{1}_{\tau_x>u}G(t-u,x-X_u)|\mathcal{G}_u)}$$

et G est définie dans la proposition 1.2.29.

L'équation décrite dans ce théorème est l'analogue en théorie de filtrage stochastique de l'équation du filtre normalisé dite équation de Kushner-Stratonovich. Comme en théorie du filtrage, la preuve de ce théorème repose sur la proposition 1.2.37 via les résultats suivants : la proposition 1.2.34, cf. [JR00], permet de passer de la filtration $\mathcal{G}$ à la filtration $\mathcal{F}^Q$.

Proposition 1.2.34 *Pour toute variable aléatoire $\mathcal{G}-$mesurable Y et pour tout $t \in \mathbb{R}_+$, on a*

$$\mathbb{E}\left(\mathbf{1}_{\tau_x>t}Y|\mathcal{G}_t\right) = \mathbf{1}_{\tau_x>t}\frac{\mathbb{E}(\mathbf{1}_{\tau_x>t}Y|\mathcal{F}_t^Q)}{\mathbb{E}(\mathbf{1}_{\tau_x>t}|\mathcal{F}_t^Q)}.$$

La proposition 1.2.35, cf. [Par91], permet de passer de la probabilité $\mathbb{P}$ à la probabilité $\mathbb{P}^0$.

Proposition 1.2.35 *Pour tout* $t \geq 0$ *et pour tout* $Y \in L^1(\Omega, \mathcal{F}_t, \mathbb{P})$, $\quad L_t Y \in L^1(\Omega, \mathcal{F}_t, \mathbb{P}^0)$ *et*

$$\mathbb{E}(Y|\mathcal{F}_t^Q) = \frac{\mathbb{E}^0(L_t Y|\mathcal{F}_t^Q)}{\mathbb{E}^0(L_t|\mathcal{F}_t^Q)}. \tag{1.2.36}$$

En filtrage, la formule (1.2.36) est souvent appelée "formule de Kallianpur-Striebel".

Proposition 1.2.37 *(Proposition 4.3.15) Pour tout* (t, a, b) *tel que*$0 < t < a < b$, *sur l'évènement* $\{\tau_x > t\}$ *:*

$$\begin{aligned}
\frac{\mathbb{E}^0(L_b \mathbf{1}_{a<\tau_x<b}|\mathcal{F}_t^Q)}{\mathbb{E}^0(\mathbf{1}_{\tau_x>t} L_t|\mathcal{F}_t^Q)} &= \frac{\mathbb{P}^0(a < \tau_x < b)}{\mathbb{P}^0(\tau_x > t)} \\
&+ \int_0^t \frac{\mathbb{E}^0(\mathbf{1}_{\tau_x>u} L_u h(X_u)[G(a-u, x-X_u) - G(b-u, x-X_u)]|\mathcal{F}_u^Q)}{\mathbb{E}^0(\mathbf{1}_{\tau_x>t} L_u|\mathcal{F}_u^Q)} dQ_u \\
&- \int_0^t \frac{\mathbb{E}^0(L_u \mathbf{1}_{a<\tau_x<b}|\mathcal{F}_u^Q)\mathbb{E}^0(\mathbf{1}_{\tau_x>u} L_u h(X_u) G(t-u, x-X_u)|\mathcal{F}_u^Q)}{[\mathbb{E}^0(\mathbf{1}_{\tau_x>t} L_u|\mathcal{F}_u^Q)]^2} dQ_u \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1.2.38) \\
&+ \int_0^t \frac{\mathbb{E}^0(L_u \mathbf{1}_{a<\tau_x<b}|\mathcal{F}_u^Q)[\mathbb{E}^0(\mathbf{1}_{\tau_x>u} L_u h(X_u) G(t-u, x-X_u)|\mathcal{F}_u^Q)]^2}{[\mathbb{E}^0(\mathbf{1}_{\tau_x>t} L_u|\mathcal{F}_u^Q)]^3} du \\
&- \int_0^t \mathbb{E}^0(\mathbf{1}_{\tau_x>u} L_u h(X_u)[G(a-u, x-X_u) - G(b-u, x-X_u)]|\mathcal{F}_u^Q) \\
&\times \frac{\mathbb{E}^0(\mathbf{1}_{\tau_x>u} L_u h(X_u) G(t-u, x-X_u)|\mathcal{F}_u^Q)}{[\mathbb{E}^0(\mathbf{1}_{\tau_x>t} L_u|\mathcal{F}_u^Q)]^2} du.
\end{aligned}$$

Remarque 1.2.39 *L'équation (1.2.38) de la proposition 1.2.37 peut être écrite à nouveau comme suit :*

$$\begin{aligned}
\bar{\Gamma}_t &= \frac{\mathbb{P}^0(a < \tau_x < b)}{\mathbb{P}^0(\tau_x > t)} + \int_0^t \sigma^1(h)(t, u) dQ_u \\
&- \int_0^t \bar{\Gamma}_u \sigma(h)(t, u)) dQ_u + \int_0^t \bar{\Gamma}_u [\sigma(h)(t, u)]^2 du \\
&- \int_0^t \sigma^1(h)(t, u) \sigma(h)(t, u) du.
\end{aligned}$$

où

$$\begin{aligned}
\bar{\Gamma}_t &= \frac{\mathbb{E}^0(L_b \mathbf{1}_{a<\tau_x<b}|\mathcal{F}_t^Q)}{\mathbb{E}^0(\mathbf{1}_{\tau_x>t}L_t|\mathcal{F}_t^Q)},\\
\sigma^1(h)(t,u) &= \mathbf{1}_{\{\tau_x>t\}}\frac{\mathbb{E}^0(\mathbf{1}_{\tau_x>u}L_u h(X_u)[G(a-u,x-X_u)-G(b-u,x-X_u)]|\mathcal{F}_u^Q)}{\mathbb{E}^0(\mathbf{1}_{\tau_x>t}L_u|\mathcal{F}_u^Q)},\\
\sigma(h)(t,u) &= \mathbf{1}_{\{\tau_x>t\}}\frac{\mathbb{E}^0(\mathbf{1}_{\tau_x>u}L_u h(X_u)G(t-u,x-X_u)|\mathcal{F}_u^Q)}{\mathbb{E}^0(\mathbf{1}_{\tau_x>t}L_u|\mathcal{F}_u^Q)}.
\end{aligned}$$

Cette équation est l'analogue en théorie du filtrage stochastique de l'équation du filtre non normalisé (Equation de Zakaï). Il s'agit de l'équation (3.43) qu'on retrouve dans [BC09].

Le résultat d'unicité est le suivant.

Proposition 1.2.40 *(Proposition 4.3.6) Si l'équation (1.2.33) admet une solution, cette dernière est unique.*

Simulations

Pour compléter les contributions apportées par ce chapitre 4, nous donnons un schéma de résolution numérique de l'équation (1.2.33). Le logiciel utilisé est 'scilab', la technique de résolution est la méthode particulaire. Il s'agit d'une des méthodes les plus utilisées pour la résolution de problèmes de filtrage. Nous nous sommes référés à [BC09] : chapitre 8 (qui introduit des méthodes numériques pour la résolution de problèmes de filtrage) et chapitre 9 (qui traite en profondeur la méthode particulaire en temps continu). Nous commençons par simuler (1.2.5) dont nous rappelons l'expression ici :

$$f(t,x) = \begin{cases} \lambda\mathbb{E}(1_{\tau_x>t}(1-F_Y)(x-X_t)) + \mathbb{E}(1_{\tau_x>T_{N_t}}\tilde{f}(t-T_{N_t}, x-X_{T_{N_t}})), & \forall t>0 \\ \frac{\lambda}{2}(2-F_Y(x)-F_Y(x_-)) + \frac{\lambda}{4}(F_Y(x)-F_Y(x_-)) & \text{if } t=0 \end{cases}$$

Nous choisissons $x>0$; la continuité de f garantit que simuler (1.2.5) revient à simuler

$$\lambda\mathbb{E}(1_{\tau_x>t}(1-F_Y)(x-X_t)) + \mathbb{E}(1_{\tau_x>T_{N_t}}\tilde{f}(t-T_{N_t}, x-X_{T_{N_t}})). \tag{1.2.41}$$

Nous choisissons comme loi des sauts la loi double exponentielle de densité donnée par

$$f_Y(y) = pe^{-\eta_1 y}\mathbf{1}_{y\geq 0} + (1-p)e^{\eta_2 y}\mathbf{1}_{y<0}, \quad p>, \ \eta_1>0, \ \eta_2>0, \tag{1.2.42}$$

alors (1.2.41) devient :

$$\frac{p\lambda}{\eta_1}\mathbb{E}\left(\mathbf{1}_{X_t^*<x}e^{-\eta_1(x-X_t)}\right) + \mathbb{E}(\mathbf{1}_{X^*_{T_{N_t}}<x}\tilde{f}(t-T_{N_t}, x-X_{T_{N_t}})). \tag{1.2.43}$$

Ainsi l'algorithme permettant d'obtenir f est

Algorithme 1 *(pour la densité f)*

- *Simuler le processus de Lévy X sur l'intervalle $[0 \;\; t]$:*
  - *Choisir le pas de la subdivision,*
  - *Simuler le mouvement sur $[0 \;\; t]$,*
  - *Simuler le nombre de sauts du processus de Lévy,*
  - *Simuler les instants de saut,*
  - *Simuler la loi des sauts,*
- *Simuler le processus de Lévy X sur l'intervalle $[0 \;\; T_{N_t}]$.*
- *Utiliser la définition de $\tilde{f}$, puis faire une approximation Monte Carlo basée sur 5000 trajectoires.*

On obtient les figures suivantes :

Pour $\lambda = 3$, la figure 4.1 est réalisée avec un temps $CPU = 438.03805$.

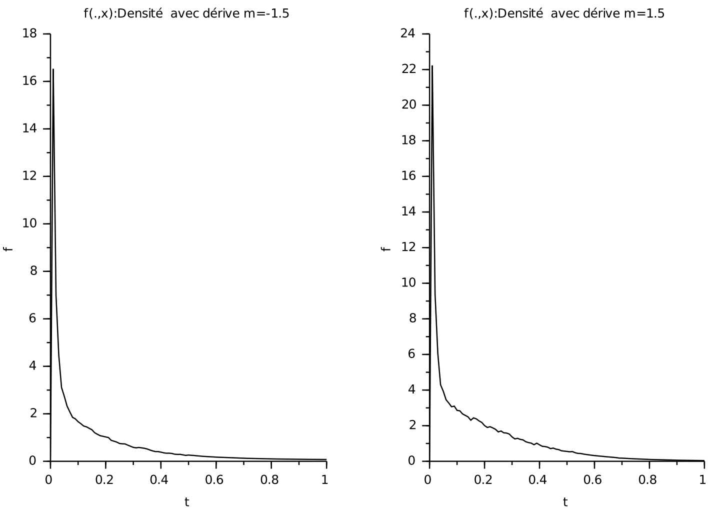

FIGURE 1.1 – A gauche, la courbe de la densité obtenue avec une dérive $m = -1.5$ et à droite celle obtenue avec une dérive $m = 1.5$.

Pour $\lambda = 0.1$, la figure 4.2 est réalisée avec un temps $CPU = 376.6704$.

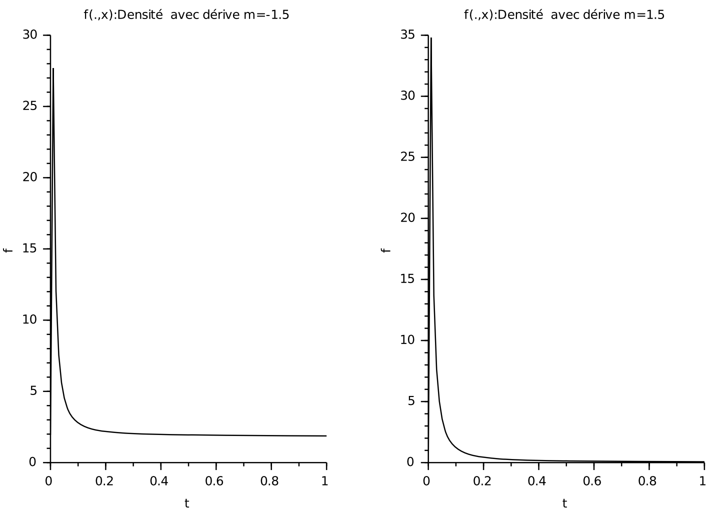

FIGURE 1.2 – A gauche la courbe de la densité obtenue avec une dérive $m = -1.5$ et à droite celle obtenue avec une dérive $m = 1.5$.

Ce qui suit est un schéma de résolution numérique de l'équation (1.2.33). En utilisant la proposition 1.2.29, l'ojectif est de visualiser

$$\bar{f}(r,t,x) = \mathbb{E}\left(f(r-t, x-X_t)|\mathcal{G}_t\right).$$

— Première étape : utiliser la formule

$$\mathbb{E}\left(f(r-t,x-X_t)|\mathcal{G}_t\right) = \mathbf{1}_{\tau_x > t} \frac{\mathbb{E}\left(\mathbf{1}_{\tau_x>t} f(r-t,x-X_t)|\mathcal{F}_t^Q\right)}{\mathbb{E}(\mathbf{1}_{\tau_x>t}|\mathcal{F}_t^Q)} \tag{1.2.44}$$

— Deuxième étape : utiliser la formule de Kallianpur-Strieblel qui permet de passer de la probabilité de référence à la probabilité d'observation :

$$\mathbb{E}\left(\mathbf{1}_{\tau_x>t} f(r-t,x-X_t)|\mathcal{F}_t^Q\right) = \frac{\mathbb{E}^0\left(L_t\mathbf{1}_{\tau_x>t} f(r-t,x-X_t)|\mathcal{F}_t^Q\right)}{\mathbb{E}^0(L_t|\mathcal{F}\text{Ăğ}t^Q)} \tag{1.2.45}$$

$$\mathbb{E}(\mathbf{1}_{\tau_x>t}|\mathcal{F}_t^Q) = \frac{\mathbb{E}^0(L_t\mathbf{1}_{\tau_x>t}|\mathcal{F}_t^Q)}{\mathbb{E}^0(L_t|\mathcal{F}\text{Ăğ}t^Q)}$$

— Simuler tour à tour $\mathbb{E}^0\left(L_t\mathbf{1}_{\tau_x>t} f(r-t,x-X_t)|\mathcal{F}_t^Q\right)$ et $\mathbb{E}^0(L_t\mathbf{1}_{\tau_x>t}|\mathcal{F}_t^Q)$. Pour ce faire, puisque sous $\mathbb{P}^0$, Q est un mouvement brownien indépendant de X, on simule alors $\mathbb{E}^0\left(L_t\mathbf{1}_{\tau_x>t} f(r-t,x-X_t)\right)$ et $\mathbb{E}^0(L_t\mathbf{1}_{\tau_x>t})$ en utilisant une approximation de Monte Carlo. La définition de f conduit à :

$$\begin{aligned} f(r-t,x-X_t) &= \lambda\mathbb{E}\left(\mathbf{1}_{\{\tau_{x-X_t}>r-t\}}[1-F_Y](x-X_t-X_{r-t})\right) \\ &\quad + \mathbb{E}\left(\mathbf{1}_{\{\tau_{x-X_t}>T_{N_{r-t}}\}} \tilde{f}(r-t-T_{N_{r-t}}, x-X_t-X_{T_{N_{r-t}}})\right). \end{aligned}$$

Nous donnons ici l'algorithme permettant d'obtenir $\mathbb{E}^0\left(L_t\mathbf{1}_{\tau_x>t} f(r-t,x-X_t)\right)$, celui qui donne $\mathbb{E}^0(L_t\mathbf{1}_{\tau_x>t})$ étant analogue. Nous choisissons pour la fonction h permettant de définir Q, la fonction sinus, pour la loi des sauts, la loi double exponentielle définie en (1.2.42).

Algorithme 2 *(Simuler $L_t\mathbf{1}_{\tau_x>t}$)*

— *Simuler le processus de Lévy X sur les intervalles respectifs $[0 \quad t]$ en utilisant l'algorithme 1,*

— *Simuler le mouvement brownien Q sur l'intervalle $[0 \quad t]$,*

— *Simuler les intégrales $\int_0^t \sin(X_s)dQ_s$ et $\int_0^t \sin^2(X_s)ds$.*

Algorithme 3 *(Simuler $f(r-t,x-X_t)$)*

— *Simuler le processus de Lévy X sur l'intervalle $[0 \quad r-t]$ comme dans l'algorithme 1,*

— *Simuler le processus de Lévy X sur l'intervalle* $[0 \quad (r-t) - T_{N_{r-t}}]$,

— *Utiliser la définition de* $\tilde{f}$ *puis une approximation Monte Carlo basée sur 122 trajectoires.*

Les figures suivantes sont celles de la densité conditionnelle $\bar{f}(.,t,x)$, pour $t = 0.1$ fixé et la variable r est dans l'intervalle $]0.1, 0.6]$.

Perspectives

L'équation obtenue dans le théorème 1.2.32 n'est pas fermée. Cependant, les résultats obtenus dans le chapitre 2 devraient permettre d'obtenir tous les outils nécessaires permettant de résoudre le problème de filtrage, c'est à dire d'obtenir l'analogue de l'équation de Kushner-Stratonovitch décrivant l'évolution de

$$t \mapsto \mathbb{E}\left(\varphi(X_t)\mathbf{1}_{\tau_x > t} | \mathcal{G}_t\right).$$

Comme Duffie et Lando [DL01] pour un modèle structurel avec un processus diffusion-saut, le cas où les investisseurs reçoivent à certaines dates prédéterminés $t_1,\ t_2,\ t_3,\ \cdots,\ t_n$ des rapports bruités sur la valeur des actifs de la firme reste à étudier.

1.3 Conclusion

Cette thèse apporte une contribution à l'étude de l'instant de défaut d'un processus de Lévy aussi bien en observation complète qu'en observation incomplète. Le chapitre 1 donne une présentation générale du thème, présente un état de l'art et un panorama des résultats apportés. Dans le chapitre 2, nous avons montré que la densité de l'instant de défaut est continue en temps et en espace sur $\mathbb{R}_+ \times \mathbb{R}_+^*$. Nous avons aussi obtenu la première expression explicite de la loi du triplet (τ_x, K_x, L_x). Dans le chapitre 3, nous avons obtenu une équation aux dérivées partielles dont la loi du quadruplet formé par la variable aléatoire X_t, le supremum du processus X à l'instant t, le supremum au dernier instant de saut avant t et le dernier instant de saut avant l'instant t est solution au sens faible. La régularité de la densité de l'instant de défaut en est aussi déduite. Ce chapitre généralise en partie les résultats de [KW03]. Ces deux premiers chapitres se placent en observation complète tandis que le dernier traite de l'observation incomplète. Dans ce dernier chapitre, nous avons obtenu une équation intégro-différentielle dont la densité de la loi conditionnelle de l'instant de défaut est solution. Ceci généralise le travail initié par D. Dorobantu dans la deuxième partie de sa thèse [Dor07]. Cependant cette équation n'est pas fermée, mais les chapitres 2 et 3 fournissent tous les outils nécessaires pour le problème de filtrage correspondant.

Pour la figure 1.3, le paramètre λ est égal à 1.5 et le temps CPU est 358.11432.

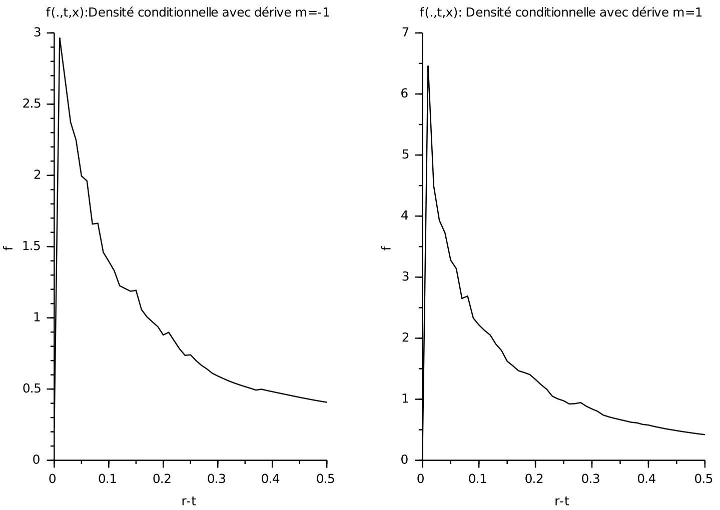

FIGURE 1.3 – A gauche la courbe de la densité conditionnelle obtenue avec une dérive $m = -1$ et à droite celle obtenue avec une dérive $m = 1$.

Pour la figure 1.4, le paramètre λ est égal à 0.1 et le temps CPU est 353.00736.

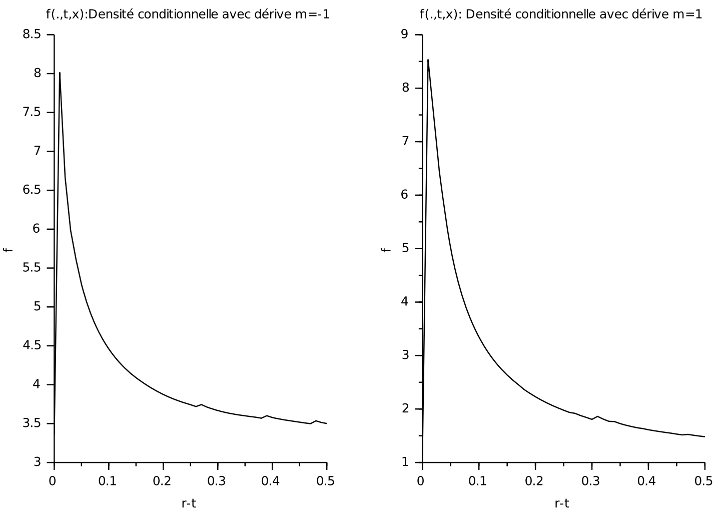

FIGURE 1.4 – A gauche la courbe de la densité conditionnelle obtenue avec une dérive $m = -1$ et à droite celle obtenue avec une dérive $m = 1$.

Chapitre 2

Joint law of the hitting time, overshoot and undershoot for a Lévy process

This chapter is an article which was submitted for publication at the journal ESAIM : Probability and Statistics.

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Abstract

Let $(X_t, t \geq 0)$ be a Lévy process which is the sum of a Brownian motion with drift and a compound Poisson process. We consider the first passage time τ_x at a fixed level $x > 0$ by $(X_t, t \geq 0)$, and $K_x := X_{\tau_x} - x$ the overshoot and $L_x := x - X_{\tau_x -}$ the undershoot. We first study the continuity of the density of τ_x. Secondly, we calculate the joint law of (τ_x, K_x, L_x).

2.1 Introduction

In the theory of risk in continuous time the surplus of an insurance company is modelled by a stochastic process $(X_t, t \geq 0)$. The positive real number x denotes the initial surplus and $\tau_x := \inf\{t \geq 0 : X_t \geq x\}$ may be interpreted as the default time. This paper deals with τ_x when X is a Lévy process, sum of a drifted Brownian motion and a compound Poisson process. Our main results lead to the regularity of the density of the hitting time and to an explicit expression characterizing the joint distribution of the triplet (first hitting time, overshoot, undershoot).
J. Bertoin [Ber98] gives a quick and concise treatment of the core theory on Lévy processes with the minimum of technical requirements. He gives some details on subordinators, fluctuation theory, Lévy processes with no positive jumps and stable processes.
P. Tankov and R. Cont [CT04] provide a self-contained overview of theoretical, numerical and empirical research on the use of Lévy processes in financial modeling.
When the process X has jumps, the first results are obtained by Zolotarev [Zol64] and Borovkov [Bor65] for X a spectrally negative Lévy process. Moreover, if X_t the probability density with respect to the Lebesgue measure $p(x,t)$ then the law of τ_x has the density with respect to the Lebesgue measure $f(t,x)$ such that $xf(t,x) = tp(x,t)$.
R. A. Doney [Don91] deals with hitting probabilities, hitting time distributions and associated quantities for Lévy processes which have only positive jumps. He gives an explicit formula for the joint Laplace transform of the hitting time τ_x and the overshoot $X_{\tau_x} - x$.
When X is a stable Lévy process, Peskir [Pes08] obtains an explicit formula for the passage time density. Moreover, if X has no negative jumps and if $S_t = \sup_{0\leq s\leq t} X_s$ is its running supremum, Bernyk et al. [BDP08] show that the density function f_t of S_t can be characterized as the unique solution to a weakly singular Voltera integral equation of the first kind.
In the case where X is a jump-diffusion process, with jump size following a double exponential law, Kou and Wang [KW03] give the law of τ_x. They obtain explicit solutions of the Laplace transform of the distribution of the first passage time. Laplace transform of the joint distribution of jump-diffusion and its running maximum, $S_t = \sup_{s\leq t} X_s$, is also obtained. Finally, they give numerical examples.
For a general Lévy process, Doney and Kyprianou [DK06] and Kyprianou [Kyp14] give the law of the quintuplet $(\bar{G}_{\tau_x}, \tau_x - \bar{G}_{\tau_{x^-}}, X_{\tau_x} - x, x - X_{\tau_{x^-}}, x - \bar{X}_{\tau_{x^-}})$ where $\bar{X}_t = \sup_{s\leq t} X_s$ and $\bar{G}_t = \sup\{s < t, X_s = X_t\}$.
For a stable Lévy process X of index $\alpha \in (1,2)$ the Lévy measure of which has the density $s(x) = cx^{-\alpha-1},\ x > \ 0$, R. A. Doney in [Don08] considers the supremum $S_t = \sup_{s\leq t} X_s$ of X. He shows that S_1 behaves as $s(x) \sim cx^{-\alpha-1}$ as $x \to +\infty$.
Recently, Pogány, Tibor K and Nadarajah in [PN15] give a shorter and more general proof of R. A. Doney's previous result [Don08]. They derive the first known closed form expression for $s(x)$ and the corresponding cumulative function, then they obtain

the order of the remainder in the asymptotic expansion of $s(x)$. With the same model, Alexey et al. [KKPW14] find the Mellin transform of the first hitting time of the origin and give an expression for its density.
Here, we show that the cumulative function of the first hitting time for one Lévy process belongs to $\mathcal{C}(\mathbb{R}_+^* \times \mathbb{R}_+^*)$ and for any $x \in \mathbb{R}_+^*$, to $\mathcal{C}(\mathbb{R}_+)$ and then we derive the first known closed form expression which characterizes the law of (τ_x, K_x, L_x).
The paper is organized as follows : Section 2.2 presents the model and the aim of the paper, Section 2.3 studies the regularity of the law of τ_x, Section 2.4 provides the joint law of (τ_x, K_x, L_x).

2.2 Model and Problem to solve

On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let X be a Lévy process, right continuous with left limit (RCLL) starting at 0. It is defined as

$$X_t = mt + W_t + \sum_{i=1}^{N_t} Y_i \tag{2.2.1}$$

where $m \in \mathbb{R}$, W is a standard Brownian motion, N a Poisson process with constant positive intensity λ and $(Y_i, i \in \mathbb{N}^*)$ is a sequence of independent identically distributed random variables with a distribution function F_Y. We suppose that the following σ-fields $\sigma(N_t, t \geq 0), \sigma(Y_i, i \in \mathbb{N}^*)$ and $\sigma(W_t, t \geq 0)$ are independent. We are interested in the first hitting time at a level $x > 0$,

$$\tau_x := \inf\{t \geq 0, X_t \geq x\}. \tag{2.2.2}$$

We also consider the overshoot and the undershoot respectively defined by

$$K_x := X_{\tau_x} - x, \tag{2.2.3}$$

$$L_x := x - X_{\tau_{x^-}}. \tag{2.2.4}$$

For $\tilde{X}_t := mt + W_t$ and $\tilde{\tau}_x := \inf\{t \geq 0; \tilde{X}_t \geq x\}$, I. Karatzas and S. E. Shreve in [KS12] shown that the law of $\tilde{\tau}_x$ is of the form $\tilde{f}(u, x)du + \mathbb{P}(\tilde{\tau}_x = \infty)\delta_\infty(du)$ where

$$\tilde{f}(u, x) = \frac{|\, x \,|}{\sqrt{2\pi u^3}} \exp[-\frac{1}{2u}(x - mu)^2] 1_{]0,+\infty[}(u) \text{ and } \mathbb{P}(\tilde{\tau}_x = \infty) = 1 - e^{mx - |mx|}. \tag{2.2.5}$$

L. Coutin and D. Dorobantu [CD+11] prove the existence of the density $f_{\tau_x}(t, x)$ of τ_x and show that :

$$f_{\tau_x}(t,x) = \begin{cases} \lambda \mathbb{E}(1_{\tau_x > t}(1-F_Y)(x-X_t)) + \mathbb{E}(1_{\tau_x > T_{N_t}} \tilde{f}(t - T_{N_t}, x - X_{T_{N_t}})), & \forall t > 0 \\ \frac{\lambda}{2}(2 - F_Y(x) - F_Y(x_-)) + \frac{\lambda}{4}(F_Y(x) - F_Y(x_-)) & \text{if } t = 0 \end{cases} \tag{2.2.6}$$

and $\mathbb{P}(\tau_x = \infty) = 0$ if and only if $m + \lambda \mathbb{E}(Y_1) \geq 0$.
For a more general jump-diffusion process, Roynette et al. [RVV08] show that the Laplace transform of (τ_x, K_x, L_x) is solution of some kind of random integral equation. The problem addressed in this paper is studying the regularity of the density of τ_x on $]0,+\infty[\times]0,+\infty[$, then at $t=0$ for a strictly positive level x fixed and compute an expression for the joint distribution of the triplet (τ_x, K_x, L_x).

2.3 Regularity of f_{τ_x}

This section deals with the regularity. The first subsection 2.3.1 treats the continuity on $]0,\infty[\times]0,\infty[$ as well as the last one 2.3.2 studies the regularity with respect to time at 0.

2.3.1 Regularity of the density f_{τ_x} on $]0,\infty[\times]0,\infty[$

Here, our goal is to prove Proposition 2.3.1 which asserts the regularity of τ_x density law on $]0,+\infty[\times]0,+\infty[$.

Proposition 2.3.1 *The application defined on* $]0,+\infty[\times]0,+\infty[$ *by*

$$(t,x) \longrightarrow f_{\tau_x}(t,x) = \lambda \mathbb{E}\left(\mathbf{1}_{\{\tau_x > t\}}(1-F_Y)(x-X_t)\right) + \mathbb{E}\left(1_{\tau_x > T_{N_t}} \tilde{f}(t - T_{N_t}, x - X_{T_{N_t}})\right)$$

is continuous.

Proof. Let be $(t_0, x_0) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$. We denote

$$\Omega' = \left\{\omega \in \Omega \text{ such that } T_{N_{t_0}(\omega)}(\omega) \neq t_0, \quad \tau_{x_0}(\omega) \neq t_0, \quad x_0 - X_{t_0}(\omega) \notin D_{F_Y}\right\}$$

where D_{F_Y} is the set of the points of discontinuity of the distribution function F_Y.
We assert that $\mathbb{P}(\Omega') = 1$: Indeed, we have

$$1 - \mathbb{P}(\Omega') \leq \mathbb{P}(\tau_x = 0) + \mathbb{P}(X_{t_0} = x_0) + \mathbb{P}(T_{N_{t_0}} = t_0) + \mathbb{P}(x_0 - X_{t_0} \in D_{F_Y}).$$

Since τ_{x_0}, $T_{N_{t_0}}$ and X_{t_0} have a densities with respect to the Lebesgue measure and D_{F_Y} is almost countable, it follows

$$\mathbb{P}(\tau_x = 0) = \mathbb{P}(X_{t_0} = x_0) = \mathbb{P}(T_{N_{t_0}} = t_0) = \mathbb{P}(x_0 - X_{t_0} \in D_{F_Y}) = 0.$$

Note that $\mathbf{1}_{\{\tau_x > T_{N_t}\}} = \mathbf{1}_{\{X^*_{T_{N_t}} < x\}}$ where $X^*_t = \sup_{u \leq t} X_u$.
The random variable $X^*_{T_{N_{t_0}}}$ is reached by the process X either before $T_{N_{t_0}}$ either at $T_{N_{t_0}}$.
Let ω be fixed in Ω'.
• On the event $\{\tau_{x_0} > T_{N_{t_0}}\} = \{X^*_{T_{N_{t_0}}} < x_0\}$, $X^*_{T_{N_{t_0}(\omega)}(\omega)}(\omega) = X_{T_{N_{t_0}(\omega)}(\omega)}(\omega) \neq x_0$.
Thus, if $X^*_{T_{N_{t_0}(\omega)}(\omega)}(\omega)$ is reached at $T_{N_{t_0}(\omega)}(\omega)$, either it is less than x_0 or more than x_0.
If $X^*_{T_{N_{t_0}(\omega)}(\omega)}(\omega) = X_{T_{N_{t_0}(\omega)}(\omega)}(\omega) < x_0$, since $T_{N_{t_0}(\omega)}(\omega) < t_0 < T_{N_{t_0}(\omega)+1}(\omega)$, there exists $\varepsilon_0(\omega) > 0$ and $\delta_0(\omega) > 0$ such that for any t satisfying $|t - t_0| \leq \delta_0(\omega)$ we have $x_0 - \varepsilon_0(\omega) < X^*_{T_{N_t(\omega)}(\omega)}(\omega) < x_0 + \varepsilon_0(\omega)$.
That means for (t, x) such that $|t - t_0| < \delta_0(\omega)$ and $|x - x_0| < \varepsilon_0(\omega)$, we have

$$\mathbf{1}_{\{\tau_x(\omega) > T_{N_t(\omega)}(\omega)\}} = \mathbf{1}_{\{\tau_{x_0}(\omega) > T_{N_{t_0}(\omega)}(\omega)\}} = 1.$$

If $X^*_{T_{N_{t_0}(\omega)}(\omega)}(\omega) = X_{T_{N_{t_0}(\omega)}(\omega)}(\omega) > x_0$, since $T_{N_{t_0}(\omega)}(\omega) < t_0 < T_{N_{t_0}(\omega)+1}(\omega)$, there exists $\varepsilon_1(\omega) > 0$ and $\delta_1(\omega) > 0$ such that for any t satisfying $|t - t_0| \leq \delta_1(\omega)$ we have $x_0 - \varepsilon_0(\omega) < X^*_{T_{N_t}} < x_0 + \varepsilon_0(\omega)$.
That means for $|t - t_0| < \delta_1(\omega)$ and $|x - x_0| < \varepsilon_1(\omega)$,

$$\mathbf{1}_{\{\tau_x(\omega) > T_{N_t(\omega)}(\omega)\}} = \mathbf{1}_{\{\tau_{x_0}(\omega) > T_{N_{t_0}(\omega)}(\omega)\}} = 0.$$

In the two above cases, we conclude that for any $\omega \in \Omega$, there exists $\delta(\omega) > 0$ and $\varepsilon(\omega) > 0$ such that :

$$|t - t_0| < \delta(\omega) \text{ and } |x - x_0| < \varepsilon(\omega) \text{ imply that } \mathbf{1}_{\{\tau_x(\omega) > T_{N_t(\omega)}(\omega)\}} = \mathbf{1}_{\{\tau_{x_0}(\omega) > T_{N_{t_0}(\omega)}(\omega)\}} \tag{2.3.2}$$

• If $X^*_{T_{N_{t_0}(\omega)}(\omega)}(\omega)$ is reached at $v < T_{N_{t_0}(\omega)}(\omega)$, either it is less than x_0 or more than x_0.
If $X^*_{T_{N_{t_0}(\omega)}(\omega)}(\omega) = X_v(\omega) < x_0$, since $T_{N_{t_0}(\omega)}(\omega) < t_0 < T_{N_{t_0}(\omega)+1}(\omega)$, there exists $\varepsilon_4(\omega) > 0$ and $\delta_4(\omega) > 0$ such that for any t satisfying $|t - t_0| \leq \delta_4(\omega)$ we have $x_0 - \varepsilon_4(\omega) < X^*_{T_{N_t(\omega)}}(\omega) < x_0 + \varepsilon_4(\omega)$.
That means for (t, x) such that $|t - t_0| < \delta_4(\omega)$ and $|x - x_0| < \varepsilon_4(\omega)$, we have

$$\mathbf{1}_{\{\tau_x > T_{N_t}\}} = \mathbf{1}_{\{\tau_{x_0} > T_{N_{t_0}}\}} = 1.$$

If $X^*_{T_{N_{t_0}(\omega)}(\omega)}(\omega) = X_v(\omega) > x_0$, since $T_{N_{t_0}(\omega)}(\omega) < t_0 < T_{N_{t_0}(\omega)+1}(\omega)$, there exists $\varepsilon_5(\omega) > 0$ and $\delta_5(\omega) > 0$ such that for any t satisfying $|t - t_0| \leq \delta_5(\omega)$ we have $x_0 - \varepsilon_5(\omega) < X^*_{T_{N_t}} < x_0 + \varepsilon_5(\omega)$. That means for $|t - t_0| < \delta_5(\omega)$ and $|x - x_0| < \varepsilon_5(\omega)$,

$$\mathbf{1}_{\{\tau_x > T_{N_t}\}} = \mathbf{1}_{\{\tau_{x_0} > T_{N_{t_0}}\}} = 0.$$

If $X^*_{T_{N_{t_0}(\omega)}(\omega)}(\omega) = X_v(\omega) = x_0$, we consider a function which is equal to $\tilde{f}$ on $]0,+\infty[\times]0,+\infty[$ and 0 on $]0,+\infty[\times\mathbb{R}_-$. We denote it $\tilde{f}$ again. This function is everywhere continuous. Since $T_{N_{t_0}(\omega)}(\omega) < t_0 < T_{N_{t_0}(\omega)+1}(\omega)$, and $(t,x) \longrightarrow \tilde{f}(t - T_{N_t(\omega)}(\omega), x - X_{T_{N_t(\omega)}(\omega)}(\omega))$ is continuous at (t_0, x_0), there exists $\varepsilon_6(\omega) > 0$ and $\delta_6(\omega) > 0$ such that for any (t,x) satisfying $|t - t_0| \leq \delta_6(\omega)$ and $|x - x_0| \leq \varepsilon_6(\omega)$, we have

$$\begin{aligned} &\lim_{(t,x)\to(t_0,x_0)} \mathbf{1}_{\{X^*_{T_{N_t(\omega)}(\omega)}(\omega)<x\}} \tilde{f}(t - T_{N_t(\omega)}(\omega), x - X_{T_{N_t(\omega)}(\omega)}(\omega)) \\ &= \mathbf{1}_{\{X^*_{T_{N_{t_0}(\omega)}(\omega)}<x_0\}} \tilde{f}(t_0 - T_{N_{t_0(\omega)}(\omega)}, x_0 - X_{T_{N_{t_0}(\omega)}(\omega)}(\omega)) = 0. \end{aligned} \tag{2.3.3}$$

Using uniform integrability of the family $\left(1_{\tau_x > T_{N_t}} \tilde{f}(t - T_{N_t}, x - X_{T_{N_t}}), \quad t > 0, \quad x > 0\right)$, obtained from Lemma 2.6.2, the continuity of

$$(t,x) \longrightarrow \mathbb{E}\left(1_{\tau_x > T_{N_t}} \tilde{f}(t - T_{N_t}, x - X_{T_{N_t}})\right)$$

at (t_0, x_0) follows.

Since the family $\left((\mathbf{1}_{\{\tau_x > t\}}(1 - F_Y)(x - X_t), t > 0, \ x > 0\right)$ is bounded by 1, it is uniformly integrable and we proceed analogously to the previous to obtain the continuity at (t_0, x_0) of

$$(t,x) \longrightarrow \mathbb{E}\left(\mathbf{1}_{\{\tau_x > t\}}(1 - F_Y)(x - X_t)\right).$$

∎

We now study the regularity with respect to time at 0.

2.3.2 Regularity of f_{τ_x} with respect to time at 0

The next two propositions show that for any fixed $x > 0$, the f_{τ_x} density law is continuous with respect to time at 0.

Proposition 2.3.4 *Let be $x > 0$ fixed, we have*

$$\lim_{t \mapsto 0} \mathbb{E}\left(\mathbf{1}_{\tau_x > t}[1 - F_Y](x - X_t)\right) = \frac{1}{2}\left(2 - F_Y(x) - F_Y(x_-)\right).$$

Proof. We have

$$\mathbb{E}(\mathbf{1}_{\tau_x > t}[1 - F_Y](x - X_t)) = \mathbb{E}(\mathbf{1}_{\{N_t = 0\}}\mathbf{1}_{\tau_x > t}[1 - F_Y](x - X_t)) + \mathbb{E}(\mathbf{1}_{\{N_t > 0\}}\mathbf{1}_{\tau_x > t}[1 - F_Y](x - X_t)).$$

But

$$\begin{aligned} (i)\quad & 0 \leq \lim_{t \mapsto 0} \mathbb{E}(\mathbf{1}_{\{N_t > 0\}}\mathbf{1}_{\tau_x > t}[1 - F_Y](x - X_t)) \\ & \leq \lim_{t \mapsto 0} \mathbb{P}(N_t \geq 1) = \lim_{t \mapsto 0} 1 - e^{-at} = 0 \\ (ii)\quad & \mathbb{E}\left(\mathbf{1}_{\{N_t = 0\}}\mathbf{1}_{\tau_x > t}[1 - F_Y](x - X_t)\right) = e^{-at}\mathbb{E}\left(\mathbf{1}_{\{\tilde{X}^*_t < x\}}[1 - F_Y](x - \tilde{X}_t)\right) \end{aligned}$$

We first remark that

$$\mathbb{P}(\{\tilde{X}_t < x \leq \tilde{X}_t^*\}) =$$
$$|\mathbb{E}\left(\mathbf{1}_{\{\tilde{X}_t^*<x\}}[1-F_Y](x-\tilde{X}_t)\right) - \mathbb{E}\left(\mathbf{1}_{\{\tilde{X}_t<x\}}[1-F_Y](x-\tilde{X}_t)\right)| \leq |\mathbb{E}(\mathbf{1}_{\{\tilde{X}_t^*<x\}} - \mathbf{1}_{\{\tilde{X}_t<x\}})|.$$

The density function of $(\tilde{X}_t^*, \tilde{X}_t)$ given by Corollary 3.2.1.2 p. 147 [JYC09] yields

$$\mathbb{P}(\{\tilde{X}_t < x \leq \tilde{X}_t^*\}) = \int_x^\infty db \int_{-\infty}^b da \frac{2(2b-a)}{\sqrt{2\pi t^3}} \exp\left[-\frac{(2b-a)^2}{2t} + ma - \frac{m^2}{2}t\right].$$

This integral is bounded (with respect to a multiplicative constant C) by

$$\mathbb{P}(\{\tilde{X}_t < x \leq \tilde{X}_t^*\}) \leq C \int_x^\infty db \int_{-\infty}^b da \frac{(2b-a)}{\sqrt{t^3}} \exp\left[-\frac{(2b-a)^2}{2t}\right].$$

Notice that the application $t \to \frac{(2b-a)}{\sqrt{t^3}} \exp\left[-\frac{(2b-a)^2}{2t}\right]$ is decreasing to 0 when $t \downarrow 0$. So Lebesgue's monotonous convergence theorem proves that

$$\lim_{t\to 0} \mathbb{P}(\{\tilde{X}_t < x \leq \tilde{X}_t^*\}) = 0. \tag{2.3.5}$$

Secondly,

$$\mathbb{E}\left(\mathbf{1}_{\{\tilde{X}_t<x\}}[1-F_Y](x-\tilde{X}_t)\right) = \mathbb{E}\left(\mathbf{1}_{\{\tilde{X}_t\leq 0\}}[1-F_Y](x-\tilde{X}_t)\right) + \mathbb{E}\left(\mathbf{1}_{\{0\leq\tilde{X}_t<x\}}[1-F_Y](x-\tilde{X}_t)\right).$$

Since F_Y is bounded and RCLL and $\tilde{X}$ continuous, Lebesgue's dominated convergence theorem yields

$$\lim_{t\mapsto 0} \mathbb{E}\left(\mathbf{1}_{\{N_t=0\}}\mathbf{1}_{\tau_x>t}[1-F_Y](x-X_t)\right) = \frac{1}{2ÃŮ}\left(2 - F_Y(x) - F_Y(x_-)\right).$$

■

Proposition 2.3.6 *Let be $x > 0$ fixed, we have*

$$\lim_{t\to 0} \mathbb{E}\left(\mathbf{1}_{\{\tau_x>T_{N_t}\}}\tilde{f}(t-T_{N_t}, x-X_{T_{N_t}})\right) = \frac{\lambda}{4}\left(F_Y(x) - F_Y(x_-)\right).$$

Proof. (i) We first deal with

$$\begin{aligned}\lim_{t\to 0} \mathbb{E}\left(\mathbf{1}_{\{N_t=0,\tau_x>T_{N_t}\}}\tilde{f}(t-T_{N_t}, x-X_{T_{N_t}})\right) &= lim_{t\to 0}\mathbb{E}(\mathbf{1}_{\{N_t=0\}})\tilde{f}(t,x)) \\ &= lim_{t\to 0}(1-e^{-at})\tilde{f}(t,x) \\ &= 0\end{aligned}$$

using Definition (2.2.5).

(ii) Then we deal with

$$\mathbb{E}\left(\mathbf{1}_{\{N_t\geq 2,\tau_x>T_{N_t}\}}\tilde{f}(t-T_{N_t},x-X_{T_{N_t}})\right).$$

We use Lemma A.1 of [CD+11] for $p=1$, law of G being the Gaussian law $\mathcal{N}(0,1)$, then

$$\mathbb{E}\left(\tilde{f}(u,\mu+\sigma G)\mathbf{1}_{\{\mu+\sigma G>0\}}\right)=\frac{1}{\sqrt{2\pi}}\frac{e^{-\frac{(\mu-mu)^2}{2(\sigma^2+u)}}}{\sqrt{u}(\sigma^2+u)}\mathbb{E}\left[\left(\sigma G+\sqrt{\frac{u}{\sigma^2+u}}(\mu-mu)+m\sqrt{u(\sigma^2+u)}\right)_+\right].$$

Using $C_{1/2}=\sup\sqrt{y}e^{-\frac{y^2}{2}}$, $y=\frac{\mu-mu}{\sqrt{\sigma^2+u}}$,

$$\mathbb{E}\left(\tilde{f}(u,\mu+\sigma G)\mathbf{1}_{\{\mu+\sigma G>0\}}\right)\leq\frac{1}{\sqrt{2\pi}}\left[\frac{1}{\sqrt{u}(\sigma^2+u)}\left(\mathbb{E}|\sigma G|+\sqrt{u}C_{1/2}+|m|\sqrt{u(\sigma^2+u)}\right)\right].$$

For $u=t-T_{N_t}$, $\sigma^2=T_{N_t}$ and $u+\sigma^2=t$, we obtain using the independence between the Poisson process and the Brownian motion

$$\mathbb{E}\left(\mathbf{1}_{\{N_t\geq 2\}}\mathbf{1}_{\{\tau_x>T_{N_t}\}}\tilde{f}(t-T_{N_t},x-X_{T_{N_t}})\right)\leq\mathbb{E}\left(\mathbf{1}_{\{N_t\geq 2\}}\frac{1}{t\sqrt{2\pi}}\left[\frac{E[|\sigma G|]}{\sqrt{t-T_{N_t}}}+C_{1/2}+|m|\sqrt{t}\right]\right).$$

So, using $\mathbb{P}(\{N_t\geq 2\})=0(t^2)$ and Lemma 2.6.1,

$$\lim_{t\to 0}\mathbb{E}\left(\mathbf{1}_{\{N_t\geq 2\}}\mathbf{1}_{\{\tau_x>T_{N_t}\}}\tilde{f}(t-T_{N_t},x-X_{T_{N_t}})\right)=0.$$

(iii) Finally we deal with

$$A_t=\mathbb{E}\left(\mathbf{1}_{\{N_t=1,\tau_x>T_{N_t}\}}\tilde{f}(t-T_{N_t},x-X_{T_{N_t}})\right)=\mathbb{E}\left(\mathbf{1}_{\{N_t=1,\tau_x>T_1\}}\tilde{f}(t-T_1,x-X_{T_1})\right).$$

Since the event $\{N_t=1,\tau_x>T_1\}=\{T_1\leq t<T_2,\tilde{X}^*_{T_1}<x,\tilde{X}_{T_1}+Y_1<x\}$ we have

$$A_t=\mathbb{E}\left(\mathbf{1}_{\{T_1\leq t<T_2,\tilde{X}^*_{T_1}<x,\tilde{X}_{T_1}+Y_1<x\}}\tilde{f}(t-T_1,x-\tilde{X}_{T_1}-Y_1)\right).$$

Using the law of T_2-T_1 and its independence from $T_1,\tilde{X}^*_{T_1},\tilde{X}_{T_1},Y_1$, it follows that

$$A_t=\mathbb{E}\left(e^{-a(t-T_1)}\mathbf{1}_{\{T_1\leq t,\tilde{X}^*_{T_1}<x,\tilde{X}_{T_1}+Y_1<x\}}\tilde{f}(t-T_1,x-\tilde{X}_{T_1}-Y_1)\right)$$

Using the law of T_1 and the independence between T_1 and $(\tilde{X},Y)$, yields

$$A_t=ae^{-at}\int_0^t du\mathbb{E}\left(\mathbf{1}_{\{\tilde{X}^*_u<x,\tilde{X}_u+Y_1<x\}}\tilde{f}(t-u,x-\tilde{X}_u-Y_1)\right).$$

Since $\tilde{X}_u$ and Y_1 are independent, conditioning by $(\tilde{X}_u, Y_1)$ and using Lemma 2.6.5 for $c = x$,

$$A_t = ae^{-at}\int_0^t du\mathbb{E}\left(\mathbf{1}_{\{\tilde{X}_u<\min(x,x-Y_1)\}}\left[1-e^{-\frac{2x^2-2x\tilde{X}_u}{u}}\right]f(t-u,x-\tilde{X}_u-Y_1)\right).$$

The change of variable $u = ts$ leads to

$$A_t = ae^{-at}\int_0^1 tds\mathbb{E}\left(\mathbf{1}_{\{\tilde{X}_{st}<\min(x,x-Y_1)\}}\left[1-e^{-\frac{2x^2-2x\tilde{X}_{ts}}{ts}}\right]f(t(1-s),x-\tilde{X}_{st}-Y_1)\right).$$

The density of $\tilde{X}_{ts}$: $\frac{1}{\sqrt{2\pi ts}}e^{-\frac{(g-mts)^2}{2ts}}$ and $\tilde{f}$ defined in (2.2.5) yield

$$A_t = ae^{-at}\int_0^1 tds$$
$$\mathbb{E}\left(\int_{-\infty}^{\min(x,x-Y_1)}\left[1-e^{-\frac{2x^2-2xg}{ts}}\right]\frac{x-Y_1-g}{\sqrt{2\pi t^3(1-s)^3}}e^{-\frac{(x-g-Y_1-mt(1-s))^2}{2t(1-s}}\frac{1}{\sqrt{2\pi ts}}e^{-\frac{(g-mts)^2}{2ts}}dg\right).$$

Let be $z = x - Y_1 - mt(1-s)$, $y = mts$, $u = t(1-s)$ and $v = ts$, with

$$\frac{vz+uy}{u+v} = \frac{ts[x-Y_1-mt(1-s)]+t(1-s)mts}{t} = s(x-Y_1),$$
$$z-y = x-Y_1-mt(1-s)-mts = x-Y_1-mt.$$

By Lemma 2.6.4,

$$A_t = ae^{-at}\int_0^1 tds$$
$$\mathbb{E}\left(\int_{-\infty}^{\min(x,x-Y_1)}\left[1-e^{-\frac{2x^2-2xg}{ts}}\right]\frac{x-Y_1-g}{\sqrt{(2\pi)^2t^4(1-s)^3s}}e^{-\frac{(g-s(x-Y_1))^2}{2ts(1-s)}-\frac{(x-Y_1-mt)^2}{2t^2s(1-s)}}dg\right).$$

A new change of variable $g' = \frac{g-s(x-Y_1)}{\sqrt{ts(1-s)}}$ meaning $g = \sqrt{ts(1-s)}g' + s(x-Y_1)$, and $x - Y_1 - g = (x-Y_1)(1-s) - \sqrt{ts(1-s)}g'$ implies

$$A_t = ae^{-at}\int_0^1 ds\int dg'\sqrt{t^3s(1-s)}$$
$$\mathbb{E}\left(\mathbf{1}_{\{g'<\frac{\min(x,x-Y_1)-s(x-Y_1)}{\sqrt{ts(1-s)}}\}}\left[1-e^{-\frac{2x^2-2x(\sqrt{ts(1-s)}g'+s(x-Y_1))}{ts}}\right]\frac{(x-Y_1)(1-s)-\sqrt{ts(1-s)}g'}{\sqrt{(2\pi)^2t^4(1-s)^3s}}e^{-\frac{(g')^2}{2}-\frac{(x-Y_1-mt)^2}{2t^2s(1-s)}}\right).$$

This would mean

$$A_t = ae^{-at}\int_0^1 ds\int dg'$$
$$\mathbb{E}\left[\mathbf{1}_{\{g'<\frac{\min(x,x-Y_1)-s(x-Y_1)}{\sqrt{ts(1-s)}}\}}\left[1-e^{-\frac{2x^2-2x(\sqrt{ts(1-s)}g'+s(x-Y_1))}{ts}}\right]e^{-\frac{(g')^2}{2}-\frac{(x-Y_1-mt)^2}{2t^2s(1-s)}}\left[\frac{(x-Y_1)}{2\pi\sqrt{t}}-\frac{g'\sqrt{s}}{2\pi\sqrt{1-s}}\right]\right].$$

On the set $\{g' < c\}$, $2x^2 - 2xc > 0$. This would mean $\{g' < \frac{\min(x, x-Y_1-s(x-Y_1)}{\sqrt{ts(1-s)}})\}$, where $c = \frac{x-Y_1-s(x-Y_1)}{\sqrt{ts(1-s)}} - \frac{2x^2-2x(\sqrt{ts(1-s)}g'+s(x-Y_1))}{ts} < 0$, and

$$\left|1 - e^{-\frac{2x^2-2x(\sqrt{ts(1-s)}g'+s(x-Y_1))}{t}}\right| \mathbf{1}_{\{g' < \frac{\min(x,x-Y_1)-s(x-Y_1)}{\sqrt{ts(1-s)}}\}} \leq 1;$$

with

$$\lim_{t\to 0} \left[1 - e^{-\frac{2x^2-2x(\sqrt{ts(1-s)}g'+s(x-Y_1))}{t}}\right|]\mathbf{1}_{\{g' < \frac{\min(x,x-Y_1)-s(x-Y_1)}{\sqrt{ts(1-s)}}\}} = 1.$$

Morever,

$$\begin{aligned}\frac{|x-Y_1|}{\sqrt{t}} e^{-\frac{(x-Y_1-mt)^2}{2t^2 s(1-s)}} &\leq \frac{|mt| + |x-Y_1-mt|}{\sqrt{t}} e^{-\frac{(x-Y_1-mt)^2}{2t^2 s(1-s)}} \\ &\leq \frac{|mt| + C_{1/2}\sqrt{t^2 s(1-s)}}{\sqrt{t}} = \sqrt{t}\left(|m| + C_{1/2}\sqrt{s(1-s)}\right)\end{aligned}$$

Thus

$$\lim_{t\to 0} \frac{|x-Y_1|}{\sqrt{t}} e^{-\frac{(x-Y_1-mt)^2}{2t^2 s(1-s)}} = 0.$$

Finally,

$$e^{-\frac{(g')^2}{2} - \frac{(x-Y_1-mt)^2}{2t^2 s(1-s)}} \frac{|g'|\sqrt{s}}{2\pi\sqrt{1-s}} \leq e^{-\frac{(g')^2}{2}} \frac{|g'|\sqrt{s}}{2\pi\sqrt{1-s}}$$

and

$$\lim_{t\to 0} \mathbf{1}_{\{g' < \frac{\min(x,x-Y_1)-s(x-Y_1)}{\sqrt{ts(1-s)}}\}} e^{-\frac{(g')^2}{2} - \frac{(x-Y_1-mt)^2}{2t^2 s(1-s)}} \frac{g'\sqrt{s}}{2\pi\sqrt{1-s}} = e^{-\frac{(g')^2}{2}} \frac{g'\sqrt{s}}{2\pi\sqrt{1-s}} \mathbf{1}_{\{Y_1=0\}} \mathbf{1}_{\{g'<0\}}.$$

By Lebesgue's dominated convergence theorem ,

$$\lim_{t\to 0} A_t = -a\mathbb{P}(Y_1 = x) \int_0^1 ds \int_{\{g'<0\}} dg' e^{-\frac{(g')^2}{2}} \frac{g'\sqrt{s}}{2\pi\sqrt{1-s}}$$

and $\lim_{t\to 0} A_t = a\mathbb{P}(Y_1 = x) \int_0^1 \frac{\sqrt{s}}{2\pi\sqrt{1-s}} ds = \frac{a\mathbb{P}(Y=x)}{4}$. Indeed, $\beta(3/2, 1/2) = \frac{1}{2}\gamma(1/2)^2$ and $\gamma(1/2) = \pi$. ■

2.4 The joint law

If the default time coincides with a jump time of the process X, it is also important to have information on the deficit, namely overshoot, right after the default and on the surplus, namely undershoot of the firm, immediately before the default time. Therefore, A. Volpi et al. [RVV08] deal with the asymptotic behavior of the triplet (τ_x, K_x, L_x) by showing after a renormalization of τ_x that it converges in distribution as x goes to ∞. To characterize the joint law of (τ_x, K_x, L_x) on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ for any $x > 0$, this section's main result is the next theorem. From now on, we consider two continuous bounded functions Φ and Ψ. The methodology used for the proof is inspired from Coutin-Dorobantu [CD$^+$11] who study the law of τ_x which is here the first marginal distribution. It consists in splitting $\mathbb{E}(1_{t<\tau_x\leq t+h}\Phi(K_x)\Psi(L_x))$ following the values of $N_{t+h} - N_t$ for $t = 0$ then for $t > 0$.

Theorem 2.4.1 *The joint law of the triplet (τ_x, K_x, L_x), conditionally on $\{\tau_x < \infty\}$, is given on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ by $p(.,.,.,x)$ such that :*

$$\begin{aligned} p(0, dk, dl) = &\frac{\lambda}{4}[F_Y(x) - F_Y(x_-)]\delta_{\{0,0,0\}}(dt, dk, dl) + \lambda F_l(dk)\delta_{\{0,x\}}(dt, dl) \\ &+ \frac{\lambda}{2}\Delta F_Y(x)\delta_{\{0,0,x\}}(dt, dk, dl) \end{aligned}$$

and for every $t > 0$,

$$\begin{aligned} p(dt, dk, dl) = &\mathbb{E}[\mathbf{1}_{\{\tau_x > T_{N_t}\}}\tilde{f}(t - T_{N_t}, x - X_{T_{N_t}})]\delta_{\{0,0\}}(dk, dl)dt \\ &+ \lambda\mathbb{E}\left[\mathbf{1}_{\{k\geq 0, l\geq 0\}}\mathbf{1}_{\{\tau_x > T_{N_t}\}}f_0(x - X_{T_{N_t}} - l)\right] F_l(dk)dldt \\ &- \lambda\mathbb{E}\left[\mathbf{1}_{\{k\geq 0, l\geq 0\}}\mathbf{1}_{\{\tau_x > T_{N_t}\}}f_0(X_{T_{N_t}} - x - l)\exp(2m(x - X_{T_{N_t}}))\right] F_l(dk)dldt \end{aligned}$$

where f_0 is the density function of a Gaussian random variable with mean $\mu = m(t - T_{N_t})$ and variance $\sigma^2 = t - T_{N_t}$, $\tilde{f}$ is defined by (2.2.5), $F_l(dk)$ is the image of $F_Y(dk)$ by the map $y \mapsto y - l$ and $\Delta F_Y(x) = F_Y(x) - F_Y(x_-)$.

Remark 2.4.2 *Referring to [RVV08], for all $x > 0$, the first passage time τ_x is finite almost surely if and only if $m + \mathbb{E}(Y_1) \geq 0$.*

To prove the theorem, we use Propositions 2.4.3 and 2.4.5. Indeed, Proposition 2.4.3 gives the law at time $t = 0$ and Proposition 2.4.5 deals with time $t > 0$. The proof of Proposition 2.4.5 is broken in three parts : Step 1, Step 2 and Proposition 2.4.8.

Proposition 2.4.3

$$\lim_{h\to 0}\frac{1}{h}\mathbb{E}(1_{\tau_x\le h}\Phi(K_x)\Psi(L_x)) = \Phi(0)\Psi(0)\frac{\lambda}{4}[F_Y(x)-F_Y(x_-)] + \lambda\mathbb{E}[\Phi(Y_1-x)\Psi(x)\mathbf{1}_{\{Y_1>x\}}] + \frac{\lambda}{2}\mathbb{E}[\Phi(0)\Psi(Y_1)\mathbf{1}_{\{Y_1=x\}}].$$

Proof. We split $\mathbb{E}(1_{\tau_x\le h}\Phi(K_x)\Psi(L_x))$ according to the values of N_h :

$$\mathbb{E}(1_{\tau_x\le h}\Phi(K_x)\Psi(L_x)) = \mathbb{E}(1_{\tau_x\le h}\mathbf{1}_{\{N_h=0\}}\Phi(K_x)\Psi(L_x)) + \mathbb{E}(1_{\tau_x\le h}\mathbf{1}_{\{N_h=1\}}\Phi(K_x)\Psi(L_x) + \mathbb{E}(1_{\tau_x\le h}\mathbf{1}_{\{N_h\ge 2\}}\Phi(K_x)\Psi(L_x).$$

By hypothesis, Φ and Ψ are bounded and on the event $\{N_h=0\}$, the law of τ_x is the one of $\tilde{\tau}_x$, so has the continuous density $\tilde{f}(.,x)$ defined in (2.2.5). Since

$$\mathbb{E}(1_{\tau_x\le h}\mathbf{1}_{\{N_h=0\}}\Phi(K_x)\Psi(L_x)) = \Phi(0)\Psi(0)\mathbb{P}(\tilde{\tau}_x\le h)$$

and

$$|\mathbb{E}(1_{\tau_x\le h}\mathbf{1}_{\{N_h\ge 2\}}\Phi(K_x)\Psi(L_x)| \le ||\Phi||_\infty||\Psi||_\infty\mathbb{P}(N_h\ge 2),$$

it follows that

$$\lim_{h\to 0}\frac{1}{h}\mathbb{E}(1_{\tau_x\le h}\mathbf{1}_{\{N_h=0\}}\Phi(K_x)\Psi(L_x)) = 0 \tag{2.4.4}$$

and

$$\lim_{h\to 0}\frac{1}{h}\mathbb{E}(1_{\tau_x\le h}\mathbf{1}_{\{N_h\ge 2\}}\Phi(K_x)\Psi(L_x) = 0.$$

It remains to study $\mathbb{E}(1_{\tau_x\le h}\mathbf{1}_{\{N_h=1\}}\Phi(K_x)\Psi(L_x)$. For this purpose, we split it as in Coutin and Dorobantu [CD+11] according to the relative positions of τ_x and T_1 the first jump time of the process N.

$$\begin{aligned}\mathbb{E}(1_{\tau_x\le h}\mathbf{1}_{\{N_h=1\}}\Phi(K_x)\Psi(L_x)) &= \mathbb{E}(1_{\tau_x\le h}\mathbf{1}_{\{N_h=1\}}\mathbf{1}_{\{\tau_x<T_1\}}\Phi(K_x)\Psi(L_x))\\ &+ \mathbb{E}(1_{\tau_x\le h}\mathbf{1}_{\{N_h=1\}}\mathbf{1}_{\{\tau_x=T_1\}}\Phi(K_x)\Psi(L_x))\\ &+ \mathbb{E}(1_{\tau_x\le h}\mathbf{1}_{\{N_h=1\}}\mathbf{1}_{\{\tau_x>T_1\}}\Phi(K_x)\Psi(L_x))\\ &= A_1(h)+A_2(h)+A_3(h).\end{aligned}$$

On the set $\{\tau_x\le h, \tau_x\ne T_1, N_h=1\}$, the process X is continuous at τ_x. and $K_x=L_x=0$. Therefore, Step 1 and Step 3 of Subsection 2.1 in [CD+11] imply that :

$$\lim_{h\to 0}\frac{1}{h}A_1(h) = 0$$

and

$$\lim_{h\to 0}\frac{1}{h}\mathbb{E}(1_{\tau_x\le h}\mathbf{1}_{\{N_h=1\}}\mathbf{1}_{\{\tau_x>T_1\}})=\frac{\lambda}{4}[F_Y(x)-F_Y(x_-)].$$

To study $A_2(h)$, we observe that :

$$\begin{gathered}\tau_x=T_1 \text{ if and only if } \tilde{X}^*_{T_1}<x \text{ and } \tilde{X}_{T_1}+Y_1>x\\ \text{and on this set } K_x=\tilde{X}_{T_1}+Y_1-x,\ L_x=x-\tilde{X}_{T_1} \text{ and } Y_1>0.\end{gathered}$$

Therefore ,

$$A_2(h)=\mathbb{E}\left(\mathbf{1}_{\{T_1\le h<T_2\}}\mathbf{1}_{\{\tilde{X}^*_{T_1}<x,\tilde{X}_{T_1}+Y>x\}}\Phi(\tilde{X}_{T_1}+Y-x)\Psi(x-\tilde{X}_{T_1})\right).$$

The independence of $(S_i, i\ge 1)$ and $(Y_1,\tilde{X},\tilde{\tau}_x)$ leads after integrating with respect to S_2, then S_1 to

$$\frac{1}{h}A_2(h)=\frac{\lambda e^{-\lambda h}}{h}\int_0^h\mathbb{E}\left(\mathbf{1}_{\{\tilde{X}^*_u<x<\tilde{X}_u+Y_1\}}\Phi(\tilde{X}_u+Y_1-x)\Psi(x-\tilde{X}_u)\right)du$$

$$|\mathbb{E}\left(\mathbf{1}_{\{\tilde{X}^*_u<x<\tilde{X}_u+Y_1\}}\Phi(\tilde{X}_u+Y_1-x)\Psi(x-\tilde{X}_u)\right)-\mathbb{E}\left(\mathbf{1}_{\{\tilde{X}_u<x<\tilde{X}_u+Y_1\}}\Phi(\tilde{X}_u+Y_1-x)\Psi(x-\tilde{X}_u)\right)|$$

is less than $||\Phi||_\infty||\Psi||_\infty\mathbb{P}\left(\tilde{X}_u<x<\tilde{X}^*_u\right)$ which, by (2.3.5), goes to zero when u goes to zero. Hence, we obtain

$$\lim_{h\to 0}\frac{1}{h}A_2(h)=\lim_{h\to 0}\frac{\lambda e^{-\lambda h}}{h}\int_0^h\mathbb{E}\left(\mathbf{1}_{\{\tilde{X}_u<x<\tilde{X}_u+Y_1\}}\Phi(\tilde{X}_u+Y_1-x)\Psi(x-\tilde{X}_u)\right)du.$$

But, we have the equality of the sets

$$\{\tilde{X}_u<x<\tilde{X}_u+Y_1\}=\Big\{\{\tilde{X}_u<x<\tilde{X}_u+Y_1\}\cap\{Y_1\neq x\}\Big\}\cup\Big\{\{\tilde{X}_u<x<\tilde{X}_u+Y_1\}\cap\{Y_1=x\}\Big\}.$$

It follows that

$$\begin{aligned}\lim_{h\to 0}\frac{1}{h}A_2(h)&=\lim_{h\to 0}\frac{\lambda e^{-\lambda h}}{h}\int_0^h\mathbb{E}\left(\mathbf{1}_{\{\{\tilde{X}_u<x<\tilde{X}_u+Y_1\}\cap\{Y_1\neq x\}\}}\Phi(\tilde{X}_u+Y_1-x)\Psi(x-\tilde{X}_u)\right)du\\&+\lim_{h\to 0}\frac{\lambda e^{-\lambda h}}{h}\int_0^h\mathbb{E}\left(\mathbf{1}_{\{\{\tilde{X}_u<x<\tilde{X}_u+Y_1\}\cap\{Y_1=x\}\}}\Phi(\tilde{X}_u+Y_1-x)\Psi(x-\tilde{X}_u)\right)du.\end{aligned}$$

Since Φ and Ψ are Borel and continuous bounded functions and $\tilde{X}$ continuous, Lebesgue's dominated convergence theorem yields

$$\lim_{h\to 0}\frac{1}{h}A_2(h)=\lambda\mathbb{E}[\Phi(Y_1-x)\Psi(x)\mathbf{1}_{\{Y_1>x\}}]+\frac{\lambda}{2}\mathbb{E}[\Phi(0)\Psi(Y_1)\mathbf{1}_{\{Y_1=x\}}].$$

■

Proposition 2.4.5 *The joint law of the triplet* (τ_x, K_x, L_x)*, conditionally on* $\{\tau_x < \infty\}$*, is defined on* $\mathbb{R}_+^* \times \mathbb{R}_+ \times \mathbb{R}_+$ *as following*

$$
\begin{aligned}
p(dt, dk, dl) = & \ \mathbb{E}(1_{\tau_x > T_{N_t}} \tilde{f}(t - T_{N_t}, x - X_{T_{N_t}}))\delta_{\{0,0\}}(dk, dl)dt \\
& + \mathbf{1}_{\{k \geq 0, l \geq 0\}} \mathbb{E}\left[\mathbf{1}_{\{\tau_x > T_{N_t}\}} \frac{\lambda e^{(-\frac{(x-l-X_{T_{N_t}} - m(t-T_{N_t}))^2}{2(t-T_{N_t})})}}{\sqrt{2\pi(t - T_{N_t})}} \right] F_l(dk)dldt \\
& - \mathbf{1}_{\{k \geq 0, l \geq 0\}} \mathbb{E}\left[\mathbf{1}_{\{\tau_x > T_{N_t}\}} \frac{\lambda e^{(-\frac{(x-X_{T_{N_t}} + l + m(t-T_{N_t}))^2}{2(t-T_{N_t})} + 2m(x - X_{T_{N_t}}))}}{\sqrt{2\pi(t - T_{N_t})}} \right] F_l(dk)dldt
\end{aligned}
$$

where $F_l(dk)$ *is the image of* $F_Y(dk)$ *by the map* $y \mapsto y - l$*.*

Proof. We calculate

$$\lim_{h \longrightarrow 0} \frac{1}{h\text{ÃŮ}} \mathbb{E}(1_{t < \tau_x \leq t+h} \Phi(K_x)\Psi(L_x)) \tag{2.4.6}$$

for $t > 0$ fixed. For this purpose, we split $\mathbb{E}(1_{t<\tau_x \leq t+h}\Phi(K_x)\Psi(L_x))$ according to the values of $N_{t+h} - N_t$ as following :

$$
\begin{aligned}
\mathbb{E}(1_{t<\tau_x \leq t+h}\Phi(K_x)\Psi(L_x)) = & \ \mathbb{E}(1_{t<\tau_x \leq t+h}\Phi(K_x)\Psi(L_x)1_{N_{t+h}-N_t=0}) \\
& + \mathbb{E}(1_{t<\tau_x \leq t+h}\Phi(K_x)\Psi(L_x)1_{N_{t+h}-N_t=1}) \\
& + \mathbb{E}(1_{t<\tau_x \leq t+h}\Phi(K_x)\Psi(L_x)1_{N_{t+h}-N_t \geq 2}).
\end{aligned} \tag{2.4.7}
$$

So we deal the proof with three steps, the third one being Proposition 2.4.8

(i) The third term of the right hand side of (2.4.7) is upper bounded as following :

$$\mathbb{E}(1_{t<\tau_x \leq t+h}\Phi(K_x)\Psi(L_x)1_{N_{t+h}-N_t \geq 2}) \leq (1 - e^{ah} - ahe^{ah})\|\Phi\|_\infty \|\Psi\|_\infty.$$

Therefore

$$\lim_{h \longrightarrow 0} \frac{1}{h\text{ÃŮ}} \mathbb{E}(1_{t<\tau_x \leq t+h}\Phi(K_x)\Psi(L_x)1_{N_{t+h}-N_t \geq 2}) = 0.$$

(ii) Let us study the first term on the right hand side of (2.4.7). On the set

$$\{\omega, \quad N_{t+h}(\omega) - N_t(\omega) = \ 0\}, \text{ we have } L_x = \ 0 = \ K_x.$$

Then

$$\mathbb{E}(1_{t<\tau_x \leq t+h}\Phi(K_x)\Psi(L_x)1_{N_{t+h}-N_t=0}) = \Phi(0)\Psi(0)\mathbb{E}(1_{t<\tau_x \leq t+h}1_{N_{t+h}-N_t=0}).$$

Refer to Equation (4) et seq at Section (2.2) in [CD+11], we have

$$\lim_{h \longrightarrow 0} \frac{1}{h\text{ÃŮ}} \mathbb{E}(1_{t<\tau_x \leq t+h}1_{N_{t+h}-N_t=0}) = \mathbb{E}(1_{\tau_x > T_{N_t}} f(t - T_{N_t}, x - X_{T_{N_t}})).$$

So, we deduce that

$$\lim_{h\longrightarrow 0}\frac{1}{h\tilde{A}\mathring{U}}\mathbb{E}(1_{t<\tau_x\leq t+h}\Phi(K_x)\Psi(L_x)1_{N_{t+h}-N_t=0}) = \Phi(0)\Psi(0)\mathbb{E}(1_{\tau_x>T_{N_t}}f(t-T_{N_t}, x-X_{T_{N_t}})).$$

The third step is Proposition 2.4.8 which deals with the middle term in (2.4.7).

Proposition 2.4.8 *When h goes to* 0*,*

$$h\mapsto\frac{1}{h}\mathbb{E}\left[1_{t<\tau_x\leq t+h}\Phi(K_x)\Psi(L_x)1_{N_{t+h}-N_t=1}\right]$$

converges to

$$\int_0^\infty\int_0^{+\infty}\Phi(k)\Psi(l)\mathbb{E}\left[\mathbf{1}_{\{\tau_x>T_{N_t}\}}\frac{\lambda e^{(-\frac{(x-l-X_{T_{N_t}}-m(t-T_{N_t}))^2}{2(t-T_{N_t})})}}{\sqrt{2\pi(t-T_{N_t})}}\right]F_l(dk)dl$$
$$-\int_0^{+\infty}\int_0^{+\infty}\Phi(k)\Psi(l)\mathbb{E}\left[\mathbf{1}_{\{\tau_x>T_{N_t}\}}\frac{\lambda e^{(-\frac{(x-X_{T_{N_t}}+l+m(t-T_{N_t}))^2}{2(t-T_{N_t})}+2m(x-X_{T_{N_t}}))}}{\sqrt{2\pi(t-T_{N_t})}}\right]F_l(dk)dl$$

where $F_l(dk)$ is the image of $F_Y(dk)$ by the map $y\mapsto y-l$.

Proof. We split the middle term of (2.4.7) according to the values of N_t. Since $\{N_t = n\} = \{T_n \leq t < T_{n+1}\}$ so

$$\begin{aligned}\mathbb{E}(1_{t<\tau_x\leq t+h}\Phi(K_x)\Psi(L_x)1_{N_{t+h}-N_t=1}) &= \mathbb{E}(1_{t<\tau_x\leq t+h}\Phi(K_x)\Psi(L_x)1_{T_{N_t}\leq t<T_{N_t+1}\leq t+h<T_{N_t+2}})\\ &= \sum_{n\geq 0}\mathbb{E}(1_{t<\tau_x\leq t+h}\Phi(K_x)\Psi(L_x)1_{T_n\leq t<T_{n+1}\leq t+h<T_{n+2}}).\end{aligned}\tag{2.4.9}$$

We split again the right term of (2.4.9) according to the relative positions between τ_x and T_{n+1}. It follows

$$\begin{aligned}\mathbb{E}(1_{t<\tau_x\leq t+h}\Phi(K_x)\Psi(L_x)1_{N_{t+h}-N_t=1}) &= \sum_{n\geq 0}\mathbb{E}(\Phi(K_x)\Psi(L_x)\mathbf{1}_{T_n\leq t<\tau_x<T_{n+1}\leq t+h<T_{n+2}})\\ &+\sum_{n\geq 0}\mathbb{E}(\Phi(K_x)\Psi(L_x)\mathbf{1}_{T_n\leq t<\tau_x=T_{n+1}\leq t+h<T_{n+2}})\\ &+\sum_{n\geq 0}\mathbb{E}(\Phi(K_x)\Psi(L_x)\mathbf{1}_{T_n\leq t<T_{n+1}<\tau_x\leq t+h<T_{n+2}}).\end{aligned}\tag{2.4.10}$$

We recall

$$T_1 = S_1, \quad T_2 = S_1 + S_2, \quad \cdots, \quad T_n = \sum_{i=1}^{n} S_i$$

where $(S_i)_{i\in\mathbb{N}^*}$ is a sequence of independent random variables following an exponential law with parameter λ.
Let be

$$\begin{aligned}
A_h^1 &= \sum_{n\geq 0} \mathbb{E}(\Phi(K_x)\Psi(L_x)\mathbf{1}_{T_n\leq t<\tau_x<T_{n+1}\leq t+h<T_{n+2}}) \\
A_h^2 &= \sum_{n\geq 0} \mathbb{E}(\Phi(K_x)\Psi(L_x)\mathbf{1}_{T_n\leq t<\tau_x=T_{n+1}\leq t+h<T_{n+2}}) \\
A_h^3 &= \sum_{n\geq 0} \mathbb{E}(\Phi(K_x)\Psi(L_x)\mathbf{1}_{T_n\leq t<T_{n+1}<\tau_x\leq t+h<T_{n+2}}).
\end{aligned}$$

STEP 1 : Here, we refer to the analysis of $\frac{1}{h}B_1(h)$, Equation (4) et seq. in [CD+11]. On the sets $\{T_n \leq t < \tau_x < T_{n+1}\}$ and $\{T_{n+1} < \tau_x \leq t+h < T_{n+2}\}$, we have $K_x = 0 = L_x$. So

$$\begin{aligned}
A_h^1 &= \Phi(0)\Psi(0)\sum_{n\geq 0} \mathbb{E}(\mathbf{1}_{T_n\leq t<\tau_x<T_{n+1}\leq t+h<T_{n+2}}) \\
&= \Phi(0)\Psi(0)\sum_{n\geq 0} \mathbb{E}(\mathbf{1}_{T_n\leq t<\tau_x<T_n+S_{n+1}\leq t+h<T_n+S_{n+1}+S_{n+2}}).
\end{aligned}$$

Strong Markov property at T_n yields

$$A_h^1 = \Phi(0)\Psi(0)\sum_{n\geq 0} \mathbb{E}\left(\mathbf{1}_{\{T_n\leq t\}}\mathbf{1}_{\{\tau_x>T_n\}}\mathbb{E}^{T_n}(\mathbf{1}_{\{t-T_n<\tilde{\tau}_{x-X_{T_n}}<S_{n+1}<t+h-T_n<S_{n+1}+S_{n+2}\}})\right)$$

where $\mathbb{E}^{T_n}(.) = \mathbb{E}(.|\mathcal{F}_{T_n})$. Since S_{n+2} is independent from S_{n+1}, $\tau_{x-X_{T_n}}$ and T_n, we obtain

$$\begin{aligned}
A_h^1 &= \Phi(0)\Psi(0)\sum_{n\geq 0} \mathbb{E}\left(\mathbf{1}_{\{T_n\leq t\}}\mathbf{1}_{\{\tau_x>T_n\}}\mathbb{E}^{T_n}(e^{-\lambda(t+h-T_n-S_{n+1})}\mathbf{1}_{\{t-T_n<\tilde{\tau}_{x-X_{T_n}}<S_{n+1}<t+h-T_n\}})\right) \\
&\leq e^{-\lambda h}\Phi(0)\Psi(0)\sum_{n\geq 0} \mathbb{E}\left(\mathbf{1}_{\{\tau_x>T_n\}}\mathbb{E}^{T_n}(e^{\lambda S_{n+1}}\mathbf{1}_{\{t-T_n<\tilde{\tau}_{x-X_{T_n}}<S_{n+1}<t+h-T_n\}})\right).
\end{aligned}$$

The random variables $\tilde{\tau}_{x-X_{T_n}}$ and S_{n+1} are independent and their laws admit a density. Therefore

$$\begin{aligned}
\mathbb{E}^{T_n}(e^{\lambda S_{n+1}}\mathbf{1}_{\{t-T_n<\tilde{\tau}_{x-X_{T_n}}<S_{n+1}<t+h-T_n\}}) &= \int_{t-T_n}^{t-T_n+h} \tilde{f}(u, x-X_{T_n}) \int_u^{t-T_n+h} \lambda ds du \\
&= \int_{t-T_n}^{t-T_n+h} \lambda[t-T_n+h-u]\tilde{f}(u, x-X_{T_n})du.
\end{aligned}$$

The change of variable $v = u - (t - T_n)$ implies

$$\mathbb{E}^{T_n}(e^{\lambda S_{n+1}}\mathbf{1}_{\{t-T_n<\tilde{\tau}_{x-X_{T_n}}<S_{n+1}<t+h-T_n\}}) = \int_0^h \lambda[h-v]\tilde{f}(t-T_n+v, x-X_{T_n})dv.$$

Thus,

$$\begin{aligned} A_h^1 &\leq \lambda e^{-\lambda h}\Phi(0)\Psi(0)\sum_{n\geq 0}\int_0^h \mathbb{E}\left(\mathbf{1}_{\{\tau_x>T_n\}}[h-v]\tilde{f}(t-T_n+v, x-X_{T_n})\right)dv \\ &\leq \lambda e^{-\lambda h}\Phi(0)\Psi(0)\int_0^h \mathbb{E}\left(\mathbf{1}_{\{\tau_x>T_{N_t}\}}[h-v]\tilde{f}(t-T_{N_t}+v, x-X_{T_{N_t}})\right)dv. \end{aligned}$$

Similarly to the computation of A_h^1, we have

$$A_h^3 \leq \lambda e^{-\lambda h}\Phi(0)\Psi(0)\int_0^h \mathbb{E}\left(\mathbf{1}_{\{\tau_x>T_{N_t}\}}v\tilde{f}(t-T_{N_t}+v, x-X_{T_{N_t}})\right)dv.$$

So,

$$\frac{1}{h}[A_h^1 + A_h^3] \leq \lambda e^{-\lambda h}\Phi(0)\Psi(0)\int_0^h \mathbb{E}\left(\mathbf{1}_{\{\tau_x>T_{N_t}\}}\tilde{f}(t-T_{N_t}+v, x-X_{T_{N_t}})\right)dv.$$

Using the fact that the application

$$v \mapsto \mathbb{E}\left(\mathbf{1}_{\{\tau_x>T_{N_t}\}}\tilde{f}(t-T_{N_t}+v, x-X_{T_{N_t}})\right) \text{ is continuous.}$$

yields

$$\lim_{h\mapsto 0}\frac{1}{h}[A_h^1 + A_h^3] = 0.$$

STEP 2 : We now deal with A_h^2. On the set $\{\tau_x = T_{n+1}\}$, we have :

$$K_x = X_{\tau_x} - x = X_{T_{n+1}} - x = X_{T_n} + [mS_1 + W_{S_1} + Y_1]\circ\theta_{T_n} - x$$

and

$$L_x = x - X_{\tau_x^-} = x - X_{T_n} - [mS_1 + W_{S_1}]\circ\theta_{T_n}$$

where θ is the shift operator. Since $\tilde{X}_{S_1} = [mS_1 + W_{S_1}]$, then

$$K_x = X_{T_n} + [\tilde{X}_{S_1} + Y_1]\circ\theta_{T_n} - x \text{ and } L_x = x - X_{T_n} - \tilde{X}_{S_1}\circ\theta_{T_n}.$$

So A_h^2 can be written as following :

$$\begin{aligned} A_h^2 = \sum_{n\geq 0}\mathbb{E}\Big(&\mathbf{1}_{\tau_x=T_n+S_{n+1}>T_n}\Phi(X_{T_n}+\tilde{X}_{S_{n+1}}+Y_{n+1}-x)\Psi(x-X_{T_n}-\tilde{X}_{S_{n+1}}) \\ &\mathbf{1}_{T_n\leq t<T_n+S_{n+1}\leq t+h<T_n+S_{n+1}+S_{n+2}}\Big). \end{aligned}$$

Strong Markov property applied at T_n leads to

$$A_h^2 = \sum_{n\geq 0} \mathbb{E}\Big(\mathbf{1}_{\tau_x > T_n}\mathbf{1}_{T_n\leq t}\mathbb{E}^{T_n}(\Phi(X_{T_n} + \tilde{X}_{S_{n+1}} + Y_{n+1} - x)\Psi(x - X_{T_n} - \tilde{X}_{S_{n+1}})\mathbf{1}_{\tilde{\tau}_{x-X_{T_n}} = S_{n+1}} \mathbf{1}_{t-T_n < S_{n+1}\leq t+h-T_n < S_{n+1}+S_{n+2}})\Big).$$

integrating with respect to S_{n+2} , we have

$$e^{\lambda h} A_h^2 = \sum_{n\geq 0} \mathbb{E}\Big(\mathbf{1}_{\tau_x > T_n}\mathbf{1}_{T_n\leq t} e^{-\lambda(t-T_n)}\mathbb{E}^{T_n}(e^{\lambda S_{n+1}}\Phi(X_{T_n} + \tilde{X}_{S_{n+1}} + Y_{n+1} - x)\Psi(x - X_{T_n} - \tilde{X}_{S_{n+1}})\mathbf{1}_{\tilde{\tau}_{x-X_{T_n}} = S_{n+1}} \mathbf{1}_{t-T_n < S_{n+1}\leq t+h-T_n})\Big).$$

We observe that on the set $\{\tilde{\tau}_{x-X_{T_n}} = S_{n+1}\}, Y_{n+1} > 0,\ K_x \geq 0,\ L_x \geq 0$ and $K_x + L_x = Y_{n+1}$. More over :

$$\{\tilde{\tau}_{x-X_{T_n}} = S_{n+1}\} = \{\sup_{s\leq S_{n+1}} \tilde{X}_s < x - X_{T_n}, \tilde{X}_{S_{n+1}} + Y_{n+1} > x - X_{T_n}\}.$$

Integrating with respect to (S_{n+1}, Y_{n+1}) implies that

$$\mathbb{E}^{T_n}(e^{\lambda S_{n+1}}\Phi(X_{T_n} + \tilde{X}_{S_{n+1}} + Y_{n+1} - x)\Psi(x - X_{T_n} - \tilde{X}_{S_{n+1}})\mathbf{1}_{\{\tilde{\tau}_{x-X_{T_n}} = S_{n+1}\}}\mathbf{1}_{t-T_n<S_{n+1}\leq t+h-T_n}) =$$
$$\int_0^{+\infty}\int_{t-T_n}^{t+h-T_n} \lambda\mathbb{E}^{T_n}[\Phi(X_{T_n} + \tilde{X}_u + y - x)\Psi(x - X_{T_n} - \tilde{X}_u)\mathbf{1}_{\{\sup_{s\leq u}\tilde{X}_s < x - X_{T_n}, \tilde{X}_u + y > x - X_{T_n}\}}]F_Y(dy)du.$$

According to Corollary 3.2.1.2 page 147 of [JYC09], $(\sup_{s\leq u}\tilde{X}_s, \tilde{X}_u)$ admits a density

$$\tilde{p}(b,a,t) = \frac{2(2b-a)}{\sqrt{2\pi t^3}}\exp\left[-\frac{(2b-a)^2}{2t} + ma - \frac{m^2}{2}t\right]\mathbf{1}_{b>\max\{0,a\}}.$$

So, A_h^2 is equal to

$$e^{-\lambda h}\sum_{n\geq 0}\mathbb{E}\left[\mathbf{1}_{\tau_x > T_n}\mathbf{1}_{T_n\leq t}e^{-\lambda(t-T_n)}\int_{\mathbb{R}^2}\int_0^{+\infty}\int_{t-T_n}^{t+h-T_n}\lambda\Phi(X_{T_n} + a + y - x)\Psi(x - X_{T_n} - a)\mathbf{1}_{\{b<x-X_{T_n}, a+y>x-X_{T_n}\}}\tilde{p}(b,a,u)\right] F_Y(dy)dudbda.$$

Since $e^{-\lambda(t-T_n)} = \mathbb{E}^{T_n}(\mathbf{1}_{T_{n+1}>t})$, and on the event $\{N_t = n\}$, $T_{N_t+1} > t$ $a.s$, we have $A_h^2 =$

$$e^{-\lambda h}\mathbb{E}\left[\mathbf{1}_{\tau_x > T_{N_t}}\int_{\mathbb{R}^2}\int_0^{+\infty}\int_{t-T_{N_t}}^{t+h-T_{N_t}}\lambda\Phi(X_{T_{N_t}} + a + y - x)\Psi(x - X_{T_{N_t}} - a)\mathbf{1}_{\{b<x-X_{T_{N_t}}, a+y>x-X_{T_{N_t}}\}}\tilde{p}(b,a,u)\right] F_Y(dy)dudbda.$$

We compute the integral with respect to db and it follows

$$\frac{A_h^2}{e^{-\lambda h}} = \mathbb{E}\left[\mathbf{1}_{\tau_x > T_{N_t}}\int_{x-y-X_{T_{N_t}}}^{x-X_{T_{N_t}}}\int_0^{+\infty}\int_{t-T_{N_t}}^{t+h-T_{N_t}}\Phi(X_{T_{N_t}} + a + y - x)\Psi(x - X_{T_{N_t}} - a)\frac{\lambda e^{-\frac{(a-mu)^2}{2u}}}{\sqrt{2\pi u}}F_Y(dy)duda\right] -$$

$$\mathbb{E}\left[\mathbf{1}_{\tau_x > T_{N_t}}\int_{x-y-X_{T_{N_t}}}^{x-X_{T_{N_t}}}\int_0^{+\infty}\int_{t-T_{N_t}}^{t+h-T_{N_t}}\Phi(X_{T_{N_t}} + a + y - x)\Psi(x - X_{T_{N_t}} - a)\frac{\lambda e^{(-\frac{(a-mu-2x+2X_{T_{N_t}})^2}{2u} + 2m(x-X_{T_{N_t}}))}}{\sqrt{2\pi u}} F_Y(dy)duda\right].$$

Change of variables $v = u - (t - T_{N_t})$ and $l = x - X_{T_{N_t}} - a$ yields

$$A_h^2 = e^{-\lambda h}\mathbb{E}\left[\int_0^y\int_0^{+\infty}\int_0^h \mathbf{1}_{\{\tau_x > T_{N_t}\}}\Phi(y-l)\Psi(l)\frac{\lambda e^{(-\frac{(x-l-X_{T_{N_t}}-m(t-T_{N_t}+v))^2}{2(t-T_{N_t}+v)})}}{\sqrt{2\pi(t-T_{N_t}+v)}}dvF_Y(dy)dl\right]$$
$$- e^{-\lambda h}\mathbb{E}\left[\int_0^y\int_0^{+\infty}\int_0^h \mathbf{1}_{\{\tau_x > T_{N_t}\}}\Phi(y-l)\Psi(l)\frac{\lambda e^{(-\frac{(x-X_{T_{N_t}}+l+m(t-T_{N_t}+v))^2}{2(t-T_{N_t}+v)}+2m(x-X_{T_{N_t}}))}}{\sqrt{2\pi(t-T_{N_t}+v)}}dvF_Y(dy)dl\right].$$

Let F_l be the image of F_Y by the map $y \mapsto y - l$. Hence,

$$\lim_{h\mapsto 0}\frac{1}{h}A_h^2 = \int_0^{+\infty}\int_0^{+\infty}\Phi(k)\Psi(l)\mathbb{E}\left[\mathbf{1}_{\{\tau_x > T_{N_t}\}}\frac{\lambda e^{(-\frac{(x-l-X_{T_{N_t}}-m(t-T_{N_t}))^2}{2(t-T_{N_t})})}}{\sqrt{2\pi(t-T_{N_t})}}\right]F_l(dk)dl$$
$$- \int_0^{+\infty}\int_0^{+\infty}\Phi(k)\Psi(l)\mathbb{E}\left[\mathbf{1}_{\{\tau_x > T_{N_t}\}}\frac{\lambda e^{(-\frac{(x-X_{T_{N_t}}+l+m(t-T_{N_t}+))^2}{2(t-T_{N_t})}+2m(x-X_{T_{N_t}}))}}{\sqrt{2\pi(t-T_{N_t})}}\right]F_l(dk)dl$$

since

$$v \mapsto \int_0^{+\infty}\int_0^{+\infty}\Phi(k)\Psi(l)\mathbb{E}\left[\mathbf{1}_{\{\tau_x > T_{N_t}\}}\frac{\lambda e^{(-\frac{(x-l-X_{T_{N_t}}-m(t-T_{N_t}+v))^2}{2(t-T_{N_t}+v)})}}{\sqrt{2\pi(t-T_{N_t}+v)}}\right]F_l(dk)dl$$
$$- \int_0^{+\infty}\int_0^{+\infty}\Phi(k)\Psi(l)\mathbb{E}\left[\mathbf{1}_{\{\tau_x > T_{N_t}\}}\frac{\lambda e^{(-\frac{(x-X_{T_{N_t}}-l+l+m(t-T_{N_t}+v))^2}{2(t-T_{N_t}+v)}+2m(x-X_{T_{N_t}}))}}{\sqrt{2\pi(t-T_{N_t}+v)}}\right]F_l(dk)dl$$

is continuous. ■

This proposition concludes the proof of Proposition 2.4.5 . ■

2.5 Conclusion

Our study relies on the default time of a Lévy process. We have first shown that the distribution function of the default time τ_x belongs to $\mathcal{C}(\mathbb{R}_+^* \times \mathbb{R}_+^*)$ and for any $x \in \mathbb{R}_+^*$, to $\mathcal{C}(\mathbb{R}_+)$. This will be very useful in our future works on default time of a Lévy process where we will use the filtering theory. Secondly, we have obtained an explicit expression to characterize the joint law of the hitting time, overshoot and undershoot of one Lévy process. In this expression, the Gaussian density is of great importance. This law gives a lot of information on the deficit and surplus at default time. In a following paper in progress, we will give a partial differential equation for a Lévy process and its running maximum.

2.6 Appendix

Lemme 2.6.1 *Let $\beta > -1$, then for $0 < t \leq 1$,*

$$\mathbb{E}\left(\mathbf{1}_{\{N_t\geq 2\}}(t-T_{N_t})^\beta\right) \leq \left(\sum_{n=1}^{\infty}\frac{\lambda^n e^t}{(n-1)!}B(n,\beta+1)\right)t^{2+\beta}$$

where $B(n,\beta+1)=\int_0^1(1-u)^\beta u^{n-1}du$.

Proof. According to the values of the process N_t, we have

$$\begin{aligned}\mathbb{E}\left(\mathbf{1}_{\{N_t\geq 2\}}(t-T_{N_t})^\beta\right) &= \sum_{n\geq 2}\mathbb{E}\left((t-T_n)^\beta\mathbf{1}_{T_n\leq t<T_n+S_{n+1}}\right)\\ &= \sum_{n\geq 2}\mathbb{E}\left(e^{-\lambda(t-T_n)}(t-T_n)^\beta\right).\end{aligned}$$

Since T_n follows a Gamma law of parameters n and λ, hence

$$\mathbb{E}\left(\mathbf{1}_{\{N_t\geq 2\}}(t-T_{N_t})^\beta\right) = \sum_{n=2}^{\infty}\frac{\lambda^n e^{-\lambda t}}{(n-1)!}t^{n+\beta}B(n,\beta+1).$$

■

Lemme 2.6.2 *For any $0 < t_1 < t_2$,*

$$\sup_{t_1\leq t\leq t_2}\mathbb{E}\left(1_{\tau_x>T_{N_t}}\tilde{f}(t-T_{N_t},x-X_{T_{N_t}})^p\right) < +\infty. \tag{2.6.3}$$

and for instance, as a consequence, the family

$$\left(1_{\tau_x>T_{N_t}}\tilde{f}(t-T_{N_t},x-X_{T_{N_t}}),\quad t\in[t_1,t_2],\quad x>0\right)$$

is uniformly integrable.

Proof. Refer to Lemma 3.1 in [CD+11], if G is a Gaussian random variable $\mathcal{N}(0,1)$, $\mu > 0$, $u > 0, \sigma \in \mathbb{R}^+$ and $p \geq 1$, then

$$\mathbb{E}[(\tilde{f}(u,\mu+\sigma G))^p 1_{\mu+\sigma G>0}] =$$
$$\frac{1}{\sqrt{(2\pi)^p}}\frac{u^{\frac{1-2p}{2}}e^{-\frac{p(\mu-mu)^2}{2(p\sigma^2+u)}}}{(p\sigma^2+u)^{\frac{p+1}{2}}}\mathbb{E}\left[\left(\sigma G+\sqrt{\frac{u}{p\sigma^2+u}}(\mu-mu)+m\sqrt{u(p\sigma^2+u)}\right)_+^p\right]$$

Using the inequality $(a+b+c)^p \leq 3^{p-1}(a^p+b^p+c^p)$ where a, b, c are positive numbers, it follows that

$$\mathbb{E}[\tilde{f}(u,\mu+\sigma G)^p 1_{\mu+\sigma G>0}] \leq$$
$$\frac{3^{p-1}}{\sqrt{(2\pi)^p}} \frac{u^{\frac{1-2p}{2}} e^{-\frac{p(\mu-mu)^2}{2(p\sigma^2+u)}}}{(p\sigma^2+u)^{\frac{p+1}{2}}} \left(\sigma^p \mathbb{E}(\mid G\mid^p) + (\frac{u}{p\sigma^2+u})^{\frac{p}{2}}(\mu-mu)^p + m^p(u(p\sigma^2+u))^{\frac{p}{2}}\right).$$

Therefore

$$\mathbb{E}[\tilde{f}(u,\mu+\sigma G)^p 1_{\mu+\sigma G>0}] \leq$$
$$\frac{3^{p-1}}{\sqrt{2\pi}^p}\left(\frac{u^{\frac{1-2p}{2}}}{(p\sigma^2+u)^{\frac{p+1}{2}}}\sigma^p\mathbb{E}(|G|^p) + \frac{u^{\frac{1-p}{2}}}{(p\sigma^2+u)^{\frac{p+1}{2}}}c_p + \mid m\mid^p \frac{u^{\frac{1-p}{2}}}{(p\sigma^2+u)^{\frac{1}{2}}}\right)$$
$$\leq \frac{3^{p-1}}{\sqrt{2\pi}^p}\left(\frac{u^{\frac{1-2p}{2}}}{(p\sigma^2+u)^{\frac{p+1}{2}}}\sigma^p\mathbb{E}(|G|^p) + \frac{c_p}{u^{\frac{p-1}{2}}(p\sigma^2+u)^{\frac{p+1}{2}}} + \frac{|m|^p}{\text{ÃŮ}u^{\frac{p-1}{2}}(p\sigma^2+u)^{\frac{1}{2}}}\right).$$

Using the independence between the Brownian motion and the Poisson process, we can apply this inequality to $\sigma = \sqrt{T_{N_t}}, u = t - T_{N_t}, p\sigma^2 + u = (p-1)T_{N_t} + t \geq t > t_1$:

$$\mid \mathbb{E}(1_{\tau_x > T_{N_t}} \tilde{f}(t-T_{N_t}, x - X_{N_t})^p) \mid \leq$$
$$\frac{3^{p-1}}{\sqrt{2\pi}^p}\mathbb{E}\left(\frac{T_{N_t}^{\frac{p}{2}}\mathbb{E}(\mid G\mid^p)}{(t-T_{N_t})^{\frac{2p-1}{2}}t^{\frac{p+1}{2}}} + \frac{c_p}{(t-T_{N_t})^{\frac{p-1}{2}}t^{\frac{2p+1}{2}}} + \frac{\mid m\mid^p}{\sqrt{t}(t-T_{N_t})^{\frac{p-1}{2}}}\right).$$

We use the following for $\alpha = 0$ or $p/2$, and $\beta = (2p-1)/2$ or $(p-1)/2$:

$$\begin{aligned}
\mathbb{E}(\frac{T_{N_t}^\alpha}{(t-T_{N_t})^\beta}) &= \frac{1}{t^\beta} + \sum_{n\geq 1}\mathbb{E}[\frac{T_n^\alpha}{(t-T_n)^\beta}1_{T_n<t<T_{n+1}}] \\
&= \frac{1}{t^\beta} + \sum_{n\geq 1}\int_0^t \frac{u^\alpha}{(t-u)^\beta}\frac{(\lambda u)^{n-1}}{(n-1)!}\lambda e^{-\lambda u}\int_{t-u}^{+\infty}\lambda e^{-\lambda v}dvdu \\
&= \frac{1}{t^\beta} + e^{-\lambda(t)}\sum_{n\geq 1}\frac{\lambda^n}{(n-1)!}\int_0^t \frac{u^{n+\alpha-1}}{(t-u)^\beta}du \\
&= \frac{1}{t^\beta} + e^{-\lambda(t)}\sum_{n\geq 1}\frac{[\lambda(t)]^n}{(n-1)!}B(\alpha+n;1-\beta)(t)^{\alpha-\beta}
\end{aligned}$$

where $B(n,\beta+1) = \int_0^1(1-u)^\beta u^{n-1}du$. Since $t_2 \geq t \geq t_1 > 0$, we conclude. ■

Lemme 2.6.4 *For any numbers u, v, y, z and a, the equality*

$$\frac{1}{u}(a-z)^2 + \frac{1}{v}(a-y)^2 = \frac{v+u}{uv}\left[a - \frac{vz+uy}{v+u}\right]^2 + \frac{1}{u+v}(z-y)^2$$

holds.

Proof. We develop both squared

$$\frac{1}{u}(a-z)^2+\frac{1}{v}(a-y)^2=(\frac{1}{u}+\frac{1}{v})a^2-2a(\frac{z}{u}+\frac{y}{v})+\frac{z^2}{u}+\frac{y^2}{v}.$$

We have the first square

$$\frac{1}{u}(a-z)^2+\frac{1}{v}(a-y)^2=(\frac{1}{u}+\frac{1}{v})\left[a^2-2a(\frac{z}{u}+\frac{y}{v})\frac{uv}{v+u}+\left[(\frac{z}{u}+\frac{y}{v})\frac{uv}{v+u}\right]^2\right]$$
$$+\frac{z^2}{u}+\frac{y^2}{v}-\left[\frac{z^2}{u^2}+2\frac{yz}{uv}+\frac{y^2}{v^2}\right]\frac{uv}{u+v}.$$

We order

$$\frac{1}{u}(a-z)^2+\frac{1}{v}(a-y)^2=(\frac{1}{u}+\frac{1}{v})\left[a^2-2a(\frac{z}{u}+\frac{y}{v})\frac{uv}{v+u}+\left[(\frac{z}{u}+\frac{y}{v})\frac{uv}{v+u}\right]^2\right]$$
$$+z^2\frac{1}{u}[1-\frac{v}{u+v}]+y^2\frac{1}{v}[1-\frac{u}{u+v}]-2\frac{yz}{uv}\frac{uv}{u+v}.$$

which concludes the proof. ■

Lemme 2.6.5 *Let be $(\tilde{X}_u, u\geq 0)$ be a Brownian motion with drift $m\in\mathbb{R}$. So, we have*

$$\mathbb{E}\left(\mathbf{1}_{\{\tilde{X}_u^*<c\}}|\tilde{X}_u\right)=$$
$$\mathbf{1}_{\{\tilde{X}_u<c\}}\left[1-\exp\left[-\frac{2c^2}{u}+\frac{2c}{u}\tilde{X}_u\right]\right]$$

for all real number $u>0$.

Proof. Refer to Corollary 3.2.1.2 page 147 of [JYC09], $(\tilde{X}_t^*,\tilde{X}_t)$ admits a density

$$\tilde{p}(b,a,t)=\frac{2(2b-a)}{\sqrt{2\pi t^3}}\exp\left[-\frac{(2b-a)^2}{2t}+ma-\frac{m^2}{2}t\right]\mathbf{1}_{b>\max\{0,a\}}$$

So, the conditional law of $\tilde{X}_u^*$ given $\tilde{X}_u=a$ has the density

$$f_{\tilde{X}^*|\tilde{X}=a}(b|a)=\frac{2(2b-a)}{u}\exp\left[-\frac{(2b-a)^2-a^2}{2u}\right]\mathbf{1}_{b>\max\{0,a\}}.$$

Thus

$$\mathbb{E}\left(\mathbf{1}_{\{\tilde{X}_u^*<c\}}|\tilde{X}_u=a\right)=\{\mathbf{1}_{a<c}\int_{\max\{a,0\}}^{c}\frac{2(2b-a)}{u}\exp\left[-\frac{(2b-a)^2-a^2}{2u}\right]\mathbf{1}_{b>\max\{0,a\}}db$$
$$=\mathbf{1}_{\{a<c\}}\left[1-\exp\left[-\frac{2c^2}{u}+\frac{2c}{u}a\right]\right].$$

■

Chapitre 3

Joint distribution of a Lévy process and its running maxima

This chapter is in progress.

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Abstract

Let be X a jump-diffusion process and X^* its running maximum. In this paper, we show that for any $t > 0$, the quadruplet formed by the random variable X_t, the running supremum X_t^* of X at time t, the supremum of X at the last jump time before t and the last jump time before t can be characterized as a solution of a weakly partial differential equation (PDE). This allows us to characterize the law of the pair (X^*, X) then the one of X^*. Finally, one recovers a strong equation for the law of (X_t^*, X_t).

3.1 Introduction

Consider a Lévy process $(X_t, t \geq 0)$, starting from zero, which is right continuous left limited. If moreover X is the sum of a drifted Brownian motion and a compound Poisson process, it is called a mixed diffusive-jump process. As any Lévy process, X has stationary and independent increments and is characterized by its Laplace transform.

The mixed diffusive-jump processes and the notion of first passage time (behavior of certain processes at first passage time) are very useful and widely studied.
The probability $\mathbb{P}(X_t \geq a, X_t^* \geq b) = \mathbb{P}(X_t \geq a, \tau_b \leq t)$ for some fixed real numbers (a, b), $a \leq b$ and $b > 0$, is of great importance, for example, in pricing barrier options while the logarithm of the underlying asset price is modeled by a jump-diffusion process. In this idea, Kou and Wang [KW03] give the explicit expression of the Laplace transform of the joint distribution of the double exponential mixed diffusive-jump process and its running maximum.
In [JYC09], Jeanblanc et al. consider the first passage time by a diffusion at a deterministic function h that depends on time and define a function of τ_h and X which satisfies the Fokker-Planck Equation.
In [App09], it is well noted (Theorem 2.2.9 and Exercise 2.2.10) that the $\frac{1}{2}-$stable subordinator is the first passage time of a standard Brownian motion and the inverse Gaussian subordinator is the first passage time of standard Brownian motion with a drift.
Mark Veillette and Murad S. Taqqu study in [VT10] the first passage time of a subordinator. Since it is in general non-Markovian with non-stationary and non-independent increments, they derive a partial differential equation for the Laplace transform of the $n-$ time tail distribution $\mathbb{P}(\tau_{t_1} > s_1, \cdots, \tau_{t_n} > s_n)$ where $\tau_{t_k} = \inf\{s : D_s > t_k\}$ for a subordinator $(D_s, s \geq 0)$. With this result, they give a recursive formula for multiple-time moments of the local time of a Markov process in terms of its transition density.
The authors of [CC11] use a Partial Differential Equation (PDE) approach to show that the calibration of an implied volatility surface and the pricing of contingent claims can be as simple in mixed diffusive-jump framework as it is in a diffusion framework.
Our goal is to fully characterize the first passage time τ_x of a mixed diffusive-jump process as well on the filtration generated by X than on others.

This work characterizes the law of the quadruplet formed by the random variable X_t, the running supremum X_t^* of X at time t, the supremum of X at the last jump time before t and the last jump time before t with a partial differential equation (PDE) and give an explicit expression for the density function of the pair formed by the random variable X_t, its running supremum X_t^*. Then the regularity of this density with respect to the space parameter is derived.

The paper is organized as follows : Section 3.2 deals the partial differential equation for the joint law. Section 3.3 is devoted to the proofs. Section 3.4 contains a lemma for the proofs in Section 3.3.

3.2 Valued measure differential equation for the joint law

We introduce some preliminary concepts for the diffusion part : for a standard Brownian motion W and a real number m, let be

$$\begin{aligned} \tilde{X}_t &= mt + W_t, \\ \tilde{X}^*_t &= \sup_{s\leq t} \tilde{X}_s. \end{aligned} \tag{3.2.1}$$

In [JYC09] page 147, Jeanblanc et al. show that the pair $(\tilde{X}^*_t, \tilde{X}_t)$ has a density with respect to Lebesgue measure on $\mathbb{R}^2$ noted $\tilde{p}(.,.;t)$ where

$$\tilde{p}(b,a;t) = \frac{2(2b-a)}{\sqrt{2\pi t^3}} \exp\left[-\frac{(2b-a)^2}{2t} + ma - m^2\frac{t}{2}\right] \mathbf{1}_{\{\max(0,a)<b\}}. \tag{3.2.2}$$

In all the following, Φ_G means the standard normal Gaussian distribution and one often uses the following :

$$1 - \Phi_G(x) = \Phi_G(-x) \leq \frac{1}{x\sqrt{2\pi}} \exp -\frac{x^2}{2}, \quad \forall x > 0. \tag{3.2.3}$$

In order to have a Lévy process with non zero jump part, let us introduce

$$\begin{aligned} X_t &= mt + W_t + \sum_{i=1}^{N_t} Y_i, \\ X^*_t &= \sup_{s\leq t} X_t, \end{aligned}$$

where N is a Poisson process with constant positive intensity λ and $(Y_i, i \in \mathbb{N}^*)$ is a sequence of independent and identically distributed random variables with a same distribution function F_Y. Let $(U_t;\ \ t \geq 0)$ be the $\mathbb{R}^4-$value process defined by

$$U_t = (X^*_t, X_t, X^*_{T_{N_t}}, T_{N_t}), \quad t \geq 0 \tag{3.2.4}$$

and θ be the shift operator. Our aim is to prove the theorem :

Theorem 3.2.5 *Let be $\varphi\ : \mathbb{R}^4 \to \mathbb{R}$ a C^3_b- bounded function.*
For any $t > 0$,

$$\begin{aligned} \mathbb{E}\left(\varphi(U_t)\right) &= \varphi(0,0,0,0) + \int_0^t \mathbb{E}\left[m\partial_2\varphi(U_s) + \frac{1}{2}\partial^2_{22}\varphi(U_s)\right] ds \\ &+ \int_0^t \frac{1}{2}\mathbb{E}\left[\mathbf{1}_{\{X^*_s > X^*_{T_{N_s}}\}}\partial_1\varphi(U_s)\frac{\tilde{p}(X^*_s - X^*_{T_{N_s}}, X^*_s - X^*_{T_{N_s}}, s - T_{N_s})}{\tilde{p}^*(X^*_s - X^*_{T_{N_s}}, s - T_{N_s})}\right] ds \\ &+ \lambda \int_0^t \mathbb{E}\left(\int_{\mathbb{R}} \left[\varphi(U_s(y)) - \varphi(U_s)\right] dF_Y(y)\right) ds. \end{aligned}$$

where

$$U_s(y) = (\max(X_s^*, X_s + y), X_s + y, max(X_s^*, X_s + y), s), \quad s \geq 0. \tag{3.2.6}$$

To prove the theorem, these preliminary results (namely Lemmas 3.2.7, 3.2.10 and 3.2.12) will be useful.

Lemme 3.2.7 *For any $t > 0$, the law of $\tilde{X}_t^*$ has the density with respect to Lebesgue measure on $\mathbb{R}$,*

$$\tilde{p}^*(b,t) := 2\left[\frac{1}{\sqrt{2\pi t}}\exp -\frac{(b-mt)^2}{2t} - me^{2bm}\Phi_G(\frac{-b-mt}{\sqrt{t}})\right]\mathbf{1}_{]0,+\infty[}(b). \tag{3.2.8}$$

Proof. We integrate the joint density of the pair $(\tilde{X}_t^*, \tilde{X}_t)$ given by (3.2.2) over the variable a on $\mathbb{R}$. The derivative of the function $a \mapsto \exp -\frac{(2b-a)^2}{2t}$ is the function $a \mapsto \frac{(2b-a)}{t}\exp -\frac{(2b-a)^2}{2t}$. Integration by parts yields to

$$\begin{aligned}\tilde{p}^*(b;t) &= 2\mathbf{1}_{]0,+\infty[}(b)\left[\frac{1}{\sqrt{2\pi t}}\exp\left[-\frac{(2b-a)^2}{2t} + ma - m^2\frac{t}{2}\right]\right]_{-\infty}^{b} \\ &\quad - 2\mathbf{1}_{]0,+\infty[}(b)m\int_{-\infty}^{b}\frac{1}{\sqrt{2\pi t}}\exp\left[-\frac{(2b-a)^2}{2t} + ma - m^2\frac{t}{2}\right]da.\end{aligned}$$

We factorize

$$(2b-a)^2 - 2mta + m^2t^2 = [a - (2b+mt)]^2 - 4bmt,$$

and it follows

$$\tilde{p}^*(b;t) = 2\mathbf{1}_{]0,+\infty[}(b)\frac{1}{\sqrt{2\pi t}}\exp\left[-\frac{(b-mt)^2}{2t}\right] - 2\mathbf{1}_{]0,+\infty[}(b)me^{2bm}\Phi_G(-\frac{b+mt}{\sqrt{t}}).$$

■

Remark 3.2.9 *This result is consistent with the fact that when $m = 0$, $\tilde{X}_t^*$ and $|\tilde{X}_t|$ have the same law (cf. Proposition 3.7, Revuz-Yor [RY13]).*

Below, Lemma 3.2.10 shows that $\frac{\tilde{X}_s^* - \tilde{X}_{T_{N_s}}^*}{s - T_{N_s}}$ is integrable, cf. (A6) in [CD+11] with $\gamma = 0$.

Lemme 3.2.10 *For any $s > 0$, the random variables $\frac{\tilde{X}_s^* - \tilde{X}_{T_{N_s}}^*}{s - T_{N_s}}$ and $(s - T_{N_s})^{-\alpha}$, $\forall \alpha \in]0,1[$ are integrable and satisfy*

$$\mathbb{E}\left((s - T_{N_s})^{-\alpha}\right) = e^{-\lambda s}\left(\frac{1}{s^\alpha} + \sum_{n\geq 1}\frac{\lambda^n s^{n-\alpha}}{\Gamma(n-\alpha)}\right),$$

$$\mathbb{E}\left(\frac{\tilde{X}_s^* - \tilde{X}_{T_{N_s}}^*}{s - T_{N_s}}\right) \leq |m| + C\mathbb{E}[(s - T_{N_s})^{-\frac{1}{2}}].$$

Proof. We split the expectation according the events $\{N_s = n\}$ and it follows

$$\mathbb{E}\left((s-T_{N_s})^{-\alpha}\right) = \frac{e^{-\lambda s}}{s^\alpha} + \sum_{n=1}^{\infty}\int_{(s_1,\dots,s_n)\in[0,s]^n, s_n<s<s_{n+1}} \frac{\lambda^{n+1}e^{-\lambda(s_1+\dots+s_{n+1})}}{(s-s_1-\dots.-s_n)^\alpha}ds_1....ds_n$$

$$= e^{-\lambda s}\left(\frac{1}{s}+\int_{(s_1,\dots,s_n)\in[0,s]^n}\frac{\lambda^n}{(s-s_1-\dots-s_n)^\alpha}ds_1....ds_{n+1}\right).$$

The change of variables $u_1 = s_1, ...u_n = s_1 + ... + s_n$, yields

$$\mathbb{E}\left((s-T_{N_s})^{-\alpha}\right) = e^{-\lambda s}\left(\frac{1}{s}+\int_{u_1<\dots<u_n<s}\frac{\lambda^n}{(s-u_n)^\alpha}du_1....du_n\right)$$

$$= e^{-\lambda s}\left(\frac{1}{s}+\frac{\lambda^n}{(n-1)!}\int_0^s u^{n-1}(s-u)^{-\alpha}du\right).$$

We do again a change of variable $u = sv$,

$$\mathbb{E}\left((s-T_{N_s})^{-\alpha}\right) = e^{-\lambda s}\left(\frac{1}{s}+\frac{\lambda^n}{(n-1)!}s^{n-\alpha}\int_0^1 v^{n-1}(1-v)^{-\alpha}dv\right)$$

$$= e^{-\lambda s}\left(\frac{1}{s}+\frac{\lambda^n}{(n-1)!}s^{n-\alpha}B(n,1-\alpha)\right)$$

where $B(a,b) := \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b-1)}$. The series $\sum_n \frac{\lambda^n s^{n-\alpha}}{\Gamma(n-\alpha)}$ is convergent, thus $\mathbb{E}\left((s-T_{N_s})^{-\alpha}\right) < +\infty$.

The process $\tilde{X}$ satisfies $0 \le \tilde{X}_h \le |m|h + W_h^*$. By Burkholder-Davis Gundy's inequality there exists a constant C such that $E[W_h^*] \le C\sqrt{h}$, hence for $h \le 1$,

$$\mathbb{E}\left(\tilde{X}_h^*\right) \le Ch^{1/2}. \tag{3.2.11}$$

Therefore

$$|\tilde{X}_s^* - \tilde{X}_{T_{N_s}}^*| = (\tilde{X}_{T_{N_s}} + \tilde{X}_{s-T_{N_s}}^*\circ\theta_{T_{N_s}} - \tilde{X}_{T_{N_s}}^*)^+ \le \tilde{X}_{s-T_{N_s}}^*\circ\theta_{T_{N_s}}$$

Hence, we deduce

$$\mathbb{E}\left(\frac{\tilde{X}_s^*-\tilde{X}_{T_{N_s}}^*}{s-T_{N_s}}\right) = \sum_{n=0}^{\infty}\mathbb{E}\left(\mathbf{1}_{\{N_s=n\}}\frac{\tilde{X}_s^*-\tilde{X}_{T_n}^*}{s-T_{N_s}}\right)$$

$$\le \sum_{n=0}^{\infty}\mathbb{E}\left(\mathbf{1}_{\{N_s=n\}}\frac{\tilde{X}_{s-T_n}^*\circ\theta_{T_n}}{s-T_n}\right)$$

$$\le \sum_{n=0}^{\infty}\mathbb{E}\left(\mathbf{1}_{\{N_s=n\}}\left[|m|+\frac{C}{(s-T_n)^{1/2}}\right]\right) = |m| + C\mathbb{E}\left((s-T_{N_s})^{-1/2}\right).$$

■

Lemme 3.2.12 *For any $t > 0$, denoting by $\tilde{p}(.,.,t)$ the joint density of $(\tilde{X}_t^*, \tilde{X}_t)$ and by $\tilde{p}^*(.,t)$ the density of $\tilde{X}_t^*$, we obtain*

$$\left|\frac{\tilde{p}(b,b,t)}{\tilde{p}^*(b,t)}\right| \leq \frac{b+\max(m,0)t}{t}, \quad b > 0.$$

Proof. Using (3.2.8) in Lemma 3.2.7 when $m \leq 0$, it follows

$$\tilde{p}^*(b,t) \geq \frac{2}{\sqrt{2\pi t}} \exp -\frac{(b-mt)^2}{2t}.$$

When $m > 0$, we use (3.2.3) which yields

$$\begin{aligned}\tilde{p}^*(b,t) &\geq 2\left[\frac{1}{\sqrt{2\pi t}} \exp -\frac{(b-mt)^2}{2t} - \frac{m\sqrt{t}e^{2bm}}{(b+mt)\sqrt{2\pi}} \exp -\frac{(b+mt)^2}{2t}\right] \\ &\geq 2\frac{b}{b+mt}\frac{t}{\sqrt{2\pi t^3}} \exp -\frac{(b-mt)^2}{2t} = \frac{t}{b+mt}\tilde{p}(b,b,t).\end{aligned}$$

■

In the next section, details of the proof of Theorem 3.2.5 are given.

3.3 Proof of Theorem 3.2.5

To prove the theorem, we proceed as follows : we compute $\lim_{h\to 0} h^{-1}A(t,h) = a(t)$ where

$$A(t,h) := \mathbb{E}\left[\varphi(U_{t+h}) - \varphi(U_t)\right] \tag{3.3.1}$$

then we make sure that this limit is bounded. This means that

$$\mathbb{E}\left(\varphi(U_t)\right) = \int_0^t a(s)ds + \varphi(0,0,0,0).$$

The idea is to split $A(t,h)$ in three parts according to the values of $N_{t+h} - N_t$:

$$A(t,h) = \sum_{i=0}^{2} A_i(t,h) \tag{3.3.2}$$

where

$$\begin{aligned}A_i(t,h) &:= \mathbb{E}\left(\left[\varphi(U_{t+h}) - \varphi(U_t)\right]\mathbf{1}_{\{N_{t+h}-N_t=i\}}\right), \quad i = 0,1 \\ A_2(t,h) &:= \mathbb{E}\left(\left[\varphi(U_{t+h}) - \varphi(U_t))\right]\mathbf{1}_{\{N_{t+h}-N_t\geq 2\}}\right).\end{aligned} \tag{3.3.3}$$

Lemme 3.3.4 *Under the hypothesis of Theorem 3.2.5,*

$$\lim_{h \to 0} h^{-1} A_2(t,h) = 0. \tag{3.3.5}$$

Proof. By hypothesis φ is bounded and we get

$$\begin{aligned} |A_2(t,h)| &\leq 2\|\varphi\|_\infty \mathbb{P}(N_{t+h} - N_t \geq 2) \\ &\leq 2\|\varphi\|_\infty (1 - e^{-\lambda h} - \lambda h e^{-\lambda h}). \end{aligned}$$

Thus, $\lim_{h \to 0} h^{-1} A_2(t,h) = 0$. ■

This lemma added to the three next propositions prove Theorem 3.2.5 : this lemma treats the term $A_2(t,h)$ while Proposition 3.3.6 treats the term $A_1(t,h)$. Propositions 3.3.24 and 3.3.28 treat the term $A_0(t,h)$.

Proposition 3.3.6 *Under the hypothesis of Theorem 3.2.5,*

$$\lim_{h \to 0} h^{-1} A_1(t,h) = \lambda \mathbb{E} \int_{\mathbb{R}} [\varphi(U_t(y)) - \varphi(U_t)] F_Y(dy).$$

where U_t is defined by (3.2.4) and $U_t(y)$ by (3.2.6).

Proof. Introducing the term $\varphi(U_t(Y_{N_{t+h}}))$, let be $A_1(t,h) := A_{1,1}(t,h) + A_{1,2}(t,h)$ where

$$\begin{aligned} A_{1,1}(t,h) &= \mathbb{E}\left(\left\{\varphi(U_t(Y_{N_{t+h}})) - \varphi(U_t)\right\} \mathbf{1}_{\{N_{t+h} = N_t + 1\}}\right), \\ A_{1,2}(t,h) &= \mathbb{E}\left(\left\{\varphi(U_{t+h}) - \varphi(U_t(Y_{N_{t+h}}))\right\} \mathbf{1}_{\{N_{t+h} = N_t + 1\}}\right). \end{aligned}$$

Since φ is $\mathcal{C}^1$ class with bounded derivative, Lemma 3.4.1 (cf. Appendix) implies that on the event $\{N_{t+h} - N_t = 1\}$ we have $|\varphi(U_{t+h}) - \varphi(U_t(Y_{N_{t+h}}))| \leq \|\nabla\varphi\|_\infty (3 \sup_{0 \leq u \leq h} |\tilde{X}_{t+u} - \tilde{X}_t| + h)$. Thus

$$h^{-1} A_{1,2}(t,h) \leq \lambda e^{-\lambda h} \|\nabla\varphi\|_\infty (3 \sup_{0 \leq u \leq h} |\tilde{X}_{t+u} - \tilde{X}_t| + h) \to 0$$

when $h \to 0$ using (3.2.11) : $E(\tilde{X}_h^*) \leq C\sqrt{h}$.
To complete the proof, let us deal with $A_{1,1}(t,h)$ to show

$$\lim_{h \to 0} h^{-1} A_{1,1}(t,h) = \lambda \mathbb{E} \int_{\mathbb{R}} [\varphi(U_t(y)) - \varphi(U_t)] dF_Y(y). \tag{3.3.7}$$

On the event $\{N_t = n, N_{t+h} = n+1\}$, the equality $U_t(Y_{N_{t+h}}) = U_t(Y_{n+1})$ holds. The independence properties arising from the structure of the process X, the use of the laws of Y_i, T_n and $T_{n+1} = T_n + S_{n+1}$, conditioning to $\mathcal{F}_t$, give the result :

$$A_{1,1}(t,h) = \sum_n \mathbb{E}\left[1_{\{T_n \leq t < T_{n+1} \leq t+h < T_{n+2}\}} (\varphi(U_t(Y_{n+1})) - \varphi(U_t))\right] =$$

$$\sum_n \mathbb{E}\left[1_{\{T_n \leq t\}} \int_{t-T_n}^{t+h-T_n} ds \int_{\mathbb{R}} (\varphi(U_t(y)) - \varphi(U_t)) F_Y(dy) \lambda e^{-\lambda(t+h-T_n)}\right].$$

By hypothesis, the function φ is bounded and when h goes to 0, Lebesgue's Dominated Convergence Theorem yields

$$\lim_{h\to 0} h^{-1} A_{1,1}(t,h) = \sum_n \mathbb{E}\left[\lambda e^{-\lambda(t-T_n)} 1_{\{T_n \leq t\}} \int_{\mathbb{R}} (\varphi(U_t(y)) - \varphi(U_t))\, dF_Y(y)\right]. \quad (3.3.8)$$

Since $e^{-\lambda(t-T_n)}\mathbf{1}_{\{T_n\leq t\}} = \mathbb{E}[\mathbf{1}_{\{T_n\leq t<T_{n+1}\}}/\mathcal{F}_t]$, it follows

$$\lim_{h\to 0} h^{-1} A_{1,1}(t,h) = \sum_n \mathbb{E}\left[\lambda 1_{\{T_n\leq t<T_{n+1}\}} \int_{\mathbb{R}} (\varphi(U_t(y)) - \varphi(U_t))\, dF_Y(y)\right] =$$
$$\mathbb{E}\left[\lambda \int_{\mathbb{R}} (\varphi(U_t(y)) - \varphi(U_t))\, dF_Y(y)\right]. \quad (3.3.9)$$

■

We note that

$$X_t^* = \max\{(\sup_{u\in[T_i,\inf(T_{i+1},t)[} X_u, i=0,...,N_t), X_t\}$$

and use the joint density of $(\tilde{X}_t^*, \tilde{X}_t)$ given by (3.2.2) to show that the pair (X_t^*, X_t) law's has a density. This one is given in Proposition 3.3.10 below.

Proposition 3.3.10 *For all $t > 0$, the law of the random vector (X_t^*, X_t) admits a density with respect to the Lebesgue measure given by*

$$p(b,a,t) = \mathbb{E}\left(\sum_{k=0}^{N_t} \tilde{p}\left(b - X_{T_k}, a - X_{T_k} - Y_{k+1}\mathbf{1}_{\{T_{k+1}\leq t\}} - (X_t - X_{T_{k+1}\wedge t}), t\wedge T_{k+1} - T_k\right) \mathbf{1}_{\Delta'_{k,t}}(b,a)\right)$$

where

$$\Delta'_{k,t} = \left\{(b,a), \mid b > \max\left(X_{T_k}^*, [a + \sup_{u\in[T_{k+1}\wedge t,t]} (X_u - X_t)]\mathbf{1}_{\{T_{k+1}<t\}}\right)\right\} \quad (3.3.11)$$

and $\tilde{p}$ is given by (3.2.2).

The proof relies on the following lemma :

Lemme 3.3.12 *Almost surely,*

$$X_t^* = \max\left(X_{T_k} + \sup_{u\in[T_k,T_{k+1}\wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_k}\right), \quad k=0,...,N_t\right). \quad (3.3.13)$$

Moreover, almost surely, for all t, there exists a unique k denoted as N_t^* such that

$$X_t^* = X_{T_k} + \sup_{u\in[T_k,T_{k+1}\wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_k}\right). \quad (3.3.14)$$

Proof. (i) Note that

$$X_t^* = \max\left\{\max\left(X_{T_k} + \sup_{u\in[T_k,T_{k+1}\wedge t[} (X_u - X_{T_k}), \quad k=0,...,N_t\right), X_t\right\}. \qquad (3.3.15)$$

For $k \in \mathbb{N}$, for all $u \in [T_k, T_{k+1}[$, $X_u - X_{T_k} = \tilde{X}_u - \tilde{X}_{T_k}$ where $\tilde{X}$ is the continuous process defined in (3.2.1), thus for $k \leq N_t$,

$$\sup_{u\in[T_k,T_{k+1}\wedge t[} (X_u - X_{T_k}) = \sup_{u\in[T_k,T_{k+1}\wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_k}\right). \qquad (3.3.16)$$

and

$$\max\left(X_{T_{N_t}} + \sup_{u\in[T_{N_t},T_{N_t+1}\wedge t[} \left(X_u - X_{T_{N_t}}\right), X_t\right) = X_{T_{N_t}} + \sup_{u\in[T_{N_t},T_{N_t+1}\wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_{N_t}}\right), \qquad (3.3.17)$$

Plugging identities (3.3.16) and (3.3.17) in equality (3.3.15) yields (3.3.13).

(ii) Let two integers $i < j$ then,

$$X_{T_j} + \sup_{u\in[T_j,T_{j+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_j}) = X_{T_i} + (\tilde{X}_{T_{i+1}} - \tilde{X}_{T_i}) + Y_{i+1} + (X_{T_j} - X_{T_{i+1}}) + \sup_{u\in[T_j,T_{j+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_j})$$

and

$$X_{T_j} + \sup_{u\in[T_j,T_{j+1}\wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_j}\right) - X_{T_i} - \sup_{u\in[T_i,T_{i+1}\wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_i}\right) =$$
$$\left\{Y_{i+1} + (X_{T_j} - X_{T_{i+1}}) + \sup_{u\in[T_j,T_{j+1}]} (\tilde{X}_u - \tilde{X}_{T_j})\right\} + \left\{(\tilde{X}_{T_{i+1}} - \tilde{X}_{T_i}) - \sup_{u\in[T_i,T_{i+1}\wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_i}\right)\right\}.$$

The two following random vectors are independent :

$$\left(\sup_{u\in[T_i,T_{i+1}\wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_i}\right); \tilde{X}_{T_{i+1}} - \tilde{X}_{T_i}\right),$$
$$Y_{i+1} + \left(X_{T_j} - X_{T_{i+1}}\right) + \sup_{u\in[T_j,T_{j+1}\wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_j},\right)$$

and the law of the vector $\left(\sup_{u\in[T_i,T_{i+1}\wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_i}\right); \tilde{X}_{T_{i+1}} - \tilde{X}_{T_i}\right)$ admits a density with respect to the Lebesgue measure, hence the law of random variable

$$\sup_{u\in[T_i,T_{i+1}\wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_i}\right) + \tilde{X}_{T_{i+1}} - \tilde{X}_{T_i}$$

has a density with respect to the Lebesgue measure and is independent of

$$Y_{i+1} + \left(X_{T_j} - X_{T_{i+1}}\right) + \sup_{u\in[T_j,T_{j+1}\wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_j}\right).$$

Therefore, $X_{T_j} + \sup_{u\in[T_j,T_{j+1}\wedge t]}\left(\tilde X_u - \tilde X_{T_j}\right) - X_{T_i} - \sup_{u\in[T_i,T_{i+1}\wedge t]}\left(\tilde X_u - \tilde X_{T_i}\right)$ is the sum of two independent random variables, each having a density, then also has a density. So almost surely,

$$X_{T_j} + \sup_{u\in[T_j,T_{j+1}\wedge t]}\left(\tilde X_u - \tilde X_{T_j}\right) \neq X_{T_i} + \sup_{u\in[T_i,T_{i+1}\wedge t]}\left(\tilde X_u - \tilde X_{T_i}\right)$$

whenever $i \neq j$.

(iii) We can exchange the almost sure and $\forall t > 0$ since the processes $(X_t^*, t \geq 0)$, $\left(\left(\max\left(X_{T_k} + \sup_{u\in[T_k,T_{k+1}\wedge t]}\left(\tilde X_u - \tilde X_{T_k}\right)\right),\quad k = 0, ..., N_t\right),\ t \geq 0\right)$ and $(N_t^*,\ t \geq 0)$ are right continuous. ■

Proof. of Proposition 3.3.10 : According to Lemma 3.3.12, let N_t^* denoting the index k where the maximum below is reached,

$$X_t^* = \max\left(X_{T_k} + \sup_{u\in[T_k,T_{k+1}\wedge t]}\left(\tilde X_u - \tilde X_{T_k}\right),\quad k = 0, ..., N_t\right).$$

The fact $N_t^* = k$ is equivalent to : the supremum is reached on the interval $[T_k, T_{k+1}\wedge t]$, actually meaning

$$\sup_{[T_k,T_{k+1}\wedge t]} X_u \geq X_{T_k}^* \vee \sup_{[T_{k+1}\wedge t,t]} X_u.$$

Remark that on the interval $[T_k, T_{k+1} \wedge t]$, $X_u = X_{T_k} + \tilde X_u - \tilde X_{T_k}$ Thus these two inequalities are equivalent to $N_t^* = k$:

$$(i) \quad X_{T_k} + \sup_{[T_k,T_{k+1}\wedge t]}(\tilde X_u - \tilde X_{T_k}) \geq X_{T_k}^*$$

and

$$(ii) \quad X_{T_k} + \sup_{[T_k,T_{k+1}\wedge t]}(\tilde X_u - \tilde X_{T_k}) \geq \sup_{[T_{k+1}\wedge t,t]} X_u = (X_{T_{k+1}} + \sup_{[T_{k+1},t]}(X_u - X_{T_{k+1}})\mathbf{1}_{\{T_{k+1}<t\}}) + X_t\mathbf{1}_{\{T_{k+1}\geq t\}}.$$

Using $X_{T_{k+1}} = X_{T_k} + \tilde X_{T_{k+1}} - \tilde X_{T_k} + Y_{k+1}$ (ii) is equivalent to

$$\sup_{[T_k,T_{k+1}\wedge t]}(\tilde X_u - \tilde X_{T_k}) \geq (\tilde X_{T_{k+1}} - \tilde X_{T_k} + Y_{k+1} + \sup_{[T_{k+1},t]}(X_u - X_{T_{k+1}}))\mathbf{1}_{\{T_{k+1}<t\}}) + (\tilde X_t - \tilde X_{T_k})\mathbf{1}_{\{T_{k+1}\geq t\}}.$$

As a conclusion we get $\{N_t^* = k\} =$

$$\{\sup_{[T_k,T_{k+1}\wedge t]}(\tilde X_u - \tilde X_{T_k}) \geq X_{T_k}^* - X_{T_k}\} \cap \{\sup_{[T_k,T_{k+1}\wedge t]}(\tilde X_u - \tilde X_{T_k}) \geq \tilde X_{t\wedge T_{k+1}} - \tilde X_{T_k} + (Y_{k+1} + \sup_{[T_{k+1},t]}(X_u - X_{T_{k+1}}))\mathbf{1}_{\{T_{k+1}\leq t\}})\}.$$

Thus

$$\{N_t^* = k\} = \left\{\left(\sup_{u\in[T_k,T_{k+1}\wedge t]}(\tilde X_u - \tilde X_{T_k}), \tilde X_{t\wedge T_{k+1}} - \tilde X_{T_k}\right) \in \Delta_{k,t}\right\}$$

where

$$\Delta_{k,t} = \left\{ (b,a), |\, b > \max\left(X^*_{T_k} - X_{T_k}, a + [Y_{k+1} + \sup_{u\in[T_{k+1},t]} (X_u - X_{T_{k+1}})]\mathbf{1}_{\{T_{k+1}\le t\}} \right) \right\}. \tag{3.3.18}$$

Moreover on $\{k \le N_t\}$ so on $\{N^*_t = k\} \subset \{k \le N_t\}$

$$X_t = X_{T_k} + (\tilde{X}_{t\wedge T_{k+1}} - \tilde{X}_{T_k}) + Y_{k+1}\mathbf{1}_{\{\mathbf{t}\ge\mathbf{T_{k+1}}\}} + (X_t - X_{t\wedge T_{k+1}}). \tag{3.3.19}$$

Let Φ be a bounded Borel function, hence

$$\mathbb{E}\left[\Phi(X^*_t, X_t)\right] = \mathbb{E}\left[\sum_{k=0}^{N_t} \Phi(X^*_t, X_t)\mathbf{1}_{\{N^*_t=k\}}\right] =$$

$$\mathbb{E}\left[\sum_{k=0}^{N_t} \mathbf{1}_{\{N^*_t=k\}}\Phi\left(X_{T_k} + \sup_{u\in[T_k,T_{k+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}), X_{T_k} + (\tilde{X}_{t\wedge T_{k+1}} - \tilde{X}_{T_k}) + Y_{k+1}\mathbf{1}_{\{\mathbf{t}\ge\mathbf{T_{k+1}}\}} + (X_t - X_{t\wedge T_{k+1}}) \right)\right].$$

The following random vectors are independent

$$(X_{T_k}, X^*_{T_k}),$$
$$Y_{k+1},$$
$$\left(X_t - X_{t\wedge T_{k+1}}, \sup_{u\in[T_{k+1}\wedge t,t[} (X_u - X_{T_{k+1}\wedge t}) \right),$$
$$\left(\sup_{u\in[T_k,T_{k+1}\wedge t]} \tilde{X}_u - \tilde{X}_{T_k}, \tilde{X}_{t\wedge T_{k+1}} - \tilde{X}_{T_k} \right)$$

and conditionally to $\sigma\left(\mathcal{F}_{T_k}, Y_{k+1}, (X_u - X_{T_{k+1}},\ \ u \ge T_{k+1}\wedge t),\ T_k, T_{k+1} \right)$, the law of the random vector

$$\left(\sup_{u\in[T_k,T_{k+1}\wedge t]} \tilde{X}_u - \tilde{X}_{T_k}, \tilde{X}_{t\wedge T_{k+1}} - \tilde{X}_{T_k} \right)$$

has a density with respect to the Lebesgue measure given by

$$\tilde{p}(b, a, T_{k+1}\wedge t - T_k)$$

where $\tilde{p}$ is defined by (3.2.2). We obtain
$\mathbb{E}\left(\Phi(X^*_t, X_t)\right) =$

$$\int \mathbb{E}\left[\sum_{k=0}^{N_t} \Phi(X_{T_k} + b, X_{T_k} + a + Y_{k+1}\mathbf{1}_{\{\mathbf{t}\ge\mathbf{T_{k+1}}\}} + (X_t - X_{T_{k+1\wedge t}}))\tilde{p}(b, a, T_{k+1}\wedge t - T_k)\mathbf{1}_{\Delta_{k,t}}(b,a)\right] dadb.$$

We conclude with the change of variable formula $v = b + X_{T_k}$ and $u = X_{T_k} + a + Y_{k+1}\mathbf{1}_{\{\mathbf{t}\ge\mathbf{T_{k+1}}\}} + (X_t - X_{T_{k+1\wedge t}})$. ■

As a corollary the law of X_t^* is deduced :

Corollary 3.3.20 *For any $t > 0$, the random variable X_t^* has a density $p^*(.,t)$ given by*

$$p^*(b,t) = 2\mathbb{E}\left(\sum_{k=0}^{N_t} e^{2m(b-X_{T_k})}\left[\frac{e^{-\frac{(b-B_{t,k})^2}{2(t\wedge T_{k+1}-T_k)}}}{\sqrt{2\pi(t\wedge T_{k+1}-T_k)}} - m\Phi_G(-\frac{b-B_{t,k}}{\sqrt{t\wedge T_{k+1}-T_k}})\right]\mathbf{1}_{\{b>X^*_{T_k}\}}\right)$$

where

$$B_{t,k} = m(t\wedge T_{k+1}-T_k) - \min\left(0; -X_{T_k} - Y_{k+1}\mathbf{1}_{\{k<N_t\}} - \sup_{u\in[T_{k+1}\wedge t,t]}(X_u - X_{t\wedge T_{k+1}})\right).$$

Proof. Let be $\tilde{q}$ the function such that such that $\tilde{p}(b,a,.) = \tilde{q}(b,a,.)\mathbf{1}_{b>a\vee 0}$
We have

$$\tilde{q}(b,a,t) = \frac{2e^{2bm}}{\sqrt{2\pi t}}\left(\frac{2b-a-mt}{t}\exp\left[-\frac{[a-(2b+mt)]^2}{2t}\right] - m\exp\left[-\frac{[a-(2b+mt)]^2}{2t}\right]\right).$$

Then

$$\int_{-\infty}^{H}\tilde{p}(b,a,t)da = 2e^{2bm}\left(\frac{1}{\sqrt{2\pi t}}\exp\left[-\frac{(2b+mt-\min(H,b))^2}{2t}\right] - m\Phi_G(\frac{b\wedge H-2b-mt}{\sqrt{t}})\right). \tag{3.3.21}$$

Let be k fixed and $P_k^*(b,t)$ be given by

$$P_k^*(b,t) := \int_{\mathbb{R}}\tilde{p}\left(b-X_{T_k}, a-X_{T_k}-Y_{k+1}\mathbf{1}_{k<N_t}-(X_t-X_{T_{k+1}\wedge t}), t\wedge T_{k+1}-T_k\right)\mathbf{1}_{\Delta'_{k,t}}(b,a)da$$

then

$$p^*(b,t) = \mathbb{E}\left(\sum_{k=0}^{N_t} P_k^*(b,t)\right).$$

With the change of variables $u = a - X_{T_k} - Y_{k+1}\mathbf{1}_{k<N_t} - (X_t - X_{t\wedge T_{k+1}})$, it follows

$$P_k^*(b,t) := \int_{\mathbb{R}}\tilde{p}\left(b-X_{T_k}, u, t\wedge T_{k+1}-T_k\right)\mathbf{1}_{\Delta'_{k,t}}(b, u+X_{T_k}+Y_{k+1}\mathbf{1}_{k<N_t}+(X_t-X_{t\wedge T_{k+1}}))du.$$

According to the definition of $\Delta'_{k,t}$ given in (3.3.11)

$$\mathbf{1}_{\Delta'_{k,t}}(b, u+X_{T_k}+Y_{k+1}\mathbf{1}_{k<N_t}+(X_t-X_{t\wedge T_{k+1}})) = \mathbf{1}_{\{b>X^*_{T_k}\}}\mathbf{1}_{]-\infty,b+C_{t,k}[}(u)$$

where

$$C_{t,k} = -X_{T_k} - Y_{k+1}\mathbf{1}_{\{k<N_t\}} - \sup_{u\in[T_{k+1}\wedge t,t]}(X_u - X_{t\wedge T_{k+1}}).$$

Applying (3.3.21) to $H = b + C_{t,k}$ and $t\wedge T_{k+1}-T_k$ instead of t and since

$$2b + m(t\wedge T_{k+1}-T_k) - \min(C_{t,k}+b,b) = b + m(t\wedge T_{k+1}-T_k) - B_{t,k}$$

we achieve the proof. ■

We now turn to the sudy of $h^{-1}A_0(t,h)$ when h goes to 0. Indeed, on the event $\{N_{t+h} - N_t = 0\}$, $T_{N_t} = T_{N_{t+h}}$, hence $X^*_{T_{N_{t+h}}} = X^*_{T_{N_t}}$ and $X_{T_{N_{t+h}}} = X_{T_{N_t}}$.

$$\begin{aligned} X_{t+h} &= X_t + \tilde{X}_h \circ \theta_t, \\ X^*_{t+h} &= \max(X^*_t, X_t + \tilde{X}^*_h \circ \theta_t). \end{aligned}$$

Using Markov property at t and the fact that N is independent from $\tilde{X}$,

$$A_0(t,h) = e^{-\lambda h}\mathbb{E}\left(\mathbb{E}\left(\varphi(\max(x^*, x + \tilde{X}^*_h), x + \tilde{X}_h, y, u) - \varphi(x^*, x, y, u)\right)_{|x^*=X^*_t, x=X_t, y=X^*_{T_{N_t}}, T_{N_t}=u}\right).$$

Let us introduce

$$a_0(h, x^*, x, y, u) := \mathbb{E}\left(\varphi(\max(x^*, x + \tilde{X}^*_h), x + \tilde{X}_h, y, u) - \varphi(x^*, x, y, u)\right).$$

To study the term $a_0(h, x^*, x, y, u)$, we make a Taylor expansion at a neighborhood of (x^*, x) (y, u are seeing as constant)

$$\begin{aligned} a_0(h, x^*, x, y, u) : &= \partial_1\varphi(x^*, x, y, u)\mathbb{E}\left(\left[\max(x^*, x + \tilde{X}^*_h) - x^*\right]\right) \\ &+ \partial^2_{1,2}\varphi(x^*, x, y, u)\mathbb{E}\left(\left[\max(x^*, x + \tilde{X}^*_h) - x^*\right]\tilde{X}_h\right) \\ &+ \frac{1}{2}\partial^2_{1,1}\varphi(x^*, x, y, u)\mathbb{E}\left(\left[\max(x^*, x + \tilde{X}^*_h) - x^*\right]^2\right) \\ &+ \partial_2\varphi(x^*, x, y, u)mh + \frac{1}{2}\partial^2_{22}\varphi(x^*, x, y, u)[m^2h^2 + h] + R_0(t, h, x^*, x, y, u), \end{aligned}$$

where, using ∇^i the tensor of order i, for all y and u :

$$|R_0(h, x^*, x, y, u)| \le 4\|\nabla^3\varphi\|_\infty\left[\mathbb{E}\left(\left|\max(x^*, x + \tilde{X}^*_h) - x^*\right|^3\right) + \mathbb{E}\left(\left|\tilde{X}_h\right|^3\right)\right].$$

This allows us to write :

$$A_0(t,h) = \sum_{i=1}^{3} A_{0,i}(t,h), \tag{3.3.22}$$

$$A_{0,i}(t,h) := \mathbb{E}\left(a_{0,i}(h, , x^*, x, y, u)_{|x^*=X^*_t, x=X_t, y=X^*_{T_{N_t}}, T_{N_t}=u}\right) \tag{3.3.23}$$

where

$$\begin{aligned} a_{0,1}(h, x^*, x, y, u) : &= \partial_2\varphi(x^*, x, y, u)mh + \frac{1}{2}\partial^2_{22}\varphi(x^*, x, y, u)[m^2h^2 + h] \\ a_{0,2}(h, x^*, x, y, u) : &= \partial^2_{1,2}\varphi(x^*, x, y, u)\mathbb{E}\left(\left[\max(x^*, x + \tilde{X}^*_h) - x^*\right]\tilde{X}_h\right) \\ &\quad + \frac{1}{2}\partial^2_{1,1}\varphi(x^*, x, y, u)\mathbb{E}\left(\left[\max(x^*, x + \tilde{X}^*_h) - x^*\right]^2\right) + R_0(h, x^*, x, y, u), \\ a_{0,3}(h, x^*, x, y, u) : &= \partial_1\varphi(x^*, x, y, u)\mathbb{E}\left(\left[\max(x^*, x + \tilde{X}^*_h) - x^*\right]\right). \end{aligned}$$

Proposition 3.3.24 *Under the hypotheses of Theorem 3.2.5,*

$$\lim_{h\to 0} h^{-1}(A_{0,1}+A_{0,2})(t,h) = \mathbb{E}\left(\partial_2\varphi(U_t)m + \frac{1}{2}\partial^2_{22}\varphi(U_t)\right) \tag{3.3.25}$$

Proof.
(i) Since $\tilde{X}$ is a drifted Brownian motion and φ a three times differentiable function with bounded differential, it follows

$$\lim_{h\to 0} h^{-1}A_{0,1}(t,h) = \mathbb{E}\left(\partial_2\varphi(U_t)m + \frac{1}{2}\partial^2_{22}\varphi(U_t)\right). \tag{3.3.26}$$

(ii) The second term satisfies : Under hypothesis of Theorem 3.2.5,

$$\lim_{h\to 0} h^{-1}A_{0,2}(t,h) = 0.$$

Indeed, we first note that $\max(x^*, x+\tilde{X}^*_h) - x^* = (\tilde{X}^*_h - (x^*-x))^+ \leq \tilde{X}^*_h \mathbf{1}_{\{\tilde{X}^*_h > x^*-x\}}$. Using (3.2.11)

$$\mathbb{E}\left(\left[\max(x^*, x+\tilde{X}^*_h) - x^*\right]^i\right) \leq C h^{i/2}\sqrt{\mathbb{P}(\tilde{X}^*_h > x^*-x)}.$$

The function φ is three times differentiable with bounded differential, we deduce from the expression of $a_{0,2}$,

$$A_{0,2}(t,h) \leq \left[\sum_{i=2}^{3}\|\nabla^i\varphi\|\right] C\sum_{i=2}^{3} h^{i/2}\mathbb{E}\left(\sqrt{\mathbb{P}(\tilde{X}^*_h > x^*-x)}_{|x^*=X_t, x=X_t}\right).$$

The law of the pair (X^*_t, X_t) has a density with respect to Lebesgue measure on $\mathbb{R}^2$, (cf. Proposition 3.3.10) almost surely $X^*_t > X_t$, it follows with Lebesgue dominated convergence Theorem

$$\lim_{h\to 0} h^{-1}A_{0,2}(t,h) = 0.$$

■

The following is needed to prove Proposition 3.3.28.

Lemme 3.3.27 *Let be $h > 0$ and $G(x) := \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} - x\Phi_G(-x)$:*

$$\frac{1}{2}\mathbb{E}\left(\left[\max(x^*, x+\tilde{X}^*_h) - x^*\right]\right) = -mh\int_0^\infty e^{2bm\sqrt{h}}\Phi_G(-b-m\sqrt{h})(b - \frac{(x^*-x)}{\sqrt{h}})_+ db$$
$$+ \sqrt{h}G(\frac{(x^*-x-mh)}{\sqrt{h}}).$$

Proof. By construction,

$$\mathbb{E}\left(\left[\max(x^*, x+\tilde{X}_h^*) - x^*\right]\right) = \mathbb{E}\left(\left[\tilde{X}_h^* - (x^* - x)\right] \mathbf{1}_{\{\tilde{X}_h^* > x^* - x\}}\right).$$

Lemma 3.2.7 gives the density of $\tilde{X}_h^*$ and the change of variable $b \to \sqrt{h}b$ yields

$$\frac{1}{2}\mathbb{E}\left(\left[\max(x^*, x+\tilde{X}_h^*) - x^*\right]\right) =$$
$$\int_{\frac{x^*-x}{\sqrt{h}}}^{\infty} \sqrt{h}\left[b - \frac{(x^*-x)}{\sqrt{h}}]\right]_+ \left[\frac{1}{\sqrt{2\pi}} e^{-\frac{(b-m\sqrt{h})^2}{2}} - m\sqrt{h}e^{2bm\sqrt{h}}\Phi_G(-b - m\sqrt{h})\right] db.$$

This can be written again as

$$\frac{1}{2}\mathbb{E}\left(\left[\max(x^*, x+\tilde{X}_h^*) - x^*\right]\right) = \int_{\frac{x^*-x}{\sqrt{h}}}^{\infty} \sqrt{h}[b - m\sqrt{h} - \frac{(x^*-x) - mh}{\sqrt{h}}]\frac{1}{\sqrt{2\pi}} e^{-\frac{(b-m\sqrt{h})^2}{2}} db$$
$$- mh\int_{\frac{x^*-x}{\sqrt{h}}}^{\infty} e^{2bm\sqrt{h}}(b - \frac{x^*-x}{\sqrt{h}})\Phi_G(-b - m\sqrt{h})db.$$

The lemma is proved using the integration by parts formula and the definition of G. ■

Proposition 3.3.28 *Under hypothesis of Theorem 3.2.5,*

$$\lim_{h\to 0}\frac{1}{2h}\mathbb{E}\left(\partial_1\varphi(U_t)\mathbb{E}\left(\max(x^*, x+\tilde{X}_h^*) - x^*\right)_{x^*=X_t^*, x=X_t}\right) =$$
$$+\frac{1}{4}\mathbb{E}\left[\mathbf{1}_{X_t^* > X_{T_{N_t}}^*}\partial_1\varphi(X_t^*, X_t^*, X_{T_{N_t}}^*, T_{N_t})\frac{\tilde{p}(X_t^* - X_{T_{N_t}}^*, X_t^* - X_{T_{N_t}}^*, t - T_{N_t})}{\tilde{p}^*(X_t^* - X_{T_{N_t}}^*, t - T_{N_t})}\right] \quad (3.3.29)$$

meaning

$$\lim_{h\to 0} A_{0,3}(t,h) = \frac{1}{2}\mathbb{E}\left[\mathbf{1}_{\{X_s^* > X_{T_{N_s}}^*\}}\partial\varphi(X_t^*, X_t^*, X_{T_{N_t}}^*, T_{N_t})\frac{\tilde{p}(X_t^* - X_{T_{N_t}}^*, X_t^* - X_{T_{N_t}}^*, t - T_{N_t})}{\tilde{p}^*(X_t^* - X_{T_{N_t}}^*, t - T_{N_t})}\right].$$

Proof. According to Lemma 3.3.27 , we have

$$\frac{1}{2}\mathbb{E}\left(\partial_1\varphi(U_t)\mathbb{E}\left(\max(x^*, x+\tilde{X}_h^*) - x^*\right)_{x^*=X_t^*, x=X_t}\right) = \sqrt{h}\mathbb{E}\left(\partial_1\varphi(U_t)G(\frac{(X_t^* - X_t - mh)}{\sqrt{h}})\right)$$
$$-mh\mathbb{E}\left(\partial_1\varphi(U_t)\int_0^\infty e^{2bm\sqrt{h}}\Phi_G(-b - m\sqrt{h})(b - \frac{X_t^* - X_t}{\sqrt{h}})_+ db\right).$$

(i) Firstly, we show that

$$\lim_{h\to 0} -m\mathbb{E}\left[\partial\varphi_1(U_t)\int_0^{+\infty} e^{2mb\sqrt{h}}\Phi_G(-b-m\sqrt{h})(b-\frac{X_t^*-X_t}{\sqrt{h}})_+ db\right] = 0.$$

The term $\partial\varphi_1(U_t)\int_0^{+\infty} e^{2mb\sqrt{h}}\Phi_G(-b-m\sqrt{h})(b-\frac{X_t^*-X_t}{\sqrt{h}})_+$ is uniformly bounded with respect to h then use the Lebesgue dominated convergence theorem to obtain the limit. Indeed, $b>0$ and let $0<h\leq 1$.

$$e^{2mb\sqrt{h}}\Phi_G(-b-m\sqrt{h})(b-\frac{X_t^*-X_t}{\sqrt{h}})_+ \leq e^{2mb\sqrt{h}}\Phi_G(-b-m\sqrt{h})b$$

The function $(h,b)\mapsto e^{2mb\sqrt{h}}\Phi_G(-b-m\sqrt{h})b$ is continuous on the compact interval $[0\quad 1]\times[0\quad 2|m|]$, then it is bounded on this interval.
Now, consider $b>2|m|$. Therefore $b+m\sqrt{h}>|m|>0$, $b-m\sqrt{h}>\frac{b}{2}$ and $\frac{b}{(b+m\sqrt{h})}\leq 2$.
We use the inequality (3.2.3) which we recall here

$$1-\Phi_G(x)=\Phi_G(-x)\leq\frac{1}{x\sqrt{2\pi}}\exp-\frac{x^2}{2},\quad \forall x>0$$

to obtain for $b>2|m|$, $h\in[0,1]$

$$e^{2mb\sqrt{h}}\Phi_G(-b-m\sqrt{h})b\leq\frac{b}{(b+m\sqrt{h})\sqrt{2\pi}}e^{2mb\sqrt{h}}e^{-\frac{(b+m\sqrt{h})^2}{2}}\leq\frac{2}{\sqrt{2\pi}}e^{-\frac{b^2}{8}}.$$

This implies that the term $\partial\varphi_1(U_t)\int_0^{+\infty} e^{2mb\sqrt{h}}\Phi_G(-b-m\sqrt{h})(b-\frac{X_t^*-X_t}{\sqrt{h}})_+db$ is uniformly bounded by a constant. The result follows by Lebesgue dominated convergence Theorem : Indeed, almost surely $X_t^*-X_t>0$ and on this set, the integrand goes almost surely to 0. .

(ii) Secondly our goal is to compute the limit when h goes to 0 of the term

$$B_1^*(t,h)=E[\partial_1\varphi(U_t)\frac{1}{\sqrt{h}}G(\frac{X_t^*-X_t-mh}{\sqrt{h}})]=\sum_{n=0}^{\infty}B_{1,n}^*(t,h)$$

where

$$B_{1,n}^*(t,h):=\mathbb{E}\left(\partial_1\varphi(U_t)\frac{1}{\sqrt{h}}G(\frac{X_t^*-X_t-mh}{\sqrt{h}})\mathbf{1}_{\{N_t=n\}}\right).$$

We now develop the proof along five steps.
Step 1 : For any n, we express $B_{1,n}^*(t,h)$ according to the density of the law of the pair

$(\tilde{X}_t^*, \tilde{X}_t)$.
On the event $\{N_t = n\}$,

$$U_t = (\max(X_{T_n}^*, X_{T_n} + \tilde{X}_{t-T_n}^* \circ \theta_t); X_{T_n} + \tilde{X}_{t-T_n} \circ \theta_t, X_{T_n}^*, T_n),$$

hence

$$B_{1,n}^*(t,h) = E[1_{T_n \le t} e^{-\lambda(t-T_n)} \partial_1 \varphi(max(X_{T_n}^*, X_{T_n} + \tilde{X}_{t-T_n}^* \circ \theta_{T_n}), X_{T_n} + \tilde{X}_{t-T_n} \circ \theta_{T_n}, X_{T_n}^*, T_n) \frac{1}{\sqrt{h}} G(\frac{max(X_{T_n}^*, X_{T_n} + \tilde{X}_{t-T_n}^* \circ \theta_{T_n}) - X_{T_n} - \tilde{X}_{t-T_n} \circ \theta_{T_n} - mh}{\sqrt{h}})].$$

The strong Markov property at T_n and the density of the pair $(\tilde{X}^*, \tilde{X})$ at time $t - T_n$ yield for any n,

$$B_{1,n}^*(t,h) = E[1_{T_n \le t} e^{-\lambda(t-T_n)} \int_{b>0, b>a} \partial_1 \varphi(max(X_{T_n}^*, X_{T_n} + b), X_{T_n} + a, X_{T_n}^*, T_n) \times \frac{1}{\sqrt{h}} G(\frac{max(X_{T_n}^*, X_{T_n} + b) - X_{T_n} - a - mh}{\sqrt{h}}) \tilde{p}(b, a, t - T_n)] dbda.$$

As a change of variable, let $\psi : D \mapsto \Delta_h$ be the diffeomorphism such that

$$\psi(a,b) = \left(x = \frac{b-a}{\sqrt{h}}, y = \frac{2b-a}{\sqrt{t-T_n}} - |m|\sqrt{t-T_n} \right),$$

$$D = \{(b,a) : b > a \vee 0\} \text{ and } \Delta_h = \{(y,x) : x > 0, y > \frac{x\sqrt{h} - |m|(t-T_n)}{\sqrt{t-T_n}}\}.$$

Note that

$$\Delta_h \subset \Delta = \{(y,x) : x > 0, y > -|m|\sqrt{t-T_n}\}. \tag{3.3.30}$$

We write again $B_{1,n}^*(t,h)$ as

$$B_{1,n}^*(t,h) = E\left[1_{T_n \le t} e^{-\lambda(t-T_n)} B_{1,n}^{**}(t,h,X_{T_n})\right]$$

where $B_{1,n}^{**}(t,h,X_{T_n}) :=$

$$\int \partial_1 \varphi(\max\{X_{T_n}^*, X_{T_n} + y\sqrt{t-T_n} + |m|(t-T_n) - x\sqrt{h}\}, X_{T_n} + y\sqrt{t-T_n} + |m|(t-T_n) - 2x\sqrt{h}, X_{T_n}, T_n)$$
$$\times \quad G\left(\frac{\max\{X_{T_n}^*, X_{T_n} + y\sqrt{t-T_n} + |m|(t-T_n) - x\sqrt{h}\} - X_{T_n} - y\sqrt{t-T_n} - |m|(t-T_n) + 2x\sqrt{h} - mh}{\sqrt{h}}\right) \tag{3.3.31}$$
$$\times \tilde{p}(y\sqrt{t-T_n} + |m|(t-T_n) - x\sqrt{h}, y\sqrt{t-T_n} + |m|(t-T_n) - 2x\sqrt{h}, t-T_n)\sqrt{t-T_n}\mathbf{1}_{\Delta_h}(y,x) dydx.$$

Step 2 : We study the almost sure limit of the integrand in (3.3.31) with respect to $db \otimes d\mathbb{P}$ when h goes to 0 :
On the event $\{X^*_{T_n} \le X_{T_n} + y\sqrt{t-T_n} + |m|(t-T_n)\}$:

$$\frac{\max\{X^*_{T_n}, X_{T_n} + y\sqrt{t-T_n} + |m|(t-T_n) - x\sqrt{h}\} - X_{T_n} - y\sqrt{t-T_n} - |m|(t-T_n) + 2x\sqrt{h} - mh}{\sqrt{h}} \to x$$

and on the event $\{X^*_{T_n} > X_{T_n} + y\sqrt{t-T_n} + |m|(t-T_n)\}$:

$$\frac{\max\{X^*_{T_n}, X_{T_n} + y\sqrt{t-T_n} + |m|(t-T_n) - x\sqrt{h}\} - X_{T_n} - y\sqrt{t-T_n} - |m|(t-T_n) + 2x\sqrt{h} - mh}{\sqrt{h}} \to \infty.$$

Since G and $\tilde{p}$ are continuous and $\lim_{x\to\infty} G(x) = 0$, and using the definition of Δ given in (3.3.30), the limit of the integrand of (3.3.31) is

$$\begin{aligned}
&\partial_1\varphi(X_{T_n} + y\sqrt{t-T_n} + |m|(t-T_n), X_{T_n} + y\sqrt{t-T_n} + |m|(t-T_n), X_{T_n}, T_n) \\
&\times G(x)\mathbf{1}_{\{X^*_{T_n} \le X_{T_n} + y\sqrt{t-T_n} + |m|(t-T_n)\}} \\
&\times \tilde{p}\left(y\sqrt{t-T_n} + |m|(t-T_n), y\sqrt{t-T_n} + |m|(t-T_n), t-T_n\right)\sqrt{t-T_n}\Big)\mathbf{1}_\Delta(y,x).
\end{aligned}$$

Step 3 : We bound the integrand of $B^*_{1,n}(t,h)$ uniformly with respect to h .
Note that for $0 < h \le 1$:

$$\begin{aligned}
&\frac{\max\{X^*_{T_n}, X_{T_n} + y\sqrt{t-T_n} + |m|(t-T_n) - x\sqrt{h}\} - X_{T_n} - y\sqrt{t-T_n} - |m|(t-T_n) + 2x\sqrt{h} - mh}{\sqrt{h}} = \\
&\frac{\max\{X^*_{T_n}, X_{T_n} + y\sqrt{t-T_n} + |m|(t-T_n) - x\sqrt{h}\} - X_{T_n} - y\sqrt{t-T_n} - |m|(t-T_n) + x\sqrt{h}}{\sqrt{h}} + x - m\sqrt{h} \\
&\ge x - |m|.
\end{aligned}$$

Since G is decreasing, then

$$G\left(\frac{\max\{X^*_{T_n}, X_{T_n} + y\sqrt{t-T_n} + |m|(t-T_n) - x\sqrt{h}\} - X_{T_n} - y\sqrt{t-T_n} - |m|(t-T_n) + 2x\sqrt{h} - mh}{\sqrt{h}}\right) \le G(x-|m|). \tag{3.3.32}$$

Furthermore, since $x > 0$, according to the definition of $\tilde{p}$ given in (3.2.2)

$$\begin{aligned}
&\tilde{p}(y\sqrt{t-T_n} + |m|(t-T_n) - x\sqrt{h}, y\sqrt{t-T_n} + |m|(t-T_n) - 2x\sqrt{h}, t-T_n)\sqrt{t-T_n} \\
&= \left(\frac{2y}{\sqrt{2\pi(t-T_n)}} + \frac{|m|}{\sqrt{2\pi}}\right)e^{-\frac{(y\sqrt{t-T_n}+|m|(t-T_n))^2}{2(t-T_n)} + m(y\sqrt{t-T_n}+|m|(t-T_n)-2x\sqrt{h}) - \frac{m^2}{2}(t-T_n)} \\
&\le \left(\frac{2|y|}{\sqrt{2\pi(t-T_n)}} + \frac{|m|}{\sqrt{2\pi}}\right)e^{-\frac{(y\sqrt{t-T_n}+|m|(t-T_n))^2}{2(t-T_n)} + |m|y\sqrt{t-T_n} + |m|^2(t-T_n) - \frac{m^2}{2}(t-T_n)} \\
&\le \left(\frac{2|y|}{\sqrt{2\pi(t-T_n)}} + \frac{|m|}{\sqrt{2\pi}}\right)e^{-\frac{y^2}{2}}.
\end{aligned} \tag{3.3.33}$$

With estimations (3.3.32) and (3.3.33) one controls the absolute value of the integrand of (3.3.31) by

$$||\partial_1\varphi||_\infty \mathbf{1}_{T_n\leq t} e^{-\lambda(t-T_n)} G(x-|m|)(\frac{2|y|}{\sqrt{2\pi(t-T_n)}}+\frac{|m|}{\sqrt{2\pi}})\exp -\frac{y^2}{2}\mathbf{1}_\Delta(x,y). \quad (3.3.34)$$

Step 4 : We show that (3.3.34) is the term of an integrable and summable series with respect to $d\mathbb{P}\otimes dy\otimes dx$.
Lemma 3.2.10 ($\forall \quad \alpha \in]-1,0]$ $(t-T_{N_t})^\alpha \in L^1$) ensures that for any n (3.3.34) is integrable with respect to $d\mathbb{P}\otimes db\otimes dx$ on $\Omega\times]0,+\infty[^2$ and its integral is bounded up by

$$\|\partial_1\varphi\|\mathbb{E}\left(\mathbf{1}_{\{N_t=n\}}\frac{2}{\sqrt{2\pi}}(t-T_n)^{-\frac{1}{2}}\int G(x-|m|)dx\int |y|e^{-\frac{y^2}{2}}dy\right)$$
$$+\|\partial_1\varphi\|\left(\mathbf{1}_{\{N_t=n\}}\frac{2}{\sqrt{2\pi}}|m|\int G(x-|m|)dx\int e^{-\frac{y^2}{2}}dy\right).$$

Lebesgue dominated convergence theorem yields

$$\lim_{h\to 0} B^*_{1,n}(t,h) = E[1_{T_n\leq t}e^{-\lambda(t-T_n)}B^{**}_{1,n}(t,X_{T_n})]$$

where

$$B^{**}_{1,n}(t,X_{T_n}) :=$$
$$\int \partial_1\varphi(X_{T_n}+y\sqrt{t-T_n}+|m|(t-T_n), X_{T_n}+y\sqrt{t-T_n}+|m|(t-T_n), X_{T_n}, T_n)$$
$$\times G(x)\mathbf{1}_{\{X^*_{T_n}\leq X_{T_n}+y\sqrt{t-T_n}+|m|(t-T_n)\}}$$
$$\times \tilde{p}(y\sqrt{t-T_n}+|m|(t-T_n), y\sqrt{t-T_n}+|m|(t-T_n), t-T_n)\sqrt{t-T_n}\mathbf{1}_\Delta(y,x)dydx.$$

Thus as a conclusion

$$\lim_{h\to 0} B^*_1(t,h) = \sum_n \lim_{h\to 0} B^*_{1,n}(t,h).$$

Step 5 : We study the $\mathbb{P}-$ almost sure limit of left hand side in (3.3.29) when h goes to 0.
Using the change of variable

$$b = y\sqrt{t-T_n}+|m|(t-T_n)$$

we obtain

$$B^{**}_{1,n}(t,X_{T_n}) =$$
$$\int \partial_1\varphi(X_{T_n}+b, X_{T_n}+b, X_{T_n}, T_n)\times G(x)\mathbf{1}_{\{X^*_{T_n}\leq X_{T_n}+b\}}\times\tilde{p}(b,b,t-T_n)\mathbf{1}_{\{x>0,b>0\}}dbdx.$$

Let be $\tilde{p}^*(b, t-T_n)$ the density of the law of $\tilde{X}^*_{t-T_n} \circ \theta_{T_n}$. Noting that $X_{T_n} + \tilde{X}^*_{t-T_n} \circ \theta_{T_n} = X^*_t$ on the event $\{N_t = n\} \cap \{X^*_{T_n} < X_{T_n} + \tilde{X}^*_{t-T_n} \circ \theta_{T_n}\}$ and letting $U(t, T_n) := (X^*_t, X^*_t, X^*_{T_n}, T_n)$, it follows

$$\lim_{h\to 0} B^*_{1,n}(t,h) =$$
$$E\left[1_{T_n \le t} e^{-\lambda(t-T_n)} \int_{x>0} 1_{X^*_{T_n} < X^*_t} \partial_1\varphi(U(t,T_n)) G(x) \frac{\tilde{p}(X^*_t - X^*_{T_n}, X^*_t - X^*_{T_n}, t-T_n)}{\tilde{p}^*(X^*_t - X^*_{T_n}, t-T_n)} dx\right].$$

Markov property (and $\int_{x>0} G(x)dx = 1/4$) implies that for any n

$$\lim_{h\to 0} B^*_{1,n}(t,h) = 1/4 E[1_{N_t=n} 1_{X^*_{T_n} < X^*_t} \partial_1\varphi(U(t,T_n))] \frac{\tilde{p}(X^*_t - X^*_{T_n}, X^*_t - X^*_{T_n}, t-T_n)}{\tilde{p}^*(X^*_t - X^*_{T_n}, t-T_n)}]$$

We take the sum from $n = 1$ to infinity

$$\lim_{h\to 0} \frac{1}{2}\mathbb{E}[\partial_1\varphi(U_t)\frac{1}{h} G(-\frac{X^*_t - X_t - mh}{\sqrt{h}})] =$$

$$1/4\mathbb{E}\left[1_{X^*_{T_{N_t}} < X^*_t} \partial_1\varphi(X^*_t, X^*_t, X^*_{T_{N_t}}, T_{N_t}) \frac{\tilde{p}(X^*_t - X^*_{T_{N_t}}, X^*_t - X^*_{T_{N_t}}, t - T_{N_t})}{\tilde{p}^*(X^*_t - X^*_{T_{N_t}}, t - T_{N_t})}\right].$$

This last step concludes the proof of the proposition. ■

3.4 Appendix

Lemme 3.4.1 *On the event* $\{N_{t+h} = N_t + 1\}$,

$$\left|U_t(Y_{N_{t+h}}) - U_{t+h}\right| \le 3 \sup_{0\le u\le h} |\tilde{X}_{t+u} - \tilde{X}_t| + h.$$

Proof. : (i) On the event $\{N_{t+h} - N_t = 1\} = \cup_n \{N_t = n, N_{t+h} - N_t = 1\}$, we compute $U_t(Y_{N_{t+h}}) = (\max(X^*_t, X_t + Y_{N_{t+h}}); X_t + Y_{N_{t+h}}; \max(X^*_t, X_t + Y_{N_{t+h}}); t)$ and $U_{t+h} = (X^*_{t+h}, X_{t+h}, X^*_{T_{N_{t+h}}}, T_{N_{t+h}})$.
(ii) On the event $\{N_t = n, N_{t+h} = n + 1\}$,

$$U_t(Y_{N_{t+h}}) = (\max(X^*_t, X_t + Y_{n+1}); X_t + Y_{n+1}; \max(X^*_t; X_t + Y_{n+1}); t),$$
$$U_{t+h} = (X^*_{t+h}, X_{t+h}, X^*_{T_{n+1}}, T_{n+1}).$$

(iii) We bound up $|U_t(Y_{N_{t+h}}) - U_{t+h}|$ component by component :

• On the event $\{N_t = n, N_{t+h} - N_t = 1\}$ the fourth component of $|U_t(Y_{N_{t+h}}) - U_{t+h}|$ is $|t - T_{n+1}|$ and we have

$$|t - T_{n+1}|\mathbf{1}_{\{N_t=n,N_{t+h}=n+1\}} \leq h. \tag{3.4.2}$$

• For the second component, note that on the event $\{N_t = n, N_{t+h} = n + 1\}$, we have

$$X_{t+h} = X_t + Y_{n+1} + (X_{t+h} - X_t - Y_{n+1}).$$

Since there is one only jump at time T_{n+1} for the process X between t and $t+h$, hence $X_{t+h} - X_t - Y_{n+1} = \tilde{X}_{t+h} - \tilde{X}_t$ and

$$|X_{t+h} - X_t - Y_{n+1}|\mathbf{1}_{\{N_t=n,N_{t+h}=n+1\}} \leq \sup_{0\leq u\leq h} |\tilde{X}_{t+u} - \tilde{X}_t|\mathbf{1}_{\{N_t=n,N_{t+h}=n+1\}}. \tag{3.4.3}$$

• The third component is $X^*_{T_{n+1}} - \max(X^*_t, X_t + Y_{n+1})$ which we bound up by introducing X^*_{t+h} :

$$|X^*_{T_{n+1}} - \max(X^*_t, X_t + Y_{n+1})| \leq |X^*_{t+h} - \max(X^*_t, X_t + Y_{n+1})| + |X^*_{t+h} - X^*_{T_{n+1}}|.$$

Recall that $\{N_t = n, N_{t+h} = n + 1\} = \{T_n \leq t < T_{n+1} \leq t + h \ < \ T_{n+2}\}$. Since

$$X^*_{t+h} \ = \ \max(X^*_{T_{n+1}}, X_{T_{n+1}} + \sup_{T_{n+1}\leq u\leq t+h} (\tilde{X}_u - \tilde{X}_{T_{n+1}})),$$

we get

$$0 \leq X^*_{t+h} - X^*_{T_{n+1}} = 0 \vee (X_{T_{n+1}} + \sup_{T_{n+1}\leq u\leq t+h} (\tilde{X}_u - \tilde{X}_{T_{n+1}}) - X^*_{T_{n+1}}).$$

But using the inequality $X_{T_{n+1}} \leq X^*_{T_{n+1}}$ which is always true, the following holds on the event $\{t < T_{n+1} \leq t + h\}$

$$X_{T_{n+1}} + \sup_{T_{n+1}\leq u\leq t+h} (\tilde{X}_u - \tilde{X}_{T_{n+1}}) - X^*_{T_{n+1}} \leq \sup_{T_{n+1}\leq u\leq t+h} (\tilde{X}_u - \tilde{X}_{T_{n+1}}) \leq \sup_{0\leq u\leq h} |\tilde{X}_{t+u} - \tilde{X}_t|.$$

Therefore

$$|X^*_{t+h} - X^*_{T_{n+1}}| \leq \sup_{0\leq u\leq h} |\tilde{X}_{t+u} - \tilde{X}_t|. \tag{3.4.4}$$

• The first component is

$$X^*_{t+h} - \max(X^*_t, X_t + Y_{n+1})$$

with

$$X^*_{t+h} = \max(X^*_t, X_t + \sup_{t\leq u\leq T_{n+1}} (\tilde{X}_u - \tilde{X}_t), X_t + (\tilde{X}_{T_{n+1}} - \tilde{X}_t) + Y_{n+1} + \sup_{T_{n+1}\leq u\leq t+h} (\tilde{X}_u - \tilde{X}_{T_{n+1}}))$$

(i) On the event $\{X_t^* \geq X_t + Y_{n+1}\}$,

$$X_{t+h}^* - \max(X_t^*, X_t + Y_{n+1}) =$$
$$0 \vee (X_t + \sup_{t \leq u < T_{n+1}} (\tilde{X}_u - \tilde{X}_t) - X_t^*) \vee (X_t + Y_{n+1} + \sup_{T_{n+1} \leq u \leq t+h} (\tilde{X}_u - \tilde{X}_t) - X_t^*).$$

Since $X_t \leq X_t^*$:

$$X_t + \sup_{t \leq u \leq T_{n+1}} (\tilde{X}_u - \tilde{X}_t) - X_t^* \leq \sup_{t \leq u \leq T_{n+1}} (\tilde{X}_u - \tilde{X}_t) \leq \sup_{t \leq u \leq t+h} (\tilde{X}_u - \tilde{X}_t)$$

and on the event $\{X_t^* \geq X_t + Y_{n+1}\}$

$$X_t + Y_{n+1} + \sup_{T_{n+1} \leq u \leq t+h} (\tilde{X}_u - \tilde{X}_t) - X_t^* \leq \sup_{T_{n+1} \leq u \leq t+h} (\tilde{X}_u - \tilde{X}_t) \leq \sup_{t \leq u \leq t+h} (\tilde{X}_u - \tilde{X}_t).$$

On this event, globally

$$0 \leq X_{t+h}^* - \max(X_t^*, X_t + Y_{n+1}) \leq \sup_{0 \leq u \leq h} |\tilde{X}_{t+u} - \tilde{X}_t|.$$

(ii) On the event $\{X_t^* < X_t + Y_{n+1}\}$, the first component is equal to

$$(X_t^* - X_t - Y_{n+1}) \vee (\sup_{t \leq u < T_{n+1}} (\tilde{X}_u - \tilde{X}_t) - Y_{n+1}) \vee (\sup_{T_{n+1} \leq u \leq t+h} (\tilde{X}_u - \tilde{X}_{T_{n+1}})). \quad (3.4.5)$$

On this event, the first element in (3.4.5) $(X_t^* - X_t - Y_{n+1}) \leq 0$ and the third one being non negative, thus the first component is $(\sup_{t \leq u < T_{n+1}} (\tilde{X}_u - \tilde{X}_t) - Y_{n+1}) \vee (\sup_{T_{n+1} \leq u \leq t+h} (\tilde{X}_u - \tilde{X}_{T_{n+1}}))$. Now look at $\sup_{t \leq u \leq T_{n+1}} (\tilde{X}_u - \tilde{X}_t) - Y_{n+1}$ using $X_t^* < X_t + Y_{n+1}$ so $-Y_{n+1} \leq -X_t^* + X_t \leq 0$:

$$\sup_{t \leq u < T_{n+1}} (\tilde{X}_u - \tilde{X}_t) - Y_{n+1} \leq \sup_{t \leq u \leq T_{n+1}} (\tilde{X}_u - \tilde{X}_t) \leq \sup_{t \leq u \leq t+h} (\tilde{X}_u - \tilde{X}_t).$$

As a conclusion, globally :

$$|X_{t+h}^* - \max(X_t^*, X_t + Y_{n+1})| \mathbf{1}_{\{N_t = n, N_{t+h} = n+1\}} \leq \sup_{t \leq u \leq t+h} |\tilde{X}_u - \tilde{X}_t|. \quad (3.4.6)$$

■

Chapitre 4

Conditional law of the hitting time for a Lévy process in incomplete information

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Abstract

We study the default risk in incomplete information. That means we model the value of a firm by a Lévy process which is the sum of a Brownian motion with drift and

a compound Poisson process. This Lévy process can not be completely observed : another process represents the available information on the firm. We obtain a stochastic Volterra equation satisfied by the conditional density of the default time given the available information. The uniqueness of solution of this equation is proved. Numerical examples of (conditional) density are also given.

4.1 Introduction

Here we consider a jump-diffusion process X which models the value of a firm. This is a Lévy process. Details on this class of processes can be found in [Ber98] and [Sat99]. Their use in financial modeling is well developed in [CT04]. We study the first passage time of process X at level $x > 0$ modeling the default time. We investigate the behavior of the default time under incomplete observation of assets. In the literature, there exists some papers in relation to this topic. Duffie and Lando [DL01] suppose that bond investors cannot observe the issuer's assets directly : instead, they only receive periodic and imperfect reports. For a setting in which the assets of the firm are geometric Brownian motion until informed equity holders optimally liquidate, they derive the conditional distribution of the assets, given the available information. In a similar model, but with complete information, Kou and Wang [KW03] study the first passage time of a jump-diffusion process whose jump sizes follow a double exponential distribution. They obtain explicit solutions of the Laplace transform of the distribution of the first passage time. Laplace transform of the joint distribution of jump-diffusion and its running maximum, $S_t = \sup_{s \leq t} X_s$, is too obtained. To finish, they give numerical examples. Bernyk et al. [BDP08], for their part, consider stable Lévy process X of index $\alpha \in]1, 2[$ with non negative jumps and its running maximum. They characterize the density function of S_t as the unique solution of a weakly singular Volterra integral equation of the first kind. This leads to an explicit representation of the density of the first passage time. To unify the noisy information in Duffie and Lando [DL01], X. Guo, R. A. Jarrow and Y. Zang [GJZ09] define a filtration which models incomplete information. By simple examples, they give the importance of this notion. Similarly to Kou and Wang, without specifying the jumps size law, Dorobantu [Dor07] provides the intensity function of the default time. That is very important for investors, but the information brought by this intensity is low. Furthermore, Roynette et al. [RVV08] prove that the Laplace transform of the random triplet (first passage time, overshoot, undershoot) satisfies an integral equation. After normalization of the first passage time, they show under some convenient assumptions that the random triplet converges in distribution as level x goes to ∞. Gapeev and Jeanblanc [GJ10] study a model of a financial market in which the dividend rates of two risky asset's initial values change when certain unobservable external events occur. The asset price dynamics are described by a geometric Brownian motion, with random drift rates switching at independent exponential random times. These random times are independent of the constantly correlated driving Brownian motion. They obtain closed expressions for rational values of European contingent claims given the available information. Moreover, estimates of the switching times and their conditional probability density are provided. Coutin and Dorobantu [CD+11] prove that the default time law has a density (defective when $\mathbb{E}(X_1) < 0$) with respect to the Lebesgue measure in case of a stationary independent increment process built on a pair (compound Poisson process, Brownian motion).

We extend this approach studying the conditional law of the first passage time of Lévy process at level x given a partial information. We solve this problem using filtering theory inspired by Zakai [Zak69], Pardoux [Par91], Coutin [Cou96], Bain and Crisan [BC09], based on the so called "reference probability measure" method. The paper is organized as follows : Section 2 sets the model ; Section 3 gives the results on the existence of the conditional density given the observed filtration and on the integro-differential equation satisfied by this conditional density ; Section 4 gives the proofs of the results. To finish, we conclude and give some auxiliary results in Appendix.

4.2 Model and motivations

This section defines the basic space in which we work and announces what we will do. Subsection 4.2.1 gives the model of the firm value and defines the default time. Subsection 4.2.2 recalls some important results in the complete information case. Subsection 4.2.3 defines the signal and observation process and the model for available information. Basically, it introduces the notion of filtering theory. Subsection 4.2.4 gives our motivation.

4.2.1 Construction of the model

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), \mathbb{P}^0)$ be a filtered probability space satisfying the usual conditions on which we define a standard Brownian motion W, a sequence of independent and identically distributed random variables $(Y_i)_{i\in\mathbb{N}^*}$ with distribution function F_Y, a Poisson process N with intensity $\lambda > 0$ and a stochastic process Q. We assume that all these elements are independent, (W, Q) is a Brownian motion and (Y, N) is a compound Poisson process with $\mathbb{P}^0$ intensity $\nu(dt, A) = \lambda \int_A F_Y(dy)dt$ for a Borel set A. On this probability space, we define a process X as follows :

$$X_t = mt + W_t + \sum_{i}^{N_t} Y_i. \tag{4.2.1}$$

X models a firm value and the default is modeled by the first passage time of X at a level $x > 0$. Hence the default time is defined as

$$\tau_x = \inf\{t \geq 0 : X_t \geq x\} \tag{4.2.2}$$

We suppose that X is not perfectly observable and that observation is modeled by process Q.

4.2.2 Some results when X is perfectly observed

Let $(\tilde{X}_t, t \geq 0)$ be a Brownian motion with drift $m \in \mathbb{R}$ $(\tilde{X}_t = mt + W_t)$. For $z > 0$, we let $\tilde{\tau}_z = \inf\{t \geq 0, \tilde{X}_t \geq z\}$. By (5.12) page 197 of [KS91], $\tilde{\tau}_z$ has the following law on $\bar{\mathbb{R}}_+$:

$$\tilde{f}(u,z)du + \mathbb{P}^0(\tilde{\tau}_z = \infty)\delta_\infty(du) \tag{4.2.3}$$

where

$$\tilde{f}(u,z) = \frac{|\ z\ |}{\sqrt{2\pi u^3}} \exp[-\frac{1}{2u}(z-mu)^2]\mathbf{1}_{]0,+\infty[}(u) \text{ and } \mathbb{P}^0(\tilde{\tau}_z = \infty) = 1 - e^{mz-|mz|}.$$

The function $\tilde{f}(.,z)$ is $\mathcal{C}^\infty$ on $[0,+\infty[$, and all its derivatives admit 0 as right limit at 0 and therefore belongs to $\mathcal{C}^\infty([0,+\infty[)$. For $\sigma > 0$, Roynette et al. [RVV08] consider as a firm value the process $X_t = mt + \sigma W_t + \sum_{i=1}^{N_t} Y_i$ and as a default time the random variable $\tau_x = \inf\{t \geq 0 : X_t \geq x\}$. They let $K_x := X_{\tau_x} - x$ namely overshoot and $L_x := x - X_{\tau_x^-}$ namely undershoot. They prove that the Laplace transform of (τ_x, K_x, L_x) satisfies an integral equation. After a suitable renormalization of τ_x that we can note here $\bar{\tau_x}$, they show that $(\bar{\tau_x}, K_x, L_x)$ converges in distribution as x goes to ∞. Overall they have obtained an asymptotic behavior of the defaut time, the overshoot and the undershoot.
For a general Lévy process, Doney and Kiprianou [DK06] give the law of the quintuplet $(\bar{G}_{\tau_x}, \tau_x - \bar{G}_{\tau_x^-}, X_{\tau_x} - x, x - X_{\tau_x^-}, x - \bar{X}_{\tau_x^-})$ where $\bar{X}_t = \sup_{s\leq t} X_s$ and $\bar{G}_t = \sup\{s < t : \bar{X}_s = X_s\}$.
Coutin and Dorobantu [CD+11] consider (4.2.1) and (4.2.2) and show that τ_x admits a density with respect to the Lebesgue measure. They give the following closed expression of this density

$$f(t,x) = \begin{cases} \lambda\mathbb{E}(1_{\tau_x>t}(1-F_Y)(x-X_t)) + \mathbb{E}(1_{\tau_x>T_{N_t}}\tilde{f}(t-T_{N_t}, x-X_{T_{N_t}})) & \text{if } t > 0 \\ \frac{\lambda}{2}(2-F_Y(x)-F_Y(x_-)) + \frac{\lambda}{4}(F_Y(x)-F_Y(x_-)) & \text{if } t = 0, \end{cases} \tag{4.2.4}$$

where $(T_i, i \in \mathbb{N}^*)$ is the sequence of the jump times of the process N.

4.2.3 The incomplete information

Our work is inspired and is in the same spirit as D. Dorobantu [Dor07]. In her thesis, Dorobantu assumes that investors wishing to detain a part of the firm do not have complete information. They don't observe perfectly the process value X of the firm but a noisy value. She defined a process Q independent of $W,\ N,\ Y$ and satisfying the following evolution equation

$$Q_t = \int_0^t h(X_s)ds + B_t, \quad t \in \mathbb{R}_+$$

with h a Borel and bounded function and B a standard Brownian motion.

Definition 4.2.5 *The process X is called the signal. The process Q is called the observation and is perfectly observed by investors.*

This leads us to a filtering model and we introduce the filtering framework inspired of Zakai [Zak69], Coutin [Cou96] or Pardoux [Par91].

Since the function h is bounded, the Novikov condition, $\forall T > 0,\ \ \mathbb{E}^0\left(e^{\frac{1}{2}\int_0^T h^2(X_s)ds}\right) < \infty$, is satisfied and we define the following exponential martingale for the filtration $(\mathcal{F}_t)_{t\geq 0}$ by

$$L_t = \exp\left(\int_0^t h(X_s)dQ_s - \frac{1}{2}\int_0^t h^2(X_s)ds\right), \quad t \in \mathbb{R}_+.$$

For a fixed maturity $T > 0$, the process $(L_{t\wedge T}, t \in \mathbb{R}_+)$ is a uniformly integrable $(\mathbb{P}^0, (\mathcal{F}_t)_{t\geq 0})-$martingale.

Definition 4.2.6 *For fixed $t > 0$, let us define a probability measure $\mathbb{P}$ on $\mathcal{F}_t$ by*

$$\mathbb{P}_{|\mathcal{F}_t} := L_t \mathbb{P}^0_{|\mathcal{F}_t}$$

We also note that the law of X, so the one of τ_x, under $\mathbb{P}^0$ are the same as under $\mathbb{P}$. Note that investors have additional information on the firm which is modeled at time t by

$$\mathcal{D}_t = \sigma(\mathbf{1}_{\tau_x \leq u}, u \leq t).$$

Then all the available information is represented by the filtration

$$\mathcal{G} := (\ \mathcal{G}_t = \mathcal{F}_t^Q \vee \mathcal{D}_t, \quad t \geq 0)$$

where the σ-algebra $\mathcal{F}_t^Q$ is generated by the observation of the process Q up to time t.

4.2.4 Motivations

Dorobantu [Dor07] obtains the $\mathcal{G}$-intensity of the default, namely the $\mathcal{G}$-predictable process $(\lambda_t)_{t\geq 0}$, such that

$$M_t = \mathbf{1}_{\tau_x > t} - \int_0^t \lambda_s ds, \quad t \geq 0$$

is a $\mathcal{G}$-martingale. With this result, using their available information, the investors can predict the default time. More precisely, given that default did not occur at time t, the probability that it occurs at time $t + dt$ is approximated by $\lambda_t dt$. But the information brought by the knowledge of $(\lambda_t)_{t\geq 0}$ is low. This motivates us to show that the conditional law of default time τ_x given $\mathcal{G}$ admits a density with respect to Lebesgue measure and to give its dynamic evolution.

4.3 The results

4.3.1 Existence of the conditional density

We recall that $\tau_x = \inf\{t \geq 0 : X_t \geq x\}$ is the default time of a firm and $\mathcal{G}_t$ is the available information of investors at time t. In this subsection, we prove that conditionally on the σ-algebra $\mathcal{G}_t$, τ_x admits a density with respect to the Lebesgue measure.

Proposition 4.3.1 *For all $t > 0$, on the set $\{\tau_x > t\}$, the $\mathcal{G}_t$ conditional law of τ_x has the following form*

$$\begin{aligned} &\bar{f}(r,t,x)dr + \mathbb{P}(\tau_x = \infty|\mathcal{G}_t)\delta_\infty(dr) \\ &\text{and } \mathbb{P}(\tau_x = \infty|\mathcal{G}_t) = \mathbf{1}_{\tau_x > t}\mathbb{E}(G(\infty, x - X_t)|\mathcal{G}_t), \end{aligned} \tag{4.3.2}$$

where

$$\bar{f}(r,t,x) := \mathbb{E}[f(r-t, x - X_t)|\mathcal{G}_t].$$

and

$$G(t,x) := \mathbb{P}(\tau_x > t) = \mathbb{P}^0(\tau_x > t) = \int_t^\infty f(u,x)du.$$

Remark 4.3.3 *Referring to [RVV08], for all $x > 0$, the passage time τ_x is finite almost surely if and only if $m + \mathbb{E}(Y_1) \geq 0$.*

4.3.2 Mixed filtering-Integro-differential equation for conditional density

In this subsection, we give our main results. Indeed, we first show that the conditional law of the hitting time τ_x given the filtration $(\mathcal{G}_t)_{t\geq 0}$ satisfies a stochastic integro-differential equation. Afterwards, we give a uniqueness result. This type of equation is the same as the one studied in [Pro85] with the only difference that here, we have more general Voltera random coefficients.

Theorem 4.3.4 *Let $t > 0$ be a real number. For any $r > t$, on the set $\{\tau_x > t\}$, the conditional density of τ_x given $\mathcal{G}_t$ satisfies the stochastic integro-differential equation :*

$$\begin{aligned}
\bar{f}(r,t,x) &= \frac{f(r,x)}{\mathbb{P}(\tau_x > t)} + \int_0^t \Pi^1(h)(r,t,u)dQ_u \\
&- \int_0^t \frac{\bar{f}(r,u,x)}{\mathbb{E}(\mathbf{1}_{\tau_x > u}G(t-u, x-X_u)|\mathcal{G}_u)}\Pi(h)(t,u)dQ_u \\
&+ \int_0^t \frac{\bar{f}(r,u,x)}{\mathbb{E}(\mathbf{1}_{\tau_x > u}G(t-u, x-X_u)|\mathcal{G}_u)}[\Pi(h)(t,u)]^2 du \\
&- \int_0^t \Pi^1(h)(r,t,u)\Pi(h)(t,u)du.
\end{aligned} \tag{4.3.5}$$

where

$$\Pi^1(\Phi)(r,t,u) = \frac{\mathbb{E}(\mathbf{1}_{\tau_x > u}\Phi(X_u)f(r-u, x-X_u)|\mathcal{G}_u)}{\mathbb{E}(\mathbf{1}_{\tau_x > u}G(t-u, x-X_u)|\mathcal{G}_u)},$$
$$\Pi(\Phi)(t,u) = \frac{\mathbb{E}(\mathbf{1}_{\tau_x > u}\Phi(X_u)G(t-u, x-X_u)|\mathcal{G}_u)}{\mathbb{E}(\mathbf{1}_{\tau_x > u}G(t-u, x-X_u)|\mathcal{G}_u)}$$

and G is defined in Proposition 4.3.1.

Proposition 4.3.6 *If Equation (4.3.5) admits a solution, this one is unique.*

4.3.3 Some technical results

Here, we give some technical and auxiliary results which are useful to prove Theorem 4.3.4 and Proposition 4.3.6.

Proposition 4.3.7 *For any bounded function φ such that $\varphi(\tau_x)$ is $\mathcal{F}_T^X$-measurable, $\forall t \leq T$*

$$\mathbb{E}^0(\varphi(\tau_x)\mathbf{1}_{\tau_x > t}L_T|\mathcal{F}_t^Q) = \mathbb{E}^0[\varphi(\tau_x)\mathbf{1}_{\tau_x > t}] + \int_0^t \mathbb{E}^0\left[L_u h(X_u)\mathbb{E}^0[\mathbf{1}_{\tau_x > t}\varphi(\tau_x)|\mathcal{F}_u]|\mathcal{F}_u^Q\right] dQ_u. \tag{4.3.8}$$

By this proposition, we establish two corollaries which give a representation more accessible of the processes $t \longmapsto \mathbb{E}^0(L_b\mathbf{1}_{a<\tau_x<b}|\mathcal{F}_t^Q)$ and $t \longmapsto \mathbb{E}^0(\mathbf{1}_{\tau_x > T}L_T|\mathcal{F}_t^Q)$: we apply Proposition 4.3.7 respectively to the functions $\phi : y \to \mathbf{1}_{\{]a,b[(y)\}}$ and $\phi : y \to \mathbf{1}_{\{]T,\infty)(y)\}}$, the second expressions being consequence of the fact that on the event $\{\tau_x > t\} \subset \{\tau_x > u\}$, $\tau_x = u + \tau_{x-X_u} \circ \theta_u$ (θ is the shift operator) and $\mathbb{E}^0[\mathbf{1}_{\tau_x>t}\mathbf{1}_{a<\tau_x<b}|\mathcal{F}_u] = \mathbf{1}_{\tau_x>u}\mathbb{E}^0[\mathbf{1}_{a-u<\tau_{x-X_u}<b-u}|\mathcal{F}_u]$.

Corollary 4.3.9 *For all $t < a < b$, we have $\mathbb{P}^0 - a.s$*

$$(i)\quad \mathbb{E}^0(L_b \mathbf{1}_{a<\tau_x<b}|\mathcal{F}_t^Q) = \mathbb{P}^0(a < \tau_x < b) + \int_0^t \mathbb{E}^0(L_u h(X_u)\mathbb{E}^0[\mathbf{1}_{\tau_x>t}\mathbf{1}_{a<\tau_x<b}|\mathcal{F}_u]|\mathcal{F}_u^Q)dQ_u. \tag{4.3.10}$$

and equivalently

$$(ii)\quad \mathbb{E}^0(L_b \mathbf{1}_{a<\tau_x<b}|\mathcal{F}_t^Q) = \mathbb{P}^0(a < \tau_x < b) + \int_0^t \mathbb{E}^0(L_u h(X_u)\mathbf{1}_{\tau_x>u}[G(a-u, x-X_u) - G(b-u, x-X_u)]|\mathcal{F}_u^Q)dQ_u. \tag{4.3.11}$$

Corollary 4.3.12 *For $t \le T$,*

$$(i)\quad \mathbb{E}^0(L_b \mathbf{1}_{\tau_x>T}|\mathcal{F}_t^Q) = \mathbb{P}^0(\tau_x > T) + \int_0^t \mathbb{E}^0(L_u h(X_u)\mathbb{E}^0[\mathbf{1}_{\tau_x>t}\mathbf{1}_{\tau_x>T}|\mathcal{F}_u]|\mathcal{F}_u^Q)dQ_u. \tag{4.3.13}$$

and equivalently

$$(ii)\quad \mathbb{E}^0(\mathbf{1}_{\tau_x>T} L_T|\mathcal{F}_t^Q) = \mathbb{P}^0(\tau_x > T) + \int_0^t \mathbb{E}^0(L_u h(X_u)\mathbf{1}_{\tau_x>u} G(T-u, x-X_u)|\mathcal{F}_u^Q)dQ_u. \tag{4.3.14}$$

Proposition 4.3.15 *For any $0 < t < a < b$, we have on the set $\{\tau_x > t\}$,*

$$\begin{aligned}
\frac{\mathbb{E}^0(L_b \mathbf{1}_{a<\tau_x<b}|\mathcal{F}_t^Q)}{\mathbb{E}^0(\mathbf{1}_{\tau_x>t} L_t|\mathcal{F}_t^Q)} &= \frac{\mathbb{P}^0(a<\tau_x<b)}{\mathbb{P}^0(\tau_x>t)} \\
&+ \int_0^t \frac{\mathbb{E}^0(\mathbf{1}_{\tau_x>u} L_u h(X_u)[G(a-u, x-X_u) - G(b-u, x-X_u)]|\mathcal{F}_u^Q)}{\mathbb{E}^0(\mathbf{1}_{\tau_x>t} L_u|\mathcal{F}_u^Q)} dQ_u \\
&- \int_0^t \frac{\mathbb{E}^0(L_u \mathbf{1}_{a<\tau_x<b}|\mathcal{F}_u^Q)\mathbb{E}^0(\mathbf{1}_{\tau_x>u} L_u h(X_u) G(t-u, x-X_u)|\mathcal{F}_u^Q)}{[\mathbb{E}^0(\mathbf{1}_{\tau_x>t} L_u|\mathcal{F}_u^Q)]^2} dQ_u \qquad (4.3.16) \\
&+ \int_0^t \frac{\mathbb{E}^0(L_u \mathbf{1}_{a<\tau_x<b}|\mathcal{F}_u^Q)[\mathbb{E}^0(\mathbf{1}_{\tau_x>u} L_u h(X_u) G(t-u, x-X_u)|\mathcal{F}_u^Q)]^2}{[\mathbb{E}^0(\mathbf{1}_{\tau_x>t} L_u|\mathcal{F}_u^Q)]^3} du \\
&- \int_0^t \mathbb{E}^0(\mathbf{1}_{\tau_x>u} L_u h(X_u)[G(a-u, x-X_u) - G(b-u, x-X_u)]|\mathcal{F}_u^Q) \\
&\times \frac{\mathbb{E}^0(\mathbf{1}_{\tau_x>u} L_u h(X_u) G(t-u, x-X_u)|\mathcal{F}_u^Q)}{[\mathbb{E}^0(\mathbf{1}_{\tau_x>t} L_u|\mathcal{F}_u^Q)]^2} du.
\end{aligned}$$

Remark 4.3.17 *Equation (4.3.16) of Proposition 4.3.15 can be rewriten as :*

$$\begin{aligned}\bar{\Gamma}_t = \frac{\mathbb{P}^0(a<\tau_x<b)}{\mathbb{P}^0(\tau_x>t)} &+ \int_0^t \sigma^1(h)(t,u)dQ_u \\ &- \int_0^t \bar{\Gamma}_u\sigma(h)(t,u))dQ_u + \int_0^t \bar{\Gamma}_u[\sigma(h)(t,u)]^2du \\ &- \int_0^t \sigma^1(h)(t,u)\sigma(h)(t,u)du.\end{aligned}$$

where

$$\begin{aligned}\bar{\Gamma}_t &= \frac{\mathbb{E}^0(L_b\mathbf{1}_{a<\tau_x<b}|\mathcal{F}_t^Q)}{\mathbb{E}^0(\mathbf{1}_{\tau_x>t}L_t|\mathcal{F}_t^Q)}, \\ \sigma^1(h)(t,u) &= \mathbf{1}_{\{\tau_x>t\}}\frac{\mathbb{E}^0(\mathbf{1}_{\tau_x>u}L_uh(X_u)[G(a-u,x-X_u)-G(b-u,x-X_u)]|\mathcal{F}_u^Q)}{\mathbb{E}^0(\mathbf{1}_{\tau_x>t}L_u|\mathcal{F}_u^Q)}, \\ \sigma(h)(t,u) &= \mathbf{1}_{\{\tau_x>t\}}\frac{\mathbb{E}^0(\mathbf{1}_{\tau_x>u}L_uh(X_u)G(t-u,x-X_u)|\mathcal{F}_u^Q)}{\mathbb{E}^0(\mathbf{1}_{\tau_x>t}L_u|\mathcal{F}_u^Q)}.\end{aligned}$$

This equation is similar to the non normalized conditional distribution equation (3.43) in A. Bain and D. Crisan [BC09], called Zakai equation.
In the same way, Equation (4.3.5) which is derived from (4.3.16) is similar to the normalized conditional distribution equation (3.57) in A. Bain and D. Crisan [BC09], called Kushner-Stratonovich equation.

4.3.4 Numerical examples

We simulate the density of the first passage time respectively in complete information and in incomplete information. We suppose that the jump size follows a double exponential distribution, i.e, the common density of Y is given by $f_Y(y) = p.\eta1.e^{-\eta1y}\mathbf{1}_{y\geq0} + q.\eta2.e^{\eta2y}\mathbf{1}_{y<0}$ where $p,q \geq 0$ are constants, $p+q=1$ and $\eta1,\ \eta2>0$. Here, $\eta1 = \frac{1}{0.02},\ \eta2 = \frac{1}{0.03},\ p=\frac{1}{2}$ and $x=0.1$. The difference between the figures is on one hand due to the information and on another hand to the values taken by the parameters m and λ.
These four first figures (Figue4.1 and Figure4.2) represent the densities of the first passage time for a jump diffusion process (case of complete information). The variable $t\in[0,1]$ and Monte Carlo results are based on 5000 simulation runs.

The following figures are those of the conditional density $\bar{f}(r,t,x)$ (case of incomplete information), for fixed $t=0.1$ and the variable r is such that $r\in]0.1,0.6]$. Part II of A. Bain and D. Crisan [BC09], namely Numerical Algorithms, where the authors give

λ	CPU time
3	438.03805

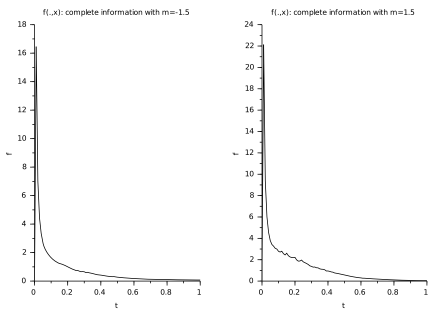

FIGURE 4.1 – Densities for $\lambda = 3$.

λ	CPU time
0.1	376.6704

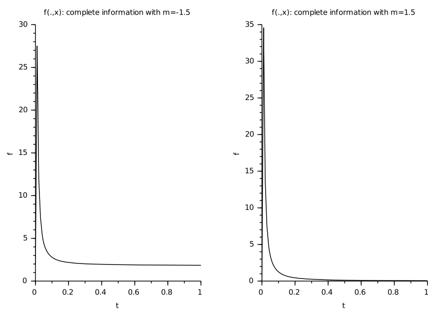

FIGURE 4.2 – Densities for $\lambda = 0.1$.

some tools to solve the filtering problem is really useful. The class of the numerical method used is the particle method for continuous time framework.Here, the Monte Carlo results are based on 120 simulation runs.

λ	CPU time
2	358.11432

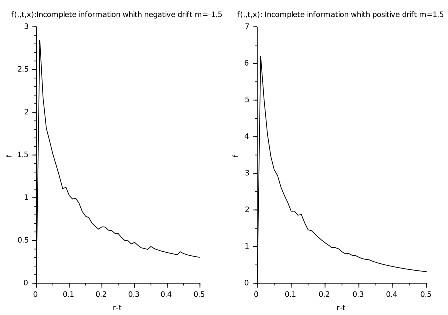

FIGURE 4.3 – Conditional densities for $\lambda = 2$.

λ	CPU time
0.1	353.00736

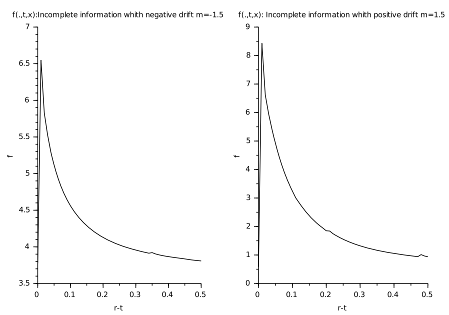

FIGURE 4.4 – Conditional densities for $\lambda = 0.1$.

We observe that in the two cases, the maximum reached is greater if the drift m is positive, meaning the positive level x is more probably reached in a shorter time.
In incomplete information, the distance between the curve and axis is greater than in complete information case, this would mean that in case of incomplete information, the level x is more difficult to be reached in a short time.
The choice of the small value of λ serves to compare the results with the limiting Brownian motion case ($\lambda = 0$). In complete information case, the formulae for the first

passage times of Brownian motion can be found in [KS91].
A large value of λ implies a lot of jumps, a large computing time and less regular curve.

λ	CPU time
0.01	373.16157

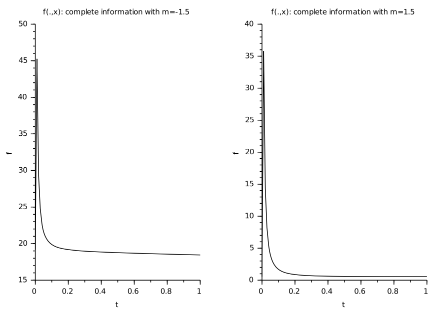

FIGURE 4.5 – Densities for $\lambda = 0.01$.

λ	CPU time
0.01	358.96784

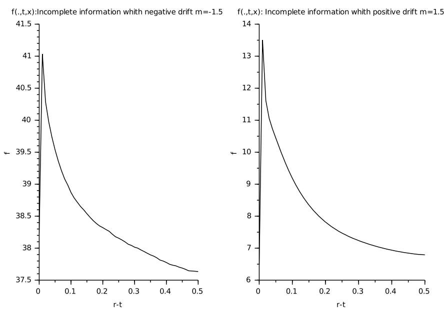

FIGURE 4.6 – Conditional densities for $\lambda = 0.01$.

In these last two figures, the maximum reached is greater if the drift m is negative, meaning the positive level x is more probably reached in a shorter time. This is due to the very small value of λ.

4.4 Proofs

Proposition 4.3.1

Proof. First note that, since X is a $(\mathcal{F}, \mathbb{P})$- Markov process and $\mathcal{G}_t \subset \mathcal{F}_t$, we have

$$\begin{aligned}\mathbb{E}(\mathbf{1}_{\tau_x=\infty}|\mathcal{G}_t) &= \mathbb{E}\left(\mathbb{E}(\mathbf{1}_{\tau_x=\infty}|\mathcal{F}_t)|\mathcal{G}_t\right)\\ &= \mathbb{E}[\mathbf{1}_{\tau_x>t}\mathbb{E}^t(\mathbf{1}_{\tau_{x-X_t}=\infty})|\mathcal{G}_t]\\ &= \mathbf{1}_{\tau_x>t}\mathbb{E}(G(\infty, x-X_t)|\mathcal{G}_t), \text{ where } \mathbb{E}^t(.) = \mathbb{E}(.|\mathcal{F}_t).\end{aligned}$$

The fact that τ_x is a $(\mathcal{G}, \mathbb{P})-$ stopping time justifies the last equality.
Secondly, for any $b \geq a > t$ the $(\mathbb{P}, \mathcal{F})$ Markov property of the process X and the fact that on the set $\{\tau_x > t\}$, $\tau_x = t + \tau_{x-X_t} \circ \theta_t$ ensure

$$\begin{aligned}\mathbb{E}(\mathbf{1}_{a\leq\tau_x<b}|\mathcal{G}_t) &= \mathbb{E}\left(\mathbb{E}(\mathbf{1}_{a\leq\tau_x<b}|\mathcal{F}_t)|\mathcal{G}_t\right)\\ &= \mathbb{E}\left(\mathbf{1}_{\tau_x>t}\mathbb{E}^t(\mathbf{1}_{a-t\leq\tau_{x-X_t}<b-t})|\mathcal{G}_t\right).\end{aligned}$$

The $\mathcal{F}_t$- conditional law of τ_{x-X_t} has the density (possibly defective) $f(.-t, x-X_t)$, thus

$$\mathbb{E}(\mathbf{1}_{a\leq\tau_x<b}|\mathcal{G}_t) = \mathbb{E}\left[\mathbf{1}_{\tau_x>t}\int_a^b f(r-t, x-X_t)dr|\mathcal{G}_t\right].$$

By hypothesis, we have $r-t \geq a-t > 0$. It follows from Lemma 4.6.3 of Appendix that

$$\mathbb{E}\left[\mathbf{1}_{\tau_x>t}\int_a^b f(r-t, x-X_t)dr\right] < \infty.$$

Then, we have for any $b \geq a > t$,

$$\mathbb{E}\left[\mathbf{1}_{\tau_x>t}\int_a^b f(r-t, x-X_t)dr|\mathcal{G}_t\right] = \int_a^b \mathbb{E}\left[\mathbf{1}_{\tau_x>t}f(r-t, x-X_t)|\mathcal{G}_t\right]dr \quad a.s.$$

Now, we show the equality almost surely for all $b \geq a > t$. Let M_1 and M_2 be the processes defined by

$$M_1 : b \longmapsto \mathbb{E}\left[\mathbf{1}_{\tau_x>t}\int_a^b f(r-t, x-X_t)dr|\mathcal{G}_t\right] \text{ and } M_2 : b \longmapsto \int_a^b \mathbb{E}\left[\mathbf{1}_{\tau_x>t}f(r-t, x-X_t)|\mathcal{G}_t\right]dr.$$

These processes are increasing, then they are sub-martingales with respect to the filtration $\tilde{\mathcal{G}}_b = \mathcal{G}_t \ \forall b \geq t$. Note that $b \longmapsto \mathbb{E}(M_1(b))$ and $b \longmapsto \mathbb{E}(M_2(b))$ are too continuous. Using Revuz-Yor Theorem 2.9 p. 61 [RY99], they have same càd-làg modification for all b, meaning that

$$\mathbb{E}\left[\mathbf{1}_{\tau_x>t}\int_a^b f(r-t, x-X_t)dr|\mathcal{G}_t\right] = \int_a^b \mathbb{E}\left[\mathbf{1}_{\tau_x>t}f(r-t, x-X_t)|\mathcal{G}_t\right]dr \qquad a.s. \ \forall b.$$

We conclude that, almost surely, for all $b \geq a > t$,

$$\mathbf{1}_{\tau_x>t}\mathbb{E}(\mathbf{1}_{a<\tau_x\leq b}|\mathcal{G}_t) = \mathbf{1}_{\tau_x>t}\int_a^b \mathbb{E}\left[\mathbf{1}_{\tau_x>t}f(r-t, x-X_t)|\mathcal{G}_t\right]dr.$$

Taking $a = t + \frac{1}{n}$, letting n going to infinity and using monotone Lebesgue Theorem yield that, $\mathbb{P}-a.s \ \forall \ b \geq t$,

$$\mathbb{E}(\mathbf{1}_{t<\tau_x\leq b}|\mathcal{G}_t) = \int_t^b \mathbb{E}\left[\mathbf{1}_{\tau_x>t}f(r-t, x-X_t)|\mathcal{G}_t\right]dr.$$

■

Proposition 4.3.6

Proof. : Let $\bar{f}$ and $\bar{g}$ be two solutions of Equation (4.3.5) and $\bar{\delta} = \bar{f} - \bar{g}$. It follows that

$$\bar{\delta}(r, t, x) = -\int_0^t \bar{\delta}(r, u, x)K(t, u, x)dQ_u + \int_0^t \bar{\delta}(r, u, x)K(t, u, x)\Pi(h)(t, u)du \quad (4.4.1)$$

where

$$K(t,u,x) = \frac{\Pi(h)(t,u)}{\mathbb{E}(\mathbf{1}_{\tau_x>u}G(t-u,x-X_u)|\mathcal{G}_u)}. \tag{4.4.2}$$

We recall the expression

$$\Pi(h)(t,u) = \frac{\mathbb{E}(\mathbf{1}_{\tau_x>u}h(X_u)G(t-u,x-X_u)|\mathcal{G}_u)}{\mathbb{E}(\mathbf{1}_{\tau_x>u}G(t-u,x-X_u)|\mathcal{G}_u)}$$

and remark that $|\Pi(h)(t,u)| \leq \|h\|_\infty$. Then

$$|K(t,u,x)| \leq \mathbf{1}_{\tau_x>u}\frac{||h||_\infty}{\mathbb{E}(\mathbf{1}_{\tau_x>u}G(t-u,x-X_u)|\mathcal{G}_u)}.$$

Markov property implies

$$|K(t,u,x)| \leq \mathbf{1}_{\tau_x>u}\frac{||h||_\infty}{\mathbb{E}(\mathbf{1}_{\tau_x>t}|\mathcal{G}_u)}.$$

We use Lemma 4.6.5 with $t = u$ $Y = \mathbf{1}_{\tau_x>t}$ and $b = t$ and it follows that

$$|K(t,u,x)| \leq ||h||_\infty \mathbf{1}_{\tau_x>u}\frac{\mathbb{E}^0(\mathbf{1}_{\tau_x>u}L_u|\mathcal{F}_u^Q)}{\mathbb{E}^0(\mathbf{1}_{\tau_x>t}L_t|\mathcal{F}_u^Q)}$$

and Lemma 4.6.10 (4.6.11) with the pair (t,u) gets

$$|K(t,u,x)| \leq ||h||_\infty \mathbf{1}_{\tau_x>u}\frac{\mathbb{E}^0(\mathbf{1}_{\tau_x>u}L_u|\mathcal{F}_u^Q)}{\mathbb{E}^0(\mathbf{1}_{\tau_x>t}L_u|\mathcal{F}_u^Q)}$$

All computations are done on the set $\{\tau_x > t\}$. We observe too $u \longrightarrow \frac{1}{\mathbb{E}^0(\mathbf{1}_{\tau_x>t}L_t|\mathcal{F}_u^Q)}$ is a positive submartingale. Then for all $T \geq t \geq u$, we obtain by Lemma 4.6.10 (4.6.11) with the pair (t,T), Doob's inequality and $\{\tau_x > T\} \subset \{\tau_x > t\}$,

$$\mathbb{E}^0\left(\left[\sup_{u\leq T}\frac{1}{\mathbb{E}^0(\mathbf{1}_{\tau_x>t}L_t|\mathcal{F}_u^Q)}\right]^2\right) \leq 4\mathbb{E}^0\left(\left[\frac{1}{\mathbb{E}^0(\mathbf{1}_{\tau_x>t}L_t|\mathcal{F}_T^Q)}\right]^2\right) \leq 4\mathbb{E}^0\left(\left[\frac{1}{\mathbb{E}^0(\mathbf{1}_{\tau_x>T}L_T|\mathcal{F}_T^Q)}\right]^2\right)$$

Thanks to Jensen inequality and Lemma 4.6.14 with $\alpha = 2$ and $t = T$, it follows that

$$\mathbb{E}^0\left(\sup_{u\leq T}\frac{1}{\mathbb{E}^0(\mathbf{1}_{\tau_x>t}L_u|\mathcal{F}_u^Q)}\right) < \infty.$$

Concerning the numerator, $\mathbb{E}^0(\mathbf{1}_{\tau_x>u}L_u|\mathcal{F}_u^Q) \leq \mathbb{E}^0(L_u|\mathcal{F}_u^Q)$. Since Novikov condition $\mathbb{E}^0\left(e^{\frac{1}{2}\int_0^T h^2(X_s)ds}\right) < \infty$ is satisfied then $\mathbb{E}^0(L_u|\mathcal{F}_u^Q)$ is a locally square integrable $(\mathbb{P}^0, \mathcal{F}^Q)$-martingale. Once again Doob's inequality gets

$$\mathbb{E}^0\left(\sup_{u\leq T}\mathbb{E}^0(\mathbf{1}_{\tau_x>u}L_u|\mathcal{F}_u^Q)\right) < \infty.$$

So finally

$$\sup_{u\le T}\frac{\mathbb{E}^0(\mathbf{1}_{\tau_x>u}L_u|\mathcal{F}^Q_u)}{\mathbb{E}^0(\mathbf{1}_{\tau_x>t}L_t|\mathcal{F}^Q_u)}<\infty\ \mathbb{P}-a.s. \tag{4.4.3}$$

Let $T_n(\omega)=\inf\left\{t\ge 0:\frac{\mathbb{E}^0(\mathbf{1}_{\tau_x>u}L_u|\mathcal{F}^Q_u)}{\mathbb{E}^0(\mathbf{1}_{\tau_x>t}L_u|\mathcal{F}^Q_u)}\ge n\right\}$, and $\Omega_n=\left\{\omega:\frac{\mathbb{E}^0(\mathbf{1}_{\tau_x>u}L_u|\mathcal{F}^Q_u)}{\mathbb{E}^0(\mathbf{1}_{\tau_x>t}L_t|\mathcal{F}^Q_u)}\le n\right\}$. On the set Ω_n, $|K(t,u,x)|\le n||h||_\infty$, $T_n(\omega)\ge t$. Moreover (4.4.3) proves that $T_n\to\infty$ so $\cup_n\Omega_n=\Omega$

It follows using (4.4.1) that

$$\begin{aligned}\bar\delta(r,t,x)\mathbf{1}_{\Omega_n}&=\mathbf{1}_{\Omega_n}\left[-\int_0^t\bar\delta(r,u,x)K(t,u,x)dQ_u+\int_0^t\bar\delta(r,u,x)K(t,u,x)\Pi(h)(t,u)du\right]\\&=\mathbf{1}_{\Omega_n}\left[-\int_0^{t\wedge T_n}\bar\delta(r,u,x)K(t,u,x)dQ_u+\int_0^{t\wedge T_n}\bar\delta(r,u,x)K(t,u,x)\Pi(h)(t,u)du\right]\\&=\mathbf{1}_{\Omega_n}\left[-\int_0^t\mathbf{1}_{u\le T_n}\bar\delta(r,u,x)K(t,u,x)dQ_u+\int_0^t\mathbf{1}_{u\le T_n}\bar\delta(r,u,x)K(t,u,x)\Pi(h)(t,u)du\right].\end{aligned}$$

Taking $\bar\Delta_n(r,t,x)=\bar\delta(r,t,x)\mathbf{1}_{\Omega_n}$, we obtain

$$\bar\Delta_n(r,t,x)=-\int_0^t\bar\Delta_n(r,u,x)K(t,u,x)dQ_u+\int_0^t\bar\Delta_n(r,u,x)K(t,u,x)\Pi(h)(t,u)du. \tag{4.4.4}$$

Then

$$\begin{aligned}\mathbb{E}[|\bar\Delta_n(r,t,x)|^2]&\le 2\mathbb{E}\left[|\int_0^t\bar\Delta_n(r,u,x)K(t,u,x)dQ_u|^2\right]+2\mathbb{E}\left[|\int_0^t\bar\Delta_n(r,u,x)K(t,u,x)\Pi(h)(t,u)du|^2\right]\\&\le 2n\|h\|^2_\infty(1+||h||^2_\infty)\int_0^t\mathbb{E}[|\bar\Delta_n(r,u,x)|^2]du\end{aligned}$$

By Gronwall's lemma, we deduce that $\bar\Delta_n(r,t,x)=0$ is the unique solution of (4.4.4) on the set Ω_n, so $\forall n\ \bar\delta(r,t,x)\mathbf{1}_{\Omega_n}=0$. Uniqueness of solution of (4.3.5) is a consequence of $\Omega=\cup_n\Omega_n$. ∎

Proposition 4.3.7

Proof. Let be a process $S\in\mathcal{S}$ where the set of processes $\mathcal{S}$ is defined in Lemma 4.6.7 and a time t. Lemma 4.6.10 applied to $Y=\ \varphi(\tau_x)\mathbf{1}_{\tau_x>t}$ which belongs to $L^\infty(\Omega,\mathbb{P}^0,\mathcal{F}^X_T)$ implies

$$\mathbb{E}^0(\varphi(\tau_x)\mathbf{1}_{\tau_x>t}L_TS_t)=\mathbb{E}^0[\varphi(\tau_x)\mathbf{1}_{\tau_x>t}]+\mathbb{E}^0\left(\int_0^t\varphi(\tau_x)\mathbf{1}_{\tau_x>t}L_uS_u\rho_uh(X_u)du\right).$$

Conditioning by $\mathcal{F}^Q_u$ under the time integral, it follows that

$$\begin{aligned}\mathbb{E}^0(\varphi(\tau_x)\mathbf{1}_{\tau_x>t}L_TS_t)=&\mathbb{E}^0[\varphi(\tau_x)\mathbf{1}_{\tau_x>t}]+\\&\mathbb{E}^0\left(\int_0^tS_u\rho_u\mathbb{E}^0(L_uh(X_u)\mathbf{1}_{\tau_x>t}\varphi(\tau_x)|\mathcal{F}^Q_u)du\right).\end{aligned}$$

Conversely compute the expectation of the product of $S_t = 1 + \int_0^t S_u \rho_u dQ_u$ by right hand of (4.3.8) :

$$\mathbb{E}^0 \left[S_t \left(\mathbb{E}^0[\varphi(\tau_x)\mathbf{1}_{\tau_x > t}] + \int_0^t \mathbb{E}^0(L_u h(X_u)\mathbf{1}_{\tau_x > t}\varphi(\tau_x)|\mathcal{F}_u^Q)dQ_u \right) \right] =$$
$$\mathbb{E}^0[\varphi(\tau_x)\mathbf{1}_{\tau_x > t}] + \mathbb{E}^0 \left(\int_0^t S_u \rho_u \mathbb{E}^0(L_u h(X_u)\mathbf{1}_{\tau_x > t}\varphi(\tau_x)|\mathcal{F}_u^Q)du \right)$$

Since $\mathcal{S}$ is dense in $L^2(\Omega, \mathcal{F}^Q, \mathbb{P}^0)$,

$$\mathbb{E}^0(\varphi(\tau_x)\mathbf{1}_{\tau_x > t}L_T|\mathcal{F}_t^Q) = \mathbb{E}^0[\varphi(\tau_x)\mathbf{1}_{\tau_x > t}] + \int_0^t \mathbb{E}^0[L_u h(X_u)\mathbf{1}_{\tau_x > t}\varphi(\tau_x)|\mathcal{F}_u^Q]dQ_u.$$

Finally we could replace $\mathbf{1}_{\tau_x > t}\varphi(\tau_x)$ by its $\mathcal{F}_u$ conditional expectation since $\mathcal{F}_u^Q \subset \mathcal{F}_u$. ■

Proposition 4.3.15

Proof. Applying Lemma 4.6.5, it follows that

$$\mathbb{E}(\mathbf{1}_{a < \tau_x < b}|\mathcal{G}_t) = \mathbf{1}_{\tau_x > t} \frac{\mathbb{E}^0(L_b \mathbf{1}_{a < \tau_x < b}|\mathcal{F}_t^Q)}{\mathbb{E}^0(\mathbf{1}_{\tau_x > t}L_t|\mathcal{F}_t^Q)}. \tag{4.4.5}$$

But, since the condition $\int_0^t \mathbb{E}^0(f^2(t-u, x - X_u)))du < \infty$ is not necessarily satisfied, we are not able to prove that $\mathbb{E}^0(\mathbf{1}_{\tau_x > t}L_t|\mathcal{F}_t^Q)$ is a semi martingale (e.g. see Protter's Theorem 65 Chapter 4 [Pro13]). This leads us to consider for $t < T \leq t+1$, the expression $\mathbb{E}^0(\mathbf{1}_{\tau_x > T}L_T|\mathcal{F}_t^Q)$ instead of $\mathbb{E}^0(\mathbf{1}_{\tau_x > t}L_t|\mathcal{F}_t^Q)$ at denominator of (4.4.5). But Lemma 4.6.10 of Appendix ensures that

$$\frac{\mathbb{E}^0(L_b \mathbf{1}_{a < \tau_x < b}|\mathcal{F}_t^Q)}{\mathbb{E}^0(\mathbf{1}_{\tau_x > T}L_T|\mathcal{F}_t^Q)} = \frac{\mathbb{E}^0(L_t \mathbf{1}_{a < \tau_x < b}|\mathcal{F}_t^Q)}{\mathbb{E}^0(\mathbf{1}_{\tau_x > T}L_t|\mathcal{F}_t^Q)}.$$

We apply Ito formula to the ratio of processes $\dfrac{\mathbb{E}^0(L_b \mathbf{1}_{a < \tau_x < b}|\mathcal{F}_t^Q)}{\mathbb{E}^0(\mathbf{1}_{\tau_x > T}L_t|\mathcal{F}_t^Q)}$. For this end, we let two processes satisfying the stochastic equations respectively (4.3.11) and (4.3.14) :

$$X_t = \mathbb{E}^0(L_t \mathbf{1}_{a < \tau_x < b}|\mathcal{F}_t^Q), \quad Y_t = \mathbb{E}^0(\mathbf{1}_{\tau_x > T}L_t|\mathcal{F}_t^Q) \text{ and } f(x, y) = \frac{x}{y}$$

The Itô's formula applied to $f(X_., Y_.)$ from 0 to t gives us

$$
\begin{aligned}
\frac{\mathbb{E}^0(L_b \mathbf{1}_{a<\tau_x<b}|\mathcal{F}^Q_t)}{\mathbb{E}^0(\mathbf{1}_{\tau_x>T} L_T|\mathcal{F}^Q_t)} = {} & \frac{\mathbb{P}^0(a<\tau_x<b)}{\mathbb{P}^0(\tau_x>T)} \\
& + \int_0^t \frac{\mathbb{E}^0(\mathbf{1}_{\tau_x>u} L_u h(X_u)[G(a-u, x-X_u) - G(b-u, x-X_u)]|\mathcal{F}^Q_u)}{\mathbb{E}^0(\mathbf{1}_{\tau_x>T} L_u|\mathcal{F}^Q_u)} dQ_u \\
& - \int_0^t \frac{\mathbb{E}^0(L_u \mathbf{1}_{a<\tau_x<b}|\mathcal{F}^Q_u)\mathbb{E}^0(\mathbf{1}_{\tau_x>u} L_u h(X_u) G(T-u, x-X_u)|\mathcal{F}^Q_u)}{[\mathbb{E}^0(\mathbf{1}_{\tau_x>T} L_u|\mathcal{F}^Q_u)]^2} dQ_u \\
& + \int_0^t \frac{\mathbb{E}^0(L_u \mathbf{1}_{a<\tau_x<b}|\mathcal{F}^Q_u)[\mathbb{E}^0(\mathbf{1}_{\tau_x>u} L_u h(X_u) G(T-u, x-X_u)|\mathcal{F}^Q_u)]^2}{[\mathbb{E}^0(\mathbf{1}_{\tau_x>T} L_u|\mathcal{F}^Q_u)]^3} du \\
& - \int_0^t \mathbb{E}^0(\mathbf{1}_{\tau_x>u} L_u h(X_u)[G(a-u, x-X_u) - G(b-u, x-X_u)]|\mathcal{F}^Q_u) \\
& \times \frac{\mathbb{E}^0(\mathbf{1}_{\tau_x>u} L_u h(X_u) G(T-u, x-X_u)|\mathcal{F}^Q_u)}{[\mathbb{E}^0(\mathbf{1}_{\tau_x>T} L_u|\mathcal{F}^Q_u)]^2} du.
\end{aligned}
$$

To achieve the proof, we can $T = t$ using monotonous Lebesgue theorem since $\mathbf{1}_{\tau_x>T}$ increases to $\mathbf{1}_{\tau_x>t}$ when $T \to t$. ■

Theorem 4.3.4

Proof.
Let us now find a mixed filtering-integro-differential equation satisfied by the conditional probability density process defined from the representation

$$\mathbb{E}(\mathbf{1}_{a<\tau_x<b}|\mathcal{G}_t) = \int_a^b \bar{f}(r, t, x)dr \text{ for some } a > t. \tag{4.4.6}$$

We fix a and t such that $a > t$. Let be $u \leq t$, recalling the $(\mathbb{P}^0, \mathcal{F})-$ Markov property of X at point u and the fact that $\mathcal{F}^Q \subset \mathcal{F}$ justify

$$\mathbb{E}^0\left(L_u \mathbf{1}_{a<\tau_x<b}|\mathcal{F}^Q_u\right) = \mathbb{E}^0\left[L_u \mathbf{1}_{\tau_x>u} \mathbb{E}^0(\mathbf{1}_{a-u<\tau_{x-X_u}<b-u}|\mathcal{G}_u)|\mathcal{F}^Q_u\right].$$

By definition of G , we have

$$\mathbb{E}^0(\mathbf{1}_{a-u<\tau_{x-X_u}<b-u}/\mathcal{G}_u) = G(a-u, x-X_u) - G(b-u, x-X_u) = \int_a^b f(r-u, x-X_u)dr.$$

Then

$$\mathbb{E}^0\left(L_u \mathbf{1}_{a<\tau_x<b}|\mathcal{F}^Q_u\right) = \mathbb{E}^0\left(L_u \mathbf{1}_{\tau_x>u} \int_a^b f(r-u, x-X_u)dr|\mathcal{F}^Q_u\right).$$

By Tonelli Theorem,

$$\mathbb{E}^0\left(L_u\mathbf{1}_{a<\tau_x<b}|\mathcal{F}_u^Q\right)=\int_a^b\mathbb{E}^0\left(L_u\mathbf{1}_{\tau_x>u}f(r-u,x-X_u)|\mathcal{F}_u^Q\right)dr.$$

Similarly

$$\mathbb{E}^0(\mathbf{1}_{\tau_x>t}L_u|\mathcal{F}_u^Q)=\mathbb{E}^0\left(L_u\mathbf{1}_{\tau_x>u}G(t-u,x-X_u)|\mathcal{F}_u^Q\right).$$

In Equation (4.3.16) of Proposition 4.3.15,

$$\mathbb{E}^0\left(L_u\mathbf{1}_{a<\tau_x<b}|\mathcal{F}_u^Q\right)\text{ and }\mathbb{E}^0(\mathbf{1}_{\tau_x>u}L_uh(X_u)[G(a-u,x-X_u)-G(b-u,x-X_u)]|\mathcal{F}_u^Q)$$

are respectively replaced by

$$\int_a^b\mathbb{E}^0\left(L_u\mathbf{1}_{\tau_x>u}f(r-u,x-X_u)|\mathcal{F}_u^Q\right)dr\text{ and }\int_a^b\mathbb{E}^0(\mathbf{1}_{\tau_x>u}L_uh(X_u)f(r-u,x-X_u)|\mathcal{F}_u^Q)dr.$$

By hypothesis, we have $r-u\geq a-u>0$.
For $T=t$, Lemma 4.6.14 of Appendix ensures that

$$\mathbb{E}^0\left(\int_0^t\frac{du}{[\mathbb{E}^0(\mathbf{1}_{\tau_x>u}L_uG(t-u,x-X_u)|\mathcal{F}_u^Q)]^2}\right)<\infty.$$

The numerators being bounded by $\|h\|_\infty L_u$, we can apply stochastic Fubini's theorem to Equation (4.3.16) Proposition 4.3.15, which can be written again as

$$\begin{aligned}
\mathbb{E}(\mathbf{1}_{a<\tau_x<b}|\mathcal{G}_t)=&\frac{1}{\mathbb{P}^0(\tau_x>t)}\int_a^b f(r,x)dr\\
&+\int_a^b\int_0^t\frac{\mathbb{E}^0(\mathbf{1}_{\tau_x>u}L_uh(X_u)f(r-u,x-X_u)|\mathcal{F}_u^Q)}{\mathbb{E}^0(\mathbf{1}_{\tau_x>t}L_u|\mathcal{F}_u^Q)}dQ_udr\\
&-\int_a^b\int_0^t\frac{\mathbb{E}^0(L_u\mathbf{1}_{\tau_x>u}f(r-u,x-X_u)|\mathcal{F}_u^Q)\mathbb{E}^0(\mathbf{1}_{\tau_x>u}L_uh(X_u)G(t-u,x-X_u)|\mathcal{F}_u^Q)}{[\mathbb{E}^0(\mathbf{1}_{\tau_x>t}L_u|\mathcal{F}_u^Q)]^2}dQ_udr\\
&+\int_a^b\int_0^t\frac{\mathbb{E}^0(L_u\mathbf{1}_{\tau_x>u}f(r-u,x-X_u)|\mathcal{F}_u^Q)[\mathbb{E}^0(\mathbf{1}_{\tau_x>u}L_uh(X_u)G(t-u,x-X_u)|\mathcal{F}_u^Q)]^2}{[\mathbb{E}^0(\mathbf{1}_{\tau_x>t}L_u|\mathcal{F}_u^Q)]^3}dudr\\
&-\int_a^b\int_0^t\mathbb{E}^0(\mathbf{1}_{\tau_x>u}L_uh(X_u)f(r-u,x-X_u)|\mathcal{F}_u^Q)\frac{\mathbb{E}^0(\mathbf{1}_{\tau_x>u}L_uh(X_u)G(t-u,x-X_u)|\mathcal{F}_u^Q)}{[\mathbb{E}^0(\mathbf{1}_{\tau_x>t}L_u|\mathcal{F}_u^Q)]^2}dudr.
\end{aligned}$$

To express this result with $\mathbb{P}$ conditional expectation instead of $\mathbb{P}^0$ conditional expectation, each fraction under the integral is multiplied and divided by the same term $\mathbb{E}^0(\mathbf{1}_{\tau_x>u}L_u|\mathcal{F}_u^Q)$. To manage the indicator function, we use the filtration $(\mathcal{G}_t,t\geq 0)$ since τ_x is a $\mathcal{G}-$stopping time.

Therefore, using (4.6.6) in Lemma 4.6.5, on the set $\{\tau_x > t\}$, we obtain

$$\begin{aligned}\int_a^b \bar{f}(r,t,x)dr =& \frac{1}{\mathbb{P}(\tau_x > t)}\int_a^b f(r,x)dr + \int_a^b \int_0^t \frac{\mathbb{E}(\mathbf{1}_{\tau_x > u} h(X_u) f(r-u, x-X_u)|\mathcal{G}_u)}{\mathbb{E}(\mathbf{1}_{\tau_x > u} G(t-u, x-X_u)|\mathcal{G}_u)} dQ_u dr \\ &- \int_a^b \int_0^t \bar{f}(r,u,x) \frac{\mathbb{E}(\mathbf{1}_{\tau_x > u} h(X_u) G(t-u, x-X_u)|\mathcal{G}_u)}{[\mathbb{E}(\mathbf{1}_{\tau_x > u} G(t-u, x-X_u)|\mathcal{G}_u)]^2} dQ_u dr \\ &+ \int_a^b \int_0^t \bar{f}(r,u,x) \frac{[\mathbb{E}(\mathbf{1}_{\tau_x > u} h(X_u) G(t-u, x-X_u)|\mathcal{G}_u)]^2}{[\mathbb{E}(\mathbf{1}_{\tau_x > u} G(t-u, x-X_u)|\mathcal{G}_u)]^3} du dr \\ &- \int_a^b \int_0^t \frac{\mathbb{E}(\mathbf{1}_{\tau_x > u} h(X_u) f(r-u, x-X_u)|\mathcal{G}_u)\mathbb{E}(\mathbf{1}_{\tau_x > u} h(X_u) G(t-u, x-X_u)|\mathcal{G}_u)}{[\mathbb{E}(\mathbf{1}_{\tau_x > u} G(t-u, x-X_u)|\mathcal{G}_u)]^2} du dr\end{aligned}$$

which finishes the proof. ■

4.5 Conclusion

This paper extends the study of the first passage time for a Lévy process in [KW03] from complete to incomplete information and D. Dorobantu's work in [Dor07] from intensity to conditional density. Here, we are proving the existence of the density of τ_x law given an information set, giving a stochastic differential integral equation satisfied by it and some numerical examples. All this gives us a behavior of the default time. In future works, we will be interested by the same studies in discrete time, in another kind of information set or under another process modeling the firm value.

4.6 Appendix

Lemma 4.6.1 *Let be μ and σ real numbers and G a Gaussian random variable with mean zero and variance one, then*

$$\mathbb{E}\left(e^{-\frac{(\mu+\sigma G)^2}{4}}\right) = \frac{\sqrt{2}e^{-\frac{\mu^2}{2(2+\sigma^2)}}}{\sqrt{2+\sigma^2}}.$$

Proof. Indeed using the law of G, we have

$$\mathbb{E}\left(e^{-\frac{(\mu+\sigma G)^2}{4}}\right) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(\mu+\sigma y)^2}{4}} e^{-\frac{y^2}{2}} dy.$$

Since $(\mu+\sigma y)^2 + 2y^2 = \left(y\sqrt{2+\sigma^2} + \frac{\mu\sigma}{\sqrt{2+\sigma^2}}\right)^2 + \frac{2\mu^2}{2+\sigma^2}$, then

$$\mathbb{E}\left(e^{-\frac{(\mu+\sigma G)^2}{4}}\right) = e^{-\frac{\mu^2}{2(2+\sigma^2)}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(y\sqrt{2+\sigma^2}+\frac{\mu\sigma}{\sqrt{2+\sigma^2}}\right)^2}{4}} dy$$

By change of variable $x = y\sqrt{2+\sigma^2}$, it follows that

$$\mathbb{E}\left(e^{-\frac{(\mu+\sigma G)^2}{4}}\right) = \frac{\sqrt{2}e^{-\frac{\mu^2}{2(2+\sigma^2)}}}{\sqrt{2+\sigma^2}} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi}} e^{-\frac{\left(x+\frac{\mu\sigma}{\sqrt{2+\sigma^2}}\right)^2}{4}} dx$$
$$= \frac{\sqrt{2}e^{-\frac{\mu^2}{2(2+\sigma^2)}}}{\sqrt{2+\sigma^2}}$$

■

Lemma 4.6.2 *If $(T_i, i \in \mathbb{N}^*)$ is the sequence of jump time of the process N, then*

$$\mathbb{E}\left(\frac{1}{\sqrt{t-T_{N_t}}}\right) < \frac{1}{\sqrt{t}} + 2\lambda\sqrt{t}.$$

Proof. We have

$$\mathbb{E}\left(\frac{1}{\sqrt{t-T_{N_t}}}\right) = \sum_{n\geq 0} \mathbb{E}\left(\frac{1}{\sqrt{t-T_n}} \mathbf{1}_{T_n<t<T_{n+1}}\right)$$
$$= \frac{e^{-\lambda t}}{\sqrt{t}} + \sum_{n\geq 1} \mathbb{E}\left(\frac{1}{\sqrt{t-T_n}} \mathbf{1}_{T_n<t<T_n+S_1}\right)$$

where S_1 is an exponential random variable with parameter λ and independent of T_n which follows a Gamma law with parameters n and λ. Therefore

$$\mathbb{E}\left(\frac{1}{\sqrt{t-T_{N_t}}}\right) = \frac{e^{-\lambda t}}{\sqrt{t}} + \sum_{n\geq 1} \int_0^t \frac{1}{\sqrt{t-u}} \frac{(\lambda u)^{n-1}}{(n-1)!} \lambda e^{-\lambda u} \int_{t-u}^{+\infty} \lambda e^{-\lambda v} dv du$$
$$\leq \frac{1}{\sqrt{t}} + \lambda e^{-\lambda t} \sum_{n\geq 1} \frac{(\lambda t)^{n-1}}{(n-1)!} \int_0^t \frac{du}{\sqrt{t-u}} = \frac{1}{\sqrt{t}} + 2\lambda\sqrt{t}.$$

■

Lemma 4.6.3 *There exists some constants $\tilde{C}$ and C such that $\forall t > 0, \quad x \geq 0$,*

$$f(t,x) \leq +\frac{C}{t} + \frac{|m|}{\sqrt{t}} + \tilde{C} + 2\lambda|m|\sqrt{t}. \tag{4.6.4}$$

Proof. The function f defined in (4.2.4) satisfies

$$f(t,x) \leq \lambda + \mathbb{E}(1_{\tau_x > T_{N_t}} \tilde{f}(t - T_{N_t}, x - X_{T_{N_t}})), \forall t > 0.$$

Using the fact that if $\tau_x > T_{N_t}$ then $x > X_{T_{N_t}}$, we have

$$\mathbb{E}(1_{\tau_x > T_{N_t}} \tilde{f}(t - T_{N_t}, x - X_{T_{N_t}})) \leq \mathbb{E}\left(\mathbf{1}_{\{x - X_{T_{N_t}} > 0\}} \tilde{f}(t - T_{N_t}, x - X_{T_{N_t}})\right).$$

Replacing $\tilde{f}$ by its expression, we obtain

$$\begin{aligned}
f(t,x) &\leq \lambda + \mathbb{E}\left(\mathbf{1}_{x - X_{T_{N_t}} > 0} \frac{|x - X_{T_{N_t}}|}{\sqrt{2\pi(t - T_{N_t})^3}} \exp\left[-\frac{(x - X_{T_{N_t}} - m(t - T_{N_t}))^2}{2(t - T_{N_t})}\right]\right) \\
&\leq \lambda + \mathbb{E}\left(\frac{[x - X_{T_{N_t}}]_+}{\sqrt{2\pi(t - T_{N_t})^3}} \exp\left[-\frac{(x - X_{T_{N_t}} - m(t - T_{N_t}))^2}{2(t - T_{N_t})}\right]\right) \\
&\leq \lambda + \mathbb{E}\left(\frac{|x - X_{T_{N_t}} - m(t - T_{N_t})|}{\sqrt{2\pi(t - T_{N_t})^3}} \exp\left[-\frac{(x - X_{T_{N_t}} - m(t - T_{N_t}))^2}{2(t - T_{N_t})}\right]\right) \\
&+ |m|\mathbb{E}\left(\frac{1}{\sqrt{2\pi(t - T_{N_t})}}\right).
\end{aligned}$$

Let $C_0 = \sup_{y \in \mathbb{R}} |y| e^{-\frac{y^2}{4}}$. We apply this bound to $y = \frac{x - X_{T_{N_t}} - m(t - T_{N_t})}{\sqrt{t - T_{N_t}}}$:

$$f(t,x) \leq \lambda + |m|\mathbb{E}\left(\frac{1}{\sqrt{2\pi(t - T_{N_t})}}\right) + \mathbb{E}\left(\frac{C_0}{(t - T_{N_t})\sqrt{2\pi}} e^{-\frac{(x - X_{T_{N_t}} - m(t - T_{N_t}))^2}{4(t - T_{N_t})}}\right).$$

Remark that conditionally to process N and the Y_i, the law of the random variable $\frac{x - X_{T_{N_t}} - m(t - T_{N_t})}{\sqrt{t - T_{N_t}}}$ is a Gaussian law with mean $\mu = \frac{x - mt - \sum_{i=1}^{N_t} Y_i}{\sqrt{t - T_{N_t}}}$ and variance $\sigma^2 = \frac{T_{N_t}}{t - T_{N_t}}$
Applying Lemma 4.6.1 we get the conditional expectation

$$\mathbb{E}\left(e^{-\frac{(x - X_{T_{N_t}} - m(t - T_{N_t}))^2}{4(t - T_{N_t})}} / N_t,\ Y_i,\ i = 1, \cdots, N_t\right) = \frac{\sqrt{2} e^{-\frac{\mu^2}{2(2+\sigma^2)}}}{\sqrt{2 + \sigma^2}}.$$

Using the fact that $\sigma^2 = \frac{T_{N_t}}{t - T_{N_t}} \Longrightarrow 2 + \sigma^2 = \frac{2t - T_{N_t}}{t - T_{N_t}}$, we obtain since $2 + \sigma^2 \geq \frac{t}{t - T_{N_t}}$

$$f(t,x) \leq \lambda + (|m| + \frac{C_0}{\sqrt{t\pi}})\mathbb{E}\left(\frac{1}{\sqrt{(t - T_{N_t})}}\right).$$

The proof is completed with Lemma 4.6.2 . ■

The next lemma is inspired of Jeanblanc and Rutkovski [JR00] and Dorobantu [Dor07].

Lemma 4.6.5 *For all $t \in \mathbb{R}_+$, for all a and b such that $t < a < b$, for all $Y \in L^1(\mathcal{F}_b, \mathbb{P})$*

$$\mathbb{E}(\mathbf{1}_{\tau_x > t} | \mathcal{F}_t^Q) > 0, \quad \mathbb{E}(Y \mathbf{1}_{t < \tau_x} | \mathcal{G}_t) = \mathbf{1}_{\tau_x > t} \frac{\mathbb{E}^0(L_b Y \mathbf{1}_{t<\tau_x} | \mathcal{F}_t^Q)}{\mathbb{E}^0(\mathbf{1}_{\tau_x > t} L_t | \mathcal{F}_t^Q)}. \tag{4.6.6}$$

For instance with $Y = \mathbf{1}_{a < \tau_x < b}$, we get

$$\mathbb{E}(\mathbf{1}_{a<\tau_x<b} | \mathcal{G}_t) = \mathbf{1}_{\tau_x > t} \frac{\mathbb{E}^0(L_b \mathbf{1}_{a<\tau_x<b} | \mathcal{F}_t^Q)}{\mathbb{E}^0(\mathbf{1}_{\tau_x > t} L_t | \mathcal{F}_t^Q)}.$$

Proof. Assume that there exists t_0 such that $\mathbb{P}(\tau_x > t_0) = 0$. Then for all $t \geq t_0$, $\mathbb{P}(\tau_x \leq t_0) = 1$. It follows that the density function of τ_x f, defined in (4.2.4), is the zero function on $[t_0, +\infty[$. This means that $\forall t \in [t_0, \infty[$,

$$f(t,x) = \lambda \mathbb{E}(1_{\tau_x > t}(1 - F_Y)(x - X_t)) + \mathbb{E}(1_{\tau_x > T_{N_t}} \tilde{f}(t - T_{N_t}, x - X_{T_{N_t}})) = 0 \quad \mathbb{P} - a.s.$$

Then, $\mathbb{P}(\tau_x \leq t) = 1$ implies that $\mathbb{E}(1_{\tau_x > t}(1 - F_Y)(x - X_t)) = 0$.
Thus $\mathbb{E}(1_{\tau_x > T_{N_t}} \tilde{f}(t - T_{N_t}, x - X_{T_{N_t}})) = 0$. But we have $t - T_{N_t} > 0 \quad \mathbb{P} - a.s$ and on the set $\{\tau_x > T_{N_t}\}$, $\quad x - X_{T_{N_t}} > 0$. Therefore, $\tilde{f}(t - T_{N_t}, x - X_{T_{N_t}}) > 0$ for all $t \geq t_0$. Hence, we obtain $\mathbf{1}_{\tau_x > T_{N_t}} = 0, \forall t \geq t_0$ what is not possible. Indeed,

$$\mathbf{1}_{\tau_x > T_{N_t}} = 0 \Longleftrightarrow \sum_{n \geq 0} \mathbf{1}_{\tau_x > T_n} \mathbf{1}_{N_t = n} = 0$$

That means for all $n \in \mathbb{N}, \mathbb{P}(T_n < t < T_{n+1}, \tau_x > T_n) = 0$. In particular, for $n = 0$,

$$\mathbb{P}(T_1 > t, \tilde{\tau}_x > 0) = \mathbb{P}(\tilde{\tau}_x > 0)\mathbb{P}(T_1 > t) = e^{\lambda t} \neq 0.$$

Thus for any t, $\mathbb{P}(\tau_x > t) > 0$ and $\mathbb{E}(\mathbf{1}_{\tau_x > t} | \mathcal{F}_t^Q) > 0$.

On the set $\{\tau_x > t\}$, any $\mathcal{G}_t-$ measurable random variable coincides with some $\mathcal{F}_t^Q-$ measurable random variable (cf. Jeanblanc and Rutkovski [JR00] p. 18). Then for all $Y \in L^1(\mathcal{F}_b, \mathbb{P})$, there exists a $\mathcal{F}_t^Q-$ measurable random variable Z such that

$$\mathbb{E}(\mathbf{1}_{\tau_x > t} Y | \mathcal{G}_t) = \mathbf{1}_{\tau_x > t} Z.$$

Taking the conditional expectation with respect to $\mathcal{F}_t^Q$, we get

$$\mathbb{E}(\mathbf{1}_{\tau_x > t} Y | \mathcal{F}_t^Q) = Z \mathbb{E}(\tau_x > t | \mathcal{F}_t^Q).$$

This implies that

$$\mathbb{E}(\mathbf{1}_{\tau_x > t} Y | \mathcal{G}_t) = \mathbf{1}_{\tau_x > t} \frac{\mathbb{E}(Y \mathbf{1}_{\tau_x > t} | \mathcal{F}_t^Q)}{\mathbb{E}(\mathbf{1}_{\tau_x > t} | \mathcal{F}_t^Q)}.$$

Using Kallianpur-Striebel formula (see Pardoux [Par91]) and $\mathbb{E}^0(L_b | \mathcal{F}_t^Q) = L_t$ we obtain

$$\mathbb{E}(Y \mathbf{1}_{\tau_x > t} | \mathcal{G}_t) = \mathbf{1}_{\tau_x > t} \frac{\mathbb{E}^0(L_b \mathbf{1}_{\tau_x > t} Y | \mathcal{F}_t^Q)}{\mathbb{E}^0(\mathbf{1}_{\tau_x > t} L_t | \mathcal{F}_t^Q)}.$$

■

The following is in [Cou96].

Lemma 4.6.7 *The family of $\mathcal{F}^Q$ adapted processes*

$$\mathcal{S} = \left\{ S_{\cdot} = \exp\left(\int_0^{\cdot} \rho_s dQ_s - \frac{1}{2} \int_0^t \rho_s^2 ds \right), \rho \in L^2([0,T], \mathbb{R}) \right\}$$

is total in the set of processes taking their values in $L^2(\Omega, \mathcal{F}_t^Q, \mathbb{P}^0)$.

Let us denote by $\mathcal{F}^W$, (resp. $\mathcal{F}^N$ and $\mathcal{F}^X$) the completed, right continuous filtration generated by W, (resp. N or X)

Lemma 4.6.8 *Let $\{U_t, t \geq 0\}$ be an $\mathcal{F}^W \otimes \mathcal{F}^N-$progressively measurable process such that for all $t \geq 0$, we have*

$$\mathbb{E}^0 \left[\int_0^t U_s^2 ds \right] < +\infty.$$

Then

$$\mathbb{E}^0 \left[\int_0^t U_s dQ_s | \mathcal{F}_t^W \otimes \mathcal{F}_t^N \right] = 0. \tag{4.6.9}$$

Proof. As in Lemma 4.6.7, the family of processes

$$\mathcal{R} = \left\{ r_{\cdot} = \mathcal{E} \left[\int_0^{\cdot} \gamma_s dW_s + \int_0^{\cdot} \int_A (e^{\beta_{s-}(x)} - 1) \tilde{N}(dsdx) \right], \gamma \in L^2([0,T], \mathbb{R}]), \beta \in L^\infty([0,T] \times A, \mathbb{R}) \right\}$$

is total in the set of processes taking their values in $L^2(\Omega, \mathcal{F}^W \otimes \mathcal{F}^N, \mathbb{P}^0)$, where $\tilde{N}$ is the compensated Poisson random measure on $\mathbb{R} \times \mathbb{R}$ and $A \subset \mathbb{R}$ is a Borel set. Therefore, since $r_t = 1 + \int_0^t r_s \gamma_s dW_s + \int_0^t \int_A r_{s-} (e^{\beta_{s-}(x)} - 1) \tilde{N}(dsdx)$, by Itô's formula, we have

$$\begin{aligned}
\mathbb{E}^0 \left(r_t \mathbb{E}^0 \left[\int_0^t U_s dQ_s | \mathcal{F}_t^W \otimes \mathcal{F}_t^N \right] \right) &= \mathbb{E}^0 \left[r_t \int_0^t U_s dQ_s \right] \\
&= \mathbb{E}^0 \left[\int_0^t r_{s-} \gamma_s U_s d < W, Q >_s \right] \\
&+ \mathbb{E}^0 \left[\int_0^t U_s \int_A r_{s-} (e^{\beta_s(x)} - 1) d < \tilde{N}, Q >_s \right] = 0.
\end{aligned}$$

The equality is obtained from the fact that under $\mathbb{P}^0$, $< Q, W > = < Q, \tilde{N} > = 0$ by independence. ∎

Lemma 4.6.10 *Let be a process $S \in \mathcal{S}$ such that for any t $S_t = \exp\left(\int_0^t \rho_s dQ_s - \frac{1}{2} \int_0^t \rho_s^2 ds \right)$, $\rho \in L^2([0,t], \mathbb{R})$. Let $Y \in L^\infty(\Omega, \mathbb{P}, \mathcal{F}_T^X)$ and $T \geq t$, then*

$$\mathbb{E}^0(Y L_T S_t) = \mathbb{E}^0(Y) + \mathbb{E}^0 \left(\int_0^t \mathbb{E}^0(Y/\mathcal{F}_u) S_u \rho_u L_u h(X_u) du \right)$$

and

$$\mathbb{E}^0(YL_T|\mathcal{F}_t^Q) = \mathbb{E}^0(YL_t|\mathcal{F}_t^Q) \ ; \ \mathbb{E}^0(YL_T|\mathcal{F}_t) = \mathbb{E}^0(YL_t|\mathcal{F}_t). \tag{4.6.11}$$

For instance

$$\mathbb{E}^0(\mathbf{1}_{\tau_x>T}L_T|\mathcal{F}_t) = \mathbb{E}^0(\mathbf{1}_{\tau_x>T}L_t|\mathcal{F}_t).$$

Proof. Let be $S_. = \int_0^. S_u\rho_u dQ_u \in \mathcal{S}$, $t \leq T$ and let us define the process K

$$K_. = 1 + \int_0^. S_u\rho_u dQ_u.$$

The integration by parts Itô formula applied to the product $L_.K_.$ between 0 and T yields

$$L_TK_T = 1 + \int_0^T L_uS_u\rho_u dQ_u + \int_0^T \mathbf{1}_{u\leq t}K_uL_uh(X_u)dQ_u + \int_0^T \mathbf{1}_{u\leq t}S_uL_u\rho_uh(X_u)du$$

and remark that $L_TK_T = L_TS_t$.
Since X and Q are independent under $\mathbb{P}^0$, we use Lemma 4.6.8 and it follows

$$\begin{aligned}\mathbb{E}^0(YL_TS_t) &= \mathbb{E}^0(Y) + \mathbb{E}^0\left(\int_0^{t\wedge T} \mathbb{E}^0(Y/\mathcal{F}_u)S_u\rho_uL_uh(X_u)du\right)\\ &= \mathbb{E}^0(Y) + \mathbb{E}^0\left(\int_0^t YS_u\rho_uL_uh(X_u)du\right).\end{aligned} \tag{4.6.12}$$

Similarly, using first $\mathbb{E}^0[YL_tS_t] = \mathbb{E}^0[\mathbb{E}^0(Y/\mathcal{F}_t)L_tS_t]$, Itô's formula on product of processes $\mathbb{E}^0(Y/\mathcal{F}_.)L_.S_.$ and the independence between X and Q under $\mathbb{P}^0$ yields

$$\mathbb{E}^0(YL_tS_t) = \mathbb{E}^0(Y) + \mathbb{E}^0\left(\int_0^t YS_u\rho_uL_uh(X_u)du\right) \tag{4.6.13}$$

Equations (4.6.12) and (4.6.13) imply that

$$\mathbb{E}^0(YL_T|\mathcal{F}_t^Q) = \mathbb{E}^0(YL_t|\mathcal{F}_t^Q).$$

Now let be $f_t(X) \in L^\infty(\Omega, \mathbb{P}^0, \mathcal{F}_t^X)$ and apply the above equality to $Yf_t(X)$:

$$\mathbb{E}^0(Yf_t(X)L_T|\mathcal{F}_t^Q) = \mathbb{E}^0(Yf_t(X)L_t|\mathcal{F}_t^Q)$$

so

$$\mathbb{E}^0(Yf_t(X)L_TS_t) = \mathbb{E}^0(Yf_t(X)L_tS_t)$$

which concludes the proof. ∎

Lemma 4.6.14 *For all $T \geq t$, $\forall \alpha > 0$, $\mathbb{E}^0(\mathbf{1}_{\{\tau_x > T\}} L_T | \mathcal{F}_t^Q) > 0$ almost surely and*

$$\mathbb{E}^0\left(\frac{1}{[\mathbb{E}^0(\mathbf{1}_{\{\tau_x>T\}}L_T|\mathcal{F}_t^Q)]^\alpha}\right) \leq \mathbb{P}^0(\tau_x > T)^{-\alpha} e^{\frac{\alpha(\alpha+1)}{2}t||h||_\infty^2}.$$

Proof. The process $(\mathbb{E}^0(\mathbf{1}_{\{\tau_x>T\}}L_T|\mathcal{F}_t^Q), \ \ t \leq T)$ is a positive $\mathcal{F}^Q$ (upper) martingale, which converges to the non null random variable $\mathbb{E}^0(\mathbf{1}_{\{\tau_x>T\}}L_T|\mathcal{F}_T^Q)$ (see Lemma 4.6.5) then it never vanishes.
From Corollary 4.3.12 (i), the process $M = (\mathbb{E}^0(\mathbf{1}_{\{\tau_x>T\}}L_T|\mathcal{F}_t^Q), \ \ t \leq T)$ is a $(\mathbb{P}^0, \mathcal{F}^Q)$ martingale with decomposition

$$\mathbb{E}^0\left(\mathbf{1}_{\{\tau_x>T\}}L_T|\mathcal{F}_t^Q\right) = \mathbb{P}^0(\tau_x > T) + \int_0^t \mathbb{E}^0\left(\mathbf{1}_{\{\tau_x>T\}}L_u h(X_u)|\mathcal{F}_u^Q\right) dQ_u.$$

Let $R_n = \inf\{t > 0, \mathbb{E}^0\left(\mathbf{1}_{\{\tau_x>T\}}L_T|\mathcal{F}_t^Q\right) < \frac{1}{n}\}$, using Itô's formula for $x \mapsto x^{-\alpha}$ between 0 and $t \wedge R_n$ and taking the expectation we derive

$$\begin{aligned}\mathbb{E}^0\left[\mathbb{E}^0\left(\mathbf{1}_{\{\tau_x>T\}}L_T|\mathcal{F}_{t\wedge R_n}^Q\right)^{-\alpha}\right] &= \mathbb{P}^0(\tau_x > T)^{-\alpha} + \frac{\alpha(\alpha+1)}{2}\mathbb{E}\left[\int_0^{t\wedge R_n} \frac{\mathbb{E}^0\left(\mathbf{1}_{\{\tau_x>T\}}L_u h(X_u)|\mathcal{F}_u^Q\right)^2}{\mathbb{E}^0\left(\mathbf{1}_{\{\tau_x>T\}}L_u|\mathcal{F}_u^Q\right)^{\alpha+2}}du\right] \\ &\leq \mathbb{P}^0(\tau_x > T)^{-\alpha} + \frac{\alpha(\alpha+1)}{2}\|h\|_\infty^2 \int_0^t \mathbb{E}^0\left[\mathbb{E}^0\left(\mathbf{1}_{\{\tau_x>T\}}L_T|\mathcal{F}_{u\wedge R_n}^Q\right)^{-\alpha}\right] du.\end{aligned}$$

Using Gronwall's Lemma

$$\mathbb{E}^0\left(\frac{1}{\mathbb{E}^0(\mathbf{1}_{\{\tau_x>T\}}L_T|\mathcal{F}_{t\wedge R_n}^Q)^\alpha}\right) \leq \mathbb{P}^0(\tau_x > T)^{-\alpha} e^{\frac{\alpha(\alpha+1)}{2}t||h||_\infty^2}.$$

The proof of Lemma 4.6.14 is achieved by letting n going to infinity. ∎

Annexe

Schémas numériques

1) Code source pour la densité f

```
clear;clc;
dt=0.01;t=1;p=0.5;eta1=1/0.02;eta2=1/0.03;lambda=0.1;
m1=-1.5;nb=5000;x=0.1;m2=1.5;
// mouvement brownien
dW=grand(t/dt,nb,'nor',0,sqrt(dt));
W=[zeros(1,nb);cumsum(dW,'r')];
//plot2d((0:dt:t)'*ones(1,nb),W);
//processus de comptage et les temps de sauts
Nt=grand(1,nb,'poi',t*lambda);
N=max(Nt);
tt=rand(N,nb);
for j=1:nb, T(:,j)= gsort(t*tt(:,j),'g','i');
end
T=floor((1/dt)*T);
for j=1:nb, for i=Nt(j)+1:N, T(i,j)=0;
    end
end
stacksize("max")
R1=T;
for j=1:nb, for i=Nt(j)+1:N, R1(i,j)=max(T(:,j));
    end
end
stacksize("max")
R=zeros(t/dt+1,nb);
for j=1:nb, for i=1:N,for k=1:t/dt+1, if k>R1(i,j) then...
                R(k,j)=dt*R1(i,j);
            end
        end
    end
end
S=zeros(t/dt+1,nb);
for j=1:nb, S(:,j)=[0:dt:t]';
end
// Les sauts
e=grand(N,nb,'bin',1,p);
U=grand(N,nb,'exp',1/eta1);
V=grand(N,nb,'exp',1/eta2);
Y=e.*U+(1-e).*V;
for j=1:nb, for i=Nt(j)+1:N, Y(i,j)=0;
    end
end
Z=zeros(t/dt,nb);
for j=1:nb, for i=1:N, if T(i,j)~=0 then Z(T(i,j),j)=Y(i,j);
        end
    end
end
//plot2d2(S,Y);
Z=[zeros(1,nb);cumsum(Z,'r')];
// Le drift
m1t=(m1*dt).*ones(t/dt,nb);
m1t=[zeros(1,nb);cumsum(m1t,'r')];
X1t=m1t+W+Z;
```

```
//plot2d((0:dt:t)'*ones(1,nb),X1t);
// le processus en TNt
for j=1:nb, for i=1:t/dt, WTNt(i,j)=grand(1,1,'nor',0,sqrt(R(i+1,j)-R(i,j)));
    end
end
WTNt=[zeros(1,nb);cumsum(WTNt,'r')];
for i=1:t/dt, for j=1:nb, m1TNt(i,j)=m1*(R(i+1,j)-R(i,j));
    end
end
m1TNt=[zeros(1,nb);cumsum(m1TNt,'r')];
X1TNt=m1TNt+WTNt+Z;
//plot2d((0:dt:t)'*ones(1,nb),X1TNt);
//-------------------------------------------------------------------------------
g1=exp(-(x-X1t)./eta1);
for i=1:nb,l1t(:,i)=max(X1t(:,i));
end
for i=1:nb,G1(:,i)=(l1t(:,i)<x).*g1(:,i);
end
stacksize("max")
G1=(lambda\p).*mean(G1,'c');
//plot2d((0:dt:t)',G1);
//-------------------------------------------------------------------------------
for i=1:nb,L1t(:,i)=max(X1TNt(:,i));
end
for i=1:t/dt+1, for j=1:nb, Hd(i,j)=(2*%pi)*(S(i,j)-R(i,j))^3;
    end
end
Hd=sqrt(Hd);
//---------------------------------------------------------------------------------
H1n=abs(x-X1TNt);
for j=1:nb,for i=1:t/dt+1,
      if S(i,j)~=R(i,j) then Ha(i,j)=H1n(i,j)/Hd(i,j); else H11(i,j)=0;
          end
    end
end
//--------------------------------------------------------------------------------
Hed=2*(S-R);
for i=1:nb,H1en(:,i)=x-X1TNt(:,i)-m1*Hed(:,i);
end
H1en=(H1en.^2)./2;
for j=1:nb, for i=1:t/dt+1,
        if S(i,j)~=R(i,j) then  Hae(i,j)=H1en(i,j)/Hed(i,j);
            else H12(i,j)=0;
        end
    end
end
H1=Ha.*exp(-Hae);
for i=1:nb,H1(:,i)=(L1t(:,i)<x).*H1(:,i);
end
H1=mean(H1,'c');
f1=G1+H1;
//---------------------------------------------------------------------
//---------------------------------------------------------------------
// Le drift
m2t=(m2*dt).*ones(t/dt,nb);
m2t=[zeros(1,nb);cumsum(m2t,'r')];
X2t=m2t+W+Z;
//--------------------------------------------------
for j=1:nb, for i=1:t/dt, m2TNt(i,j)=m2*(R(i+1,j)-R(i,j));
        end
end
m2TNt=[zeros(1,nb);cumsum(m2TNt,'r')];
```

```
X2TNt=m2TNt+WTNt+Z;
//------------------------------------------------------------
g2=exp(-(x-X2t)./eta1);
for i=1:nb,l2t(:,i)=max(X2t(:,i));
end
for i=1:nb,G2(:,i)=(l2t(:,i)<x).*g2(:,i);
end
stacksize("max")
G2=(lambda\p).*mean(G2,'c');
//-----------------------------------------------------------------
for i=1:nb,L2t(:,i)=max(X2TNt(:,i));
end
//------------------------------------------------------------------
H2n=abs(x-X2TNt);
for j=1:nb,for i=1:t/dt+1,
      if S(i,j)~=R(i,j) then Hb(i,j)=H2n(i,j)/Hd(i,j); else Hb(i,j)=0;
          end
    end
end
//-------------------------------------------------------------------
for i=1:nb,H2en(:,i)=x-X2TNt(:,i)-m2*Hed(:,i);
end
H2en=(H2en.^2)./2;
for j=1:nb, for i=1:t/dt+1,
        if S(i,j)~=R(i,j) then  Hbe(i,j)=H2en(i,j)/Hed(i,j); else Hbe(i,j)=0;
        end
    end
end
H2=Hb.*exp(-Hbe);
for i=1:nb,H2(:,i)=(L1t(:,i)<x).*H2(:,i);
end
H2=mean(H2,'c');
//plot2d((0:dt:t)',H2);
f2=G2+H2;
subplot(121);
plot2d((0:dt:t),f1);
xtitle("f(.,x):Densit\'e  avec dÃľrive m=-1.5", "t","f")
subplot(122);
plot2d((0:dt:t),f2);
xtitle('f(.,x):Densit\'e  avec dÃľrive m=1.5','t','f')
timer()
```

2) Code source pour la densité $\bar{f}$

```
clear;clc;stacksize("max")
//Nos donn\'ees et param\`etres
h=0.01;s=0.6;t=0.1;p=0.5;eta1=1/0.02;eta2=1/0.03;lambda=0.1;
m1=1;nb=122;x=0.1;m2=-1;
//processus de comptage
Nst=grand(1,nb,'poi',(s-t)*lambda); // d\'etermine les nombres de sauts
//les instants de sauts
N=max(Nst);
sstt=rand(N,nb);
for j=1:nb, ST(:,j)= gsort((s-t)*sstt(:,j),'g','i');
end
ST=floor((1/h)*ST);
for j=1:nb, for i=(Nst(j)+1):N, ST(i,j)=0;
    end
end
stacksize("max")
```

```
R1=ST;
for j=1:nb, for i=Nst(j)+1:N, R1(i,j)=max(ST(:,j));
    end
end
stacksize("max")
R=zeros((s-t)/h+1,nb);
for j=1:nb, for i=1:N,for k=1:(s-t)/h+1, if k>R1(i,j) then...
                R(k,j)=h*R1(i,j);
            end
        end
    end
end

//Subdivision dde l'intervalle d'\'evolution du processus
S=zeros((s-t)/h+1,nb);
for j=1:nb, S(:,j)=[0:h:(s-t)]'; //
end

// Les sauts
e=grand(N,nb,'bin',1,p);
U=grand(N,nb,'exp',1/eta1);
V=grand(N,nb,'exp',1/eta2);
Y=e.*U+(1-e).*V;
for j=1:nb, for i=Nst(j)+1:N, Y(i,j)=0;
    end
end

//Le processus de Poisson compos\'e
Z=zeros((s-t)/h,nb);
for j=1:nb, for i=1:N, if ST(i,j)~=0 then Z(ST(i,j),j)=Y(i,j);
        end
    end
end
Z=[zeros(1,nb);cumsum(Z,'r')];

// Mouvement brownien
dWst=grand((s-t)/h,nb,'nor',0,sqrt(h));
Wst=[zeros(1,nb);cumsum(dWst,'r')];

// Le premier drift
dm1st=(m1*h).*ones((s-t)/h,nb);
m1st=[zeros(1,nb);cumsum(dm1st,'r')];

//Le premier processus de L\'evy
X1st=m1st+Wst+Z;

// Deuxi\`eme drift
dm2st=(m2*h).*ones((s-t)/h,nb);
m2st=[zeros(1,nb);cumsum(dm2st,'r')];

//Deuxieme processus de L\'evy
X2st=m2st+Wst+Z;
//plot2d((0:h:s-t)'*ones(1,nb),X1st)
//------------------------------------------------------------------------
//------------------------------------------------------------------------

//Processus de comptage entre 0 et t
Nt=grand(1,nb,'poi',t*lambda); // d\'etermine le nombre de sauts entre 0 et t

//Les instants de sauts entre 0 et t
N1=max(Nt);
tt=rand(N1,nb);
```

```
for j=1:nb, T(:,j)= gsort(t*tt(:,j),'g','i');
end
T=floor((1/h)*T);
for j=1:nb, if i>Nt(j) then T(i,j)=0;
    end
end
// Les sauts de processus de Poisson entre 0 et t
e1=grand(N1,nb,'bin',1,p);
U1=grand(N1,nb,'exp',1/eta1);
V1=grand(N1,nb,'exp',1/eta2);
Y1=e1.*U1+(1-e1).*V1;
for j=1:nb, for i=Nt(j)+1:N1, Y1(i,j)=0;
    end
end

//Le processus de Poisson composÃľ entre 0 et t
stacksize("max")
Z1=zeros(t/h,nb);
for j=1:nb, for i=1:N1, if T(i,j)~=0 then Z1(T(i,j),j)=Y1(i,j);
        end
    end
end
Z1=[zeros(1,nb);cumsum(Z1,'r')];

// Mouvement brownien jusqu'Ãă t
dWt=grand(t/h,nb,'nor',0,sqrt(h));
Wt=[zeros(1,nb);cumsum(dWt,'r')];

// premier drift entre 0 et t
dm1t=(m1*h).*ones(t/h,nb);
m1t=[zeros(1,nb);cumsum(dm1t,'r')];

// deuxieme drift entre 0 et t
dm2t=(m2*h).*ones(t/h,nb);
m2t=[zeros(1,nb);cumsum(dm2t,'r')];

//Les deux processus de Poisson entre 0 et t
X1t=m1t+Wt+Z1;
X2t=m2t+Wt+Z1;
//plot2d((0:h:t)'*ones(1,nb),X1t);
//--------------------------------------------------------------
//---------------------------------------------------------------
stacksize("max")
W=ones((s-t)/h+1,nb); //matrice unitaire
for j=1:nb,l1st(:,j)=max(X1st(:,j)).*W(:,j);
end
for i=1:t/h+1, for j=1:nb, L1st(:,:,i,j)=l1st;
        end
end
stacksize("max")
for i=1:t/h+1, for j=1:nb, IH1st(:,:,i,j)=L1st(:,:,i,j)<(x-X1t(i,j)).*W;
        end
end
for i=1:t/h+1, for j=1:nb, a1(:,:,i,j)=x-X1t(i,j)-X1st;
end
end
g1st=(p/lambda).*exp(-a1./eta1);
G1ast=g1st.*IH1st;
for i=1:t/h+1, for j=1:nb,G1st(:,:,i,j)=mean(G1ast(:,:,i,j),'c');
    end
end
//---------------------------------------------------------------------
```

```
//---------------------------------------------------------------
stacksize("max")
for j=1:nb,l2st(:,j)=max(X2st(:,j)).*W(:,j);end
for i=1:t/h+1, for j=1:nb, L2st(:,:,i,j)=l2st;end
end
for i=1:t/h+1, for j=1:nb, IH2st(:,:,i,j)=L2st(:,:,i,j)<(x-X2t(i,j)).*W;
        end
end
for i=1:t/h+1, for j=1:nb, a2(:,:,i,j)=x-X2t(i,j)-X2st;end
end
g2st=(p/lambda).*exp(-a2./eta1);
G2ast=g2st.*IH2st;
for i=1:t/h+1, for j=1:nb,G2st(:,:,i,j)=mean(G2ast(:,:,i,j),'c');end
end
//-------------------------------------------------------------------
//--------------------------------------------------------------------
stacksize("max")
for j=1:nb, for i=1:(s-t)/h, dWTNst(i,j)=grand(1,1,'nor',0,sqrt(R(i+1,j)-R(i,j)));
    end
end
WTNst=[zeros(1,nb);cumsum(dWTNst,'r')];
for i=1:(s-t)/h, for j=1:nb, dm1TNst(i,j)=m1*(R(i+1,j)-R(i,j));
    end
end
m1TNst=[zeros(1,nb);cumsum(dm1TNst,'r')];
X1TNst=m1TNst+WTNst+Z;
for i=1:(s-t)/h, for j=1:nb, dm2TNst(i,j)=m2*(R(i+1,j)-R(i,j));
    end
end
m2TNst=[zeros(1,nb);cumsum(dm2TNst,'r')];
X2TNst=m2TNst+WTNst+Z;
//-----------------------------------------------------------------
//----------------------------------------------------------------
stacksize("max")
for j=1:nb,l1Nst(:,j)=max(X1TNst(:,j)).*W(:,j);
end
for i=1:t/h+1, for j=1:nb, L1Nst(:,:,i,j)=l1Nst;end
end
for i=1:t/h+1, for j=1:nb, IH1Nst(:,:,i,j)=L1Nst(:,:,i,j)<(x-X1t(i,j)).*W;
        end
end
//----------------------------------
stacksize("max")
for j=1:nb,l2Nst(:,j)=max(X2TNst(:,j)).*W(:,j);
end
for i=1:t/h+1, for j=1:nb, L2Nst(:,:,i,j)=l2Nst;end
end
for i=1:t/h+1, for j=1:nb, IH2Nst(:,:,i,j)=L2Nst(:,:,i,j)<(x-X2t(i,j)).*W;
        end
end
//----------------------------------------------------------------
//----------------------------------------------------------------
Da1e=2.*(S-R);
for i=1:(s-t)/h+1,for j=1:nb, if Da1e(i,j)==0 then D1e(i,j)=0;
        else D1e(i,j)=1/Da1e(i,j);
        end
    end
end
for i=1:t/h+1,for j=1:nb,De(:,:,i,j)=D1e;
        end
end
Da1=sqrt((2*%pi).*(S-R).^3);
```

```
for  i=1:(s-t)/h+1,for  j=1:nb,  if  Da1(i,j)==0 then D1(i,j)=0;
        else D1(i,j)=1/Da1(i,j);
        end
    end
end
for  i=1:t/h+1,for  j=1:nb,D(:,:,i,j)=D1;
        end
end
stacksize("max")
for  i=1:t/h+1,  for  j=1:nb,  N1TNst(:,:,i,j)=abs(x-X1t(i,j)-X1TNst);
        end
end
for  i=1:t/h+1,for  j=1:nb,  N1eTNst(:,:,i,j)=(x-X1t(i,j)-X1TNst-m1.*(S-R)).^2;
end
end
//------------------------------
stacksize("max")
f1st=N1TNst.*D;
stacksize("max")
F1st=exp(-N1eTNst.*De);
//------------------------------
stacksize("max")
ftild1st=f1st.*F1st;
stacksize("max")
zeta1ast=IH1Nst.*ftild1st;
stacksize("max")
for  i=1:t/h+1,  for  j=1:nb,zeta1st(:,:,i,j)=mean(zeta1ast(:,:,i,j),'c');
        end
end
gamma1st=G1st+zeta1st;
//------------------------------------------------------
for  i=1:t/h+1,for  j=1:nb,  N2TNst(:,:,i,j)=abs(x-X2t(i,j)-X2TNst);
        end
end
for  i=1:t/h+1,for  j=1:nb,N2eTNst(:,:,i,j)=(x-X2t(i,j)-X2TNst-m2.*(S-R)).^2;
    end,
 end
//--------------------------------------------------------------
//--------------------------------------------------------------
stacksize("max")
f2st=N2TNst.*D;
stacksize("max")
F2est=N2eTNst.*De;
F2st=exp(-F2est);
//--------------------------------------------
//--------------------------------------------
ftild2st=f2st.*F2st;
stacksize("max")
zeta2ast=IH2Nst.*ftild2st;
stacksize("max")
for  i=1:t/h+1,  for  j=1:nb,zeta2st(:,:,i,j)=mean(zeta2ast(:,:,i,j),'c');
        end
end
//---------------------------------------------
//---------------------------------------------
stacksize("max")
gamma2st=G2st+zeta2st;
//----------------------------------------------------
//----------------------------------------------------
//------------------------------------------------------
dQ=grand(t/h,nb,'nor',0,sqrt(h));
Q=[zeros(1,nb);cumsum(dQ,'r')];
```

```
//------------------------------------------------------------
for j=1:nb, SinQ1(1,j)=0; end
for j=1:nb, for i=1:t/h, SinQ1(i+1,j)=SinQ1(i,j)+sin(X1t(i,j)).*(Q(i+1,j)-Q(i,j));
    end
end
for j=1:nb, SinQ2(1,j)=0; end
for j=1:nb, for i=1:t/h, SinQ2(i+1,j)=SinQ2(i,j)+sin(X2t(i,j)).*(Q(i+1,j)-Q(i,j));
    end
end
//------------------------------------------------------------
//------------------------------------------------------------
stacksize("max");
for j=1:nb, SinQ21(1,j)=0; end
for j=1:nb, for i=1:t/h, SinQ21(i+1,j)=SinQ21(i,j)+h.*(sin(X1t(i,j)))^2;
    end
end
for j=1:nb, SinQ22(1,j)=0; end
for j=1:nb, for i=1:t/h, SinQ22(i+1,j)=SinQ22(i,j)+h.*(sin(X2t(i,j)))^2;
    end
end
//------------------------------------------------------------
L1t=exp(SinQ1-(1/2).*SinQ21);
L2t=exp(SinQ2-(1/2).*SinQ22);
//------------------------------------------------------------
//------------------------------------------------------------
for j=1:nb, iH1(:,j)=max(X1t(:,j))<x;end
for j=1:nb, iH2(:,j)=max(X2t(:,j))<x;end
for j=1:nb,pi1(:,j)=iH1(:,j).*L1t(:,j);end
for j=1:nb,pi2(:,j)=iH2(:,j).*L2t(:,j);end
for i=t/h+1,for j=1:nb,iHL1t(:,:,i,j)=pi1(i,j).*W(:,j); end
end
for i=t/h+1,for j=1:nb,iHL2t(:,:,i,j)=pi2(i,j).*W(:,j); end
end
Pi1=mean(pi1,'c');
Pi2=mean(pi2,'c');
//------------------------------------------------------------
//------------------------------------------------------------
//------------------------------------------------------------
stacksize("max")
for i=t/h+1, for j=1:nb Gamma1st(:,:,i,j)=pi1(i,j).*gamma1st(:,:,i,j);
        end
end
for i=t/h+1, for j=1:nb Gamma2st(:,:,i,j)=pi2(i,j).*gamma2st(:,:,i,j);
        end
end
for i=t/h+1, for j=1:nb, Gamma1(:,:,i,j)=Gamma1st(:,:,i,j)./Pi1(i,1);
    end
end
for i=t/h+1, for j=1:nb, Gamma2(:,:,i,j)=Gamma2st(:,:,i,j)./Pi2(i,1);
    end
end
//------------------------------------------------------------
stacksize("max")
V1=matrix(Gamma1,((s-t)/h+1)*(t/h+1),nb);
barV1=mean(V1,'c');
V2=matrix(Gamma2,((s-t)/h+1)*(t/h+1),nb);
barV2=mean(V2,'c');
barf1=sum(matrix(barV1,(s-t)/h+1,t/h+1),'c');
barf2=sum(matrix(barV2,(s-t)/h+1,t/h+1),'c');
//------------------------------------------------------------
subplot(121)
plot2d((0:h:s-t),barf2)
```

```
xtitle('f(.,t,x):Densit\'e conditionnelle avec d\'erive m=-1','r-t','f')
subplot(122)
plot2d((0:h:s-t),barf1)
xtitle('f(.,t,x): Densit\'e conditionnelle avec d\'erive m=1','r-t','f')
timer()
```

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Abstract

In this Ph.D thesis, we consider a jump-diffusion process which the diffusion part is a drifted Brownian motion and the jump part is a compound Poisson process. We assume that a firm value is modelling by a stochastic process $V = V_0 \exp -X$. This firm goes to default whenever its value is below a specified threshold b which is exogenously determined. For $x = \ln(V_0) - \ln(b) > 0$, the default time is of the form $\tau_x = \inf\{t \geq 0 : X_t \geq x\}$.
First, we suppose that agents observe perfectly the firm value. In this model, we showed in chapter 2 that the density of the default time is continuous, then study the joint law of the default time, overshoot an undershoot. We obtained in chapter 3 a valued measure differential equation which the solution is the quadruplet formed by the random variable X_t, the running supremum X_t^* of X at time t, the supremum of X at the last jump time before t and the last jump time before t.
Secondly, we assume that investors wishing detain a part of the firm can not observe the firm value. They observe a noisy value of the firm and their information is modelling by the filtration $\mathcal{G} = (\mathcal{G}_t, t \geq 0)$ generated by their observation. In this model, we have shown that the conditional density of τ_x with respect to $\mathcal{G}$ has a density which is solution of one stochastic integral-differential equation The knowledge of this density allows investors to predict the default time after time t. This second part is the chapter 4.

Keywords : Lévy processes, default time, Partial differential equation, Filtering theory, Complete observation, incomplete observation.

Résumé

Dans nos travaux, nous avons considéré un processus de Lévy X avec une composante brownienne non nulle et dont la partie à sauts est un processus de Poisson composé. Nous avons supposé que la valeur d'une entreprise est modélisée par un processus stochastique de la forme $V = V_0 \exp -X$ et que cette entreprise est mise à défaut dès lors que sa valeur passe sous un certain seuil b déterminé de façon exogène et qui donc, est une donnée du problème. L'instant de défaut τ est alors de la forme τ_x pour $x = \ln(V_0) - \ln((b)$ où $x > 0, \quad \tau_x = \inf\{t \geq 0 : X_t \geq x\}$.
Dans un premier temps, nous supposons que des agents observant la valeur V des actifs de la firme souhaitent connaître le comportement de l'instant de défaut. Dans ce modèle, au chapitre 2, nous avons étudié d'une part la régularité de la densité de la loi de l'instant de défaut. D'autre part, nous avons étudié la loi conjointe de l'instant de défaut, de l'overshoot et de l'undershoot. Au chapitre 3, nous avons obtenu une équation à valeurs mesures dont le quadriplet formé par la variable aléatoire X_t, le supremum du processus X à l'instant t, le supremum du processus X au dernier instant de saut avant l'instant t et le dernier instant de saut à l'instant t est solution au sens faible, puis une équation dont ce quadriplet est une solution forte. Dans un second temps, au chapitre 4, nous avons supposé que des investisseurs souhaitant détenir une part de cette entreprise ne disposent pas de l'information complète. Ils n'observent pas la valeur des actifs de la firme V, mais sa valeur bruitée. Leur information est modélisée par la filtration $\mathcal{G} = (\mathcal{G}_t, t \geq 0)$ engendrée par cette observation. Dans ce modèle, nous avons montré que la loi conditionnelle de l'instant de défaut sachant la tribu $\mathcal{G}_t$ admet une densité par rapport à la mesure de Lebesgue et obtenu une équation de Volttera dont cette densité est solution. Cette connaissance permet aux investisseurs de prévoir au vu de leur information, quand est-ce que l'instant de défaut va intervenir après l'instant t. Nous avons complété ce travail par des simulations numériques.

Mots clés : Processus de Lévy, Instant de défaut, Equations aux dérivées partielles, Théorie du filtrage, Observation complète, Observation incomplète.