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Jian YE

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Régularité de l'application du transport optimal sur des variétés riemanniennes compactes

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Thèse dirigée par
YuXin GE et Xinan MA

Jury
M. Xiaoping YANG, Rapporteur
M. Young-Heon KIM, Rapporteur
M. Weimin SHENG, Examinateur
Mme Xiande ZHANG, Examinateur
M. Yuxin GE, Directeur de thèse
M. Xinan MA, Co-directeur de thèse

Résumé

Dans cette thèse on s'intéressons à la régularité de l'application du transport optimal sur des variétés riemanniennes compactes.

Dans le premier chapitre, on rappelle certaines définitions sur une variété riemannienne.

Dans le deuxième chapitre, on décrit la variation de la courbure sur des géodésiques.

Dans le troisième chapitre, on étudie le tenseur de MTW sur une variété riemannienne compacte. On montre qu'une condition de MTW améliorée est satisfaite sur une variété presque sphérique. La preuve consiste à une analyse minutieuse, combinée avec les arguments de perturbation sur des sphères.

Dans le quatrième chapitre, on étudie le comportement de l'inverse de la matrice Hessienne de la distance au carré.

Dans le cinquième chapitre, on prouve la régularité du transport optimale sur deux classes des variétés riemanniennes compactes– des variétés presque sphériques et des produits riemanniens des variétés presque sphériques.

Dans le dernier chapitre, on déscrit quelques perspectives sur le transport optimal dans la littérature.

Mots-clés

Transport optimal, équation de Monge-Ampère, fonction c-convexe, tenseur de MTW, condition de MTW, convexité du domaine d'injectivité, principe de maximum, méthode de continuité.

Abstract

In this thesis, we are concerned with the regularity of optimal transport maps on compact Riemannian manifolds.

In the first chapter, we give some definitions and recall some facts in Riemannian geometry.

In the second chapter, we examine the variation of the curvature on the geodesics.

In the third chapter, we study the MTW tensor on compact Riemannian manifold. We show that an improved MTW condition is satisfied on nearly spherical manifold. The proof goes by a careful analysis combined with the perturbative arguments on the spheres.

In the fourth chapter, we study the inverse of the Hessian matrix of the squared distance.

In the fifth chapter, we prove the smoothness of the optimal transport maps on two classes of compact Riemannian manifold–nearly spherical manifolds and Riemannian products of nearly spherical manifolds.

In the last chapter, we provide some perspectives about the optimal transportation in the literature.

Keywords

Optimal transport map, Monge-Ampère equation, c-convex function, MTW tensor, MTW condition, convexity of injectivity domain, maximum principle, continuity method.

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Contents

Synthèse de la thèse

Dans cette thèse, on se concentre sur la régularité de l'application du transport optimal. Ce sujet a été amplement étudié ces dernières années. On prouve la régularité de l'application du transport optimale sur deux classes de variétés riemanniennes compactes– des variétés presque sphériques et des produits riemanniens des variétés presque sphériques.

0.1 Le problème du transport optimal

Le transport optimal est un sujet ancien. Il est étudié pour la première fois par Monge en 1781 [79] avec le coût de la distance euclidienne. Depuis, il est apparu dans de nombreux domaines tels que la théorie de probabilité, l'économie, l'optimisation, la météorologie, etc... L'introduction générale à la théorie du transport optimal peut être trouvée dans des livres [98] [99].

Le problème du transport optimal s'exprime comme suit: soient (X, μ_0) et (Y, μ_1) deux espaces métriques avec des mesures de probabilité μ_0 et μ_1 respectivement. Soit $c : X \times Y \to \mathbb{R}$ une fonction du coût. Le problème du transport optimal consiste à minimiser la fonctionnelle du coût total

$$\mathrm{Cost}(G) = \int_X c(x, G(x)) d\mu_0$$

parmi toutes les applications mesurables $G : X \to Y$, telles que $G_\# \mu_0 = \mu_1$, cela signifie que pour tout ensemble $E \subset X$ mesurable, on a

$$\mu_1(E) = \mu_0(G^{-1}(E)).$$

Les minimiseurs sont appelés les applications du transport optimal.

L'existence de l'application du transport optimal n'est pas triviale. D'une part, il peut y avoir aucune application tel que $G_\# \mu_0 = \mu_1$. Par exemple, quand μ_0 est égale à la mesure de Dirac alors que μ_1 ne l'est pas. D'autre part, le problème est non linéaire.

Cent soixante ans plus tard après Monge, Kantorovich [60] a réduit le problème ci-dessus à un programme linéaire à dimension infinie. Plus précisément, on cherche une mesure de probabilité μ sur $X \times Y$ tel que

$$\int_{X \times Y} c(x, y) d\mu(x, y) = \inf_{\gamma \in \Pi(\mu_0, \mu_1)} \int_{X \times Y} c(x, y) d\gamma(x, y), \tag{1}$$

où

$$\begin{aligned}\Pi(\mu_0, \mu_1) = \{\gamma : \gamma &\text{ est une mesure de probabilité sur } X \times Y. \text{ Pour les} \\ &\text{applications } \pi_X : X \times Y \to X, (x, y) \mapsto x \text{ et} \\ &\pi_Y : X \times Y \to X, (x, y) \mapsto y, \text{on a} \\ &(\pi_X)_\# \gamma = \mu_0, (\pi_Y)_\# \gamma = \mu_1\}.\end{aligned}$$

La mesure μ est appelée le couplage optimal. Ce problème de minimisation est plus général. L'existence est connue sous certaines hypothèses. Par exemple, soient X

et Y deux espaces métriques complets séparables. Soit c une fonction semi-continue inférieure. Le couplage optimal existe [98] [87].

Kantorovich a montré que le problème (1) a une formation duale. Concrètement, on cherche une paire de fonctions (u, v) telle que

$$-\int_X u d\mu_0 - \int_Y v d\mu_1 \quad = \sup_{(\phi,\psi)\in\mathfrak{S}} \{-\int_X \phi d\mu_0 - \int_Y \psi d\mu_1\}. \tag{2}$$

où

$$\mathfrak{S} = \{(\phi, \psi) : \forall (x, y) \in X \times Y, \phi(x) + \psi(y) \geq -c(x, y)\}.$$

Lorsque le maximum est atteint par (u, v), on a

$$u(x) = \sup_{y\in Y}[-c(x, y) - v(y)], v(y) = \sup_{x\in X}[-c(x, y) - u(x)].$$

0.2 L'existence de l'application du transport optimal

Avant d'introduire l'existence de l'application du transport optimal, on donne d'abord la définition de la c-convexité [98].

Définition 0.1. *(Fonction c-convexe) Soit $c : X \times Y \to \mathbb{R}$ une fonction. La fonction $u : X \to \mathbb{R} \cup \{\infty\}$ est c-convexe, si pour tout $x_0 \in X$, il existe un $y_0 \in Y$ tel que*

$$\forall x \in X, u(x) \geq u(x_0) + c(x_0, y_0) - c(x, y_0).$$

La c-sousdifférentiel de la fonction u au point x_0 est définie comme

$$\partial_c u(x_0) = \{y_0 \in Y : \forall x \in X, u(x) \geq u(x_0) + c(x_0, y_0) - c(x, y_0)\}.$$

On donne des exemples des fonctions c-convexes. Lorsque $X = Y = \mathbb{R}^n$ et $c(x, y) = -\langle x, y\rangle$, la c-convexité est équivalente à la convexité usuelle. Lorsque $X = Y = \mathbb{R}^n$ et $c(x, y) = \frac{1}{2}|x - y|^2$, une fonction u est c-convexe si et seulement si la fonction $u + \frac{1}{2}|x|^2$ est convexe. Lorsque $X = Y = M$ un espace métrique et $c(x, y) = d(x, y)$, la fonction u est c-convexe si et seulement si $|u(x) - u(y)| \leq c(x, y), \forall x, y \in M$.

Il est clair que la fonction u est c-convexe si et seulement s'il existe une fonction $v : Y \to \mathbb{R}$ telle que $u(x) = \sup_{y\in Y}[-c(x, y) - v(y)]$. De plus, pour une solution du probléme (2), des fonctions u et v sont c-convexes.

Maintenant, on décrit l'existence de l'application du transport optimal. Lorsque $c(x, y) = |x - y|$, Sudakov [88], Evans-Gangbo [33], Caffarelli-Feldman-Mccann [15], Trudinger-Wang [93] ont démontré l'existence de l'application du transport optimal. En général, l'application du transport optimal n'est pas unique. Lorsque $c(x, y) = \frac{1}{2}|x-y|^2$, si μ_0 est absolument continu par rapport à la mesure de Lebesgue, Brenier [4] a montré l'existence et l'unicité de l'application du transport optimal. Il a également prouvé que l'application du transport optimal est le gradient d'une fonction convexe. Pour une fonction du coût général, l'existence de l'application du transport optimal peut être trouvée dans [8] [51] [98].

Lorsque c est égal à la moitié de la distance géodésique au carré sur une variété riemannienne compacte, McCann [77] a généralisé la théorie de Brenier. Si μ_0 est absolument continu par rapport à l'élément de volume, McCann a prouvé qu'il existe une unique application du transport optimal G. Il a également montré que $G(m) = \exp_m(\nabla_m u)$ pour une certaine c-convexe fonction u. La fonction u est appelée le potentiel du transport optimal. Sur une variété riemannienne non-compacte, l'existence de l'application du transport optimale peut être trouvée dans [34].

0.3 La régularité de l'application du transport optimal

On étudie ici la régularité de l'application du transport optimal sur une variété riemannienne compacte. On a déjà vu que l'application du transport optimal s'écrit comme $G(m) = \exp_m(\nabla_m u)$. L'étude de la régularité de l'application du transport optimal revient à étudier la régularité du potentiel. Le potentiel u vérifie une équation elliptique complètement non-linéaire.

0.3.1 L'équation du transport optimal

Soient $\rho_0 dvol$ et $\rho_1 dvol$ deux mesures de probabilité sur une variété riemannienne compacte (M, g) à densité continue strictement positive. Si le potentiel u est de classe C^2, alors l'application du transport optimal G est un difféomorphisme de classe C^1.

La condition $G_{\#}\mu_0 = \mu_1$, signifie que pour tout ensemble mesurable $E \subset M$, on a

$$\int_{G^{-1}(E)} \rho_0(x) dvol(x) = \int_E \rho_1(y) dvol(y).$$

Par un changement de variable $y = G(x)$, on en déduit

$$\int_{G^{-1}(E)} \rho_0(x) dvol(x) = \int_{G^{-1}(E)} \rho_1(G(x)) |\det d_x G| dvol(x).$$

ce qui donne

$$\forall x \in M, |\det \ d_x G| = \frac{\rho_0(x)}{\rho_1(G(x))}.$$

En utilisant la propriété de la fonction c-convexe u, on a

$$\forall x \in M, \nabla_x u + \nabla_x c(x, G(x)) = 0.$$

En différenciant par rapport à x, on obtient

$$\forall x \in M, \det(\nabla_x^2 u + \nabla_x^2 c(x, G(x))) = \det(-\nabla_{x,y} c(x, G(x)) d_x G).$$

Encore par la propriété de la fonction c-convexe u, $\nabla_x^2 u + \nabla_x^2 c(x, G(x))$ est positif. En conséquence,

$$\begin{aligned} \det(\nabla_x^2 u + \nabla_x^2 c(x, G(x))) &= |\det \nabla_{x,y} c(x, G(x))| |\det d_x G| \\ &= \frac{\rho_0(x)}{\rho_1(G(x))} |\det \nabla_{x,y} c(x, G(x))|. \end{aligned}$$

En vertu de $\exp_x(-\nabla_x c(x, y)) = y$, on en déduit

$$\det(\nabla_x^2 u + \nabla_x^2 c(x, G(x))) = \frac{\rho_0(x)}{|\det d_{\nabla_x u} \exp_x |\rho_1(G(x))}.$$

En rappelant $\det d_{\nabla_x u} \exp_x > 0$, on obtient l'équation

$$\forall x \in M, \det(\nabla^2 u + \nabla_x^2 c(x, G(x))) = \frac{\rho_0(x)}{\rho_1(G(x)) \det d_{\nabla u} \exp}. \tag{3}$$

On donne ici quelques exemples.

Dans le cas euclidien, l'équation (3) est de type Monge-Ampère généralisé

$$\det(\nabla^2 u + I) = \frac{\rho_0(x)}{\rho_1(G(x))},$$

Lorsque M est un tore $\mathbb{T}^n$, l'équation (3) s'écrit

$$\det(\nabla^2 u + g) = \frac{\rho_0(x)}{\rho_1(G(x))}.$$

Lorsque M est une sphère $\mathbb{S}^n$, l'équation (3) devient

$$\left|\frac{\sin|\nabla u|}{|\nabla u|}\right|^{(n-1)} \det(\nabla^2 u + \bar{\mathcal{S}}(x, \nabla u)) = \frac{\rho_0(x)}{\rho_1(G(x))},$$

où

$$\bar{\mathcal{S}}(m, \nu)(\xi) = \xi - (1 - |\nu| \cot |\nu|)(\xi - g_m(\xi, \frac{\nu}{|\nu|})\frac{\nu}{|\nu|}).$$

0.3.2 Le tenseur de Ma-Trudinger-Wang

L'application du transport optimal n'est pas forcément continue ou lisse. Afin de garantir une certaine régularité, des hypothèses supplémentaires sont nécessaires. Dans l'espace euclidien, Ma-Trudinger-Wang [76] ont introduit une quantité en utilisant les dérivées de la fonction du coût c jusqu'à l'ordre 4, dite le tenseur de Ma-Trudinger-Wang. Ils ont montré la régularité C^2 du potentiel sous la condition A3S, c'est-à-dire, le tenseur de MTW est strictement positive. Plus tard, Kim et McCann [64] ont interprété de nouveau le tenseur de MTW comme la courbure de Riemann sur certains 2-plans d'une métrique pseudo-riemannienne issue de la fonction du coût sur l'espace produit $M \times M$. A propos du tenseur de MTW ou plus généralement la courbure croisée, voir les références [75] [45] [67] [43] [73] [28] [29].

Avant de définir le tenseur de MTW, on donne quelques notations. Soit (M, g) une variété riemannienne compacte de dimension $n \geq 2$. d et $dvol$ désignent respectivement la distance géodésique et l'élément de volume sur M. Etant donné $m \in M$, $\mathrm{Cut}(m)$ note le lieu de coupure du point m. Le domaine d'injectivité du point m est noté par $I(m)$.

Définition 0.2. *(Le tenseur de MTW) Soient $m \in M, \nu \in I(m), \xi, \eta \in T_m M$. Le tenseur de MTW est défini par*

$$\mathcal{C}(m, \nu)(\xi, \eta) = -\frac{3}{2} \frac{\partial^4}{\partial s^2 \partial t^2}\bigg|_{s=t=0} \frac{d^2}{2}(\exp_m(t\xi), \exp_m(\nu + s\eta)).$$

Cette définition a un sens. En fait, lorsque t et s sont suffisamment petits, $\exp_m(\nu + s\eta) \notin \mathrm{Cut}(\exp_m(t\xi))$. Et donc $d^2(\exp_m(t\xi), \exp_m(\nu + s\eta))$ est lisse par rapport à t et s. On énonce quelques propriétés élémentaires: quand $\xi = 0$ ou $\eta = 0$, le tenseur de MTW s'annule. En général, si le rang de la famille de vecteurs $\{\nu, \xi, \eta\}$ est plus petit que 1, alors le tenseur de MTW s'annule. Il est clair que le tenseur de MTW est homogène de degré 2 par rapport à ξ ou η, et homogène de degré 1 par rapport à la métrique g, c'est-à-dire

$$\mathcal{C}_{(m,\nu)}(\lambda\xi, \eta) = \lambda^2 \mathcal{C}_{(m,\nu)}(\xi, \eta), \mathcal{C}_{(m,\nu)}(\xi, \lambda\eta) = \lambda^2 \mathcal{C}_{(m,\nu)}(\xi, \eta);$$

$$\mathcal{C}^{\lambda g}_{(m,\nu)}(\xi, \eta) = \lambda \mathcal{C}^{g}_{(m,\nu)}(\xi, \eta).$$

Lorsque (M, g) est plate, le tenseur de MTW s'annule.

Le tenseur de MTW a des liens étroits avec la courbure de Riemann de la variété. Loeper [73] a trouvé que le tenseur de MTW sur le diagonal coïncide avec la courbure sectionnelle. En effet, on a

$$\begin{aligned} d^2(\exp_m t\xi, \exp_m s\eta) &= |\xi|^2_m t^2 - 2g_m(\xi, \eta)ts + |\eta|^2_m s^2 - \\ &\quad \frac{1}{3} R_m(\xi, \eta, \xi, \eta) t^2 s^2 + o((t^2 + s^2)^2). \end{aligned}$$

ce qui donne

$$\mathcal{C}(m, 0)(\xi, \eta) = R_m(\xi, \eta, \xi, \eta). \tag{4}$$

De plus, le tenseur de MTW a un développement (voir [67])

$$\begin{aligned}\mathcal{C}(m,\nu)(\xi,\eta) &= R_m(\xi,\eta,\xi,\eta)+\frac{1}{2}(\nabla_\eta R)_m(\xi,\nu,\xi,\eta)+\\ &\quad \frac{1}{4}(\nabla_\nu R)_m(\xi,\eta,\xi,\eta)+o(|\nu|_m).\end{aligned}$$

0.3.3 La condition de Ma-Trudinger-Wang

La régularité de l'application du transport optimal est liée à la positivité du tenseur de MTW. On introduit des conditions de courbure suivantes.

Définition 0.3. *(La condition de MTW)*

(i) On dit que le tenseur de MTW est positif, si pour tout $m \in M, \xi, \eta \in T_mM$, on a

$$\mathcal{C}(m,\nu)(\xi,\eta) \geq 0.$$

(ii) On dit que la condition A3W est vérifiée, si pour tout $m \in M$ et pour tous $\xi, \eta \in T_mM$ avec $g_m(\xi,\eta)=0$, on a

$$\mathcal{C}(m,\nu)(\xi,\eta) \geq 0.$$

(iii) On dit que la condition A3S est vérifiée, si pour tout $m \in M$ et pour tous $\xi, \eta \in T_mM \setminus \{0\}$ avec $g_m(\xi,\eta)=0$, on a

$$\mathcal{C}(m,\nu)(\xi,\eta) > 0.$$

Il est intéressant de trouver des variétés riemanniennes qui satisfont la condition de MTW. On donne quelques exemples. Lorsque M est plate (par exemple $\mathbb{R}^n, \mathbb{T}^n$), la condition A3W est satisfaite, mais la condition A3S n'est pas satisfaite. Loeper [74] a prouvé que la condition A3S est satisfaite sur la sphère $\mathbb{S}^n$. Kim-McCann [65] ont montré que la submersion riemannienne de la sphère $\mathbb{S}^n$(par exemple $\mathbb{CP}^n, \mathbb{HP}^n$) satisfait la condition A3S. Delanoë-Rouvière [31] ont montré que la variété riemannienne symétrique à courbure sectionnelle strictement positive satisfait la condition A3S. Figalli et Rifford [44] ont prouvé que la condition A3S est vérifiée sur une surface simplement connexe dont la métrique est une perturbation de classe C^4 par rapport à celle de la sphère $\mathbb{S}^2$. Lorsque la courbure de Gauss de la surface s'approche de 1 en norme C^2, Delanoë-Ge [28] ont obtenu la condition A3S là-dessus. Figalli-Rifford-Villani [45] ont démontré que la condition A3S est satisfaite sur une variétés riemannienne compacte de dimension n dont la métrique est une perturbation de classe C^4 par rapport à la sphère canonique $\mathbb{S}^n$. Du-Li [32] ont donné une condition suffisante pour que la condition A3S soit vérifiée sur une surface fermée.

On donne quelques remarques sur la condition A3W et la condition A3S.

Tout d'abord, il est clair que la condition A3S implique la condition A3W, mais la réciproque est fausse en général.

Et puis, la condition A3W implique la positivité de la courbure sectionnelle (voir [73]). En revanche, la réciproque n'est pas vraie(voir [62]). De même, la condition A3S implique que la courbure sectionnelle est strictement positive.

Pour les autres fonctions du coût, la condition A3S n'implique pas forcément que la courbure sectionnelle est positive. Par exemple, on considère la fonction du coût $c(\cdot,\cdot) = -\cosh d(\cdot,\cdot)$ sur l'espace hyperbolique $\mathbb{H}^n$. La condition A3S est satisfaite, mais la courbure sectionnelle de $\mathbb{H}^n$ est toujours égale à -1.

En outre, la condition A3W et la condition A3S sont préservées pour une submersion de Riemann [65]. La condition A3S est également stable sous la limite de Gromov-Hausdorff [100].

On a d'autres caractérisations pour les conditions A3S et A3W. Loeper [73] a prouvé que, la c-convexité des ensembles de contact, la condition A3W et la connexité de la c-sousdifférentiel du potentiel c-convexe sont toutes équivalentes. Si le lieu de coupure d'un point n'est pas un lieu conjugué, Loeper-Villani [75] ont prouvé que la condition A3S implique la convexité uniforme de les domaines d'injectivité. Figalli-Gallouët-Rifford [38] ont montré que la condition A3W implique la convexité des domaines d'injectivité sous des hypothèses convenables. Cependant, ce problème n'est pas complètement résolu.

Les conditions A3W et A3S jouent un rôle important dans la théorie de la régularité de l'application du transport optimal. Loeper [73], Villani [99], Figalli-Rifford-Villani [46] ont prouvé que la condition A3W est nécessaire pour la continuité de l'application du transport optimal.

On considère la fonction du coût égale à la distance au carré. Dans l'espace euclidien, la régularité de l'application du transport optimal a été entièrement résolue. Dans ce cas là, l'équation (3) est équivalente à l'équation de Monge-Ampère classique. L'issue de la régularité est obtenue par Caffarelli [11–13], Delanoë [24] et Urbas [96]. Sur une variété riemannienne, Cordero-Erausquin [22] a prouvé que l'application du transport optimal sur un tore $\mathbb{T}^n$ est lisse(voir également [27]). Loeper [74] a montré que l'application du transport optimal sur une sphère standard $\mathbb{S}^n$ est lisse. Si le lieu de coupure ne rencontre pas le lieu conjugué et si M satisfait la condition A3S, Loeper et Villani [75] ont obtenu le résultat de régularité. Pour le produit des sphères standards, la régularité de l'application du transport optimal est montrée par Figalli-Kim-McCann [42]. Delanoë et Ge ont étudié ce problème de régularité sur des variétés riemanniennes dont la courbure est proche de celle de la sphère standard $\mathbb{S}^n$ en C^2 norme. Delanoë [27] a prouvé que l'application du transport optimale est lisse sur la variété symétrique à courbure sectionnelle strictement positive, et sur une surface dont la courbure de gauss est proche de 1 en C^2 norme.

0.4 Résultats principaux

Avant d'énoncer les résultats, on rappelle quelques notations et définitions. Soit (M,g) une variété riemannienne compacte connexe lisse sans bord de la dimension $n \geq 2$. On dit brièvement une variété riemannienne fermée. Soit K la courbure sectionnelle de (M,g). La courbure de Riemann de (M,g) est notée par Riem. Le carré de la superficie du parallélogramme engendré par deux vecteurs tangents $\xi, \eta \in T_mM$ est égal à $|\xi \wedge \eta|_m^2 = |\xi|_m^2 |\eta|_m^2 - g_m(\xi,\eta)^2$.

Soient X, Y, Z, W des champs de vecteur lisses sur M. Le produit de Kulkarni-Nomizu $T_1 \owedge T_2$ de deux champs de 2-tenseurs symétriques T_1 et T_2 est défini par

$$\begin{aligned} T_1 \owedge T_2(X,Y,Z,W) &= T_1(X,Z)T_2(Y,W) + T_1(Y,W)T_2(X,Z) - \\ &\quad T_1(X,W)T_2(Y,Z) - T_1(Y,Z)T_2(X,W). \end{aligned}$$

On suppose toujours que la courbure sectionnelle de (M,g) satisfait

$$\min_{\mathrm{Gr}_2(M)} K = 1. \tag{5}$$

et la courbure de Riemann satisfait

$$\|\mathrm{Riem} - \frac{1}{2} g \owedge g\|_{C^2(M,g)} < \varepsilon \tag{6}$$

Pour $\nu \neq 0$, on considère

$$\bar{\mathcal{S}}(m,\nu,1)(\xi) = \xi - (1 - |\nu|_m \cot |\nu|_m)(\xi - g_m(\xi, \frac{\nu}{|\nu|_m})\frac{\nu}{|\nu|_m}) \tag{7}$$

et on note $\bar{\mathcal{C}}(m,\nu)(\xi,\eta)$ le tenseur de MTW sur la sphère standard $\mathbb{S}^n$, c'est-à-dire,

$$\bar{\mathcal{C}}(m,\nu)(\xi,\eta) = -\frac{3}{2}\frac{d^2}{ds^2}|_{s=0} g_m(\bar{\mathcal{S}}(m,\nu+s\eta,1)(\xi),\xi). \tag{8}$$

Il est clair que $\lim_{\nu\to 0} \bar{\mathcal{S}}(m,\nu,1)(\xi) = |\xi|_m^2$ et que

$$\lim_{\nu\to 0} \bar{\mathcal{C}}(m,\nu)(\xi,\eta) = \bar{R}_m(\xi,\eta,\xi,\eta) = |\xi|_m^2|\eta|_m^2 - g_m(\xi,\eta)^2. \tag{9}$$

Les résultats principaux dans cette thèse sont inclus dans deux prépublications (voir [52, 106]).

Le premier résultat consiste à la stabilité de positivité du tenseur de MTW sur une variété presque sphérique.

Théorème 0.1. *Soit (M,g) une variété riemannienne fermée de dimension $n \geq 2$. Suppose que (M,g) satisfait (5). Alors il existe deux constantes strictement positives $\varepsilon_0, \kappa_0 > 0$ qui ne dépendent que de n, telles que si (6) est vérifée avec $\varepsilon \leq \varepsilon_0$, alors pour tout $m \in M, \nu \in I(m)$ et pour tous vecteurs tangents $\xi, \eta \in T_mM$, on a*

$$\mathcal{C}(m,\nu)(\xi,\eta) \geq \kappa_0(|\xi\wedge\eta|_m^2 + |\xi|_m^2|\eta\wedge\nu|_m^2 + |\xi\wedge\nu|_m^2|\eta|_m^2). \tag{10}$$

Une conséquence immédiate du théorème est la suivante:

Corollaire 0.1. *Soient M_1 et M_2 deux variétés riemanniennes fermées de dimension $n_1 \geq 2$ et $n_2 \geq 2$ respectivement. Suppose qu'il existe un nombre petit $\varepsilon_0 > 0$ dépendant de n tel que (5) et (6) avec $\varepsilon \leq \varepsilon_0$ soient satisfaites sur M_1 et M_2. Alors le tenseur de MTW est positif sur la variété produit $M_1 \times M_2$. En particulier, la condition A3W est satisfaite.*

Dans la preuve, on utilise des estimations suivantes:

Proposition 0.1. *Soit (M,g) une variété riemannienne fermée de dimension $n \geq 2$. Suppose que (M,g) satisfait (5) et qu'il existe un nombre petit $\varepsilon_0 > 0$ dépendant de n , telle que (6) avec $\varepsilon \leq \varepsilon_0$ soit satisfaite. Alors il existe une constante strictement positive $C > 0$ qui ne dépend que de n, telle que pour tout $m \in M, v \in I(m), |v| \geq \frac{3\pi}{4}$.*

1) $|\mathcal{S}^{-1}(m,v,1) - \bar{\mathcal{S}}^{-1}(m,v,1)| \leq C\varepsilon$;

2) $|\partial_x\mathcal{S}^{-1}(m,v,1) - \partial_x\bar{\mathcal{S}}^{-1}(m,v,1)| \leq C\varepsilon$,
$|D_v\mathcal{S}^{-1}(m,v,1) - D_v\bar{\mathcal{S}}^{-1}(m,v,1)| \leq C\varepsilon$;

3) $|\partial_{xx}^2\mathcal{S}^{-1}(m,v,1) - \partial_{xx}^2\bar{\mathcal{S}}^{-1}(m,v,1)| \leq C\varepsilon$,
$|\partial_x D_v\mathcal{S}^{-1}(m,v,1) - \partial_x D_v\bar{\mathcal{S}}^{-1}(m,v,1)| \leq C\varepsilon$,
$|D_{vv}^2\mathcal{S}^{-1}(m,v,1) - D_{vv}^2\bar{\mathcal{S}}^{-1}(m,v,1)| \leq C\varepsilon$.

A l'aide de la méthode de continuité, on prouve le résultat de la régularité de l'application du transport optimal.

Théorème 0.2. *Soit (M,g) une variété riemannienne fermée de dimension $n \geq 2$. Suppose que (M,g) satisfait (5). Alors il existe une constante strictement positive $\varepsilon_0 > 0$ qui ne dépend que de n, telle que, si*

$$\|Riem - \frac{1}{2}g \owedge g\|_{C^2(M,g)} < \varepsilon_0,$$

alors pour tout $(k,\alpha) \in \mathbb{N}\times(0,1)$, avec $k \geq 2$, le potentiel du transport optimale est de classe $C^{k+2,\alpha}$ pour des mesures de probabilité sur M $\rho_0 dvol$ et $\rho_1 dvol$ à densité strictement positive de classe $C^{k,\alpha}$.

Le Théorème 0.2 implique que sur une variété presque sphérique lisse, si la densité des mesures est régulière et strictement positive, alors l'application du transport optimal est régulière.

Corollaire 0.2. *Sous les mêmes hypothèses que le théorème 0.2, suppose que les densités ρ_0, ρ_1 sont de classe C^∞. Alors l'application du transport optimal est de classe C^∞.*

De la même manière, on obtient la régularité de l'application du transport optimal sur une variété produit des variétés presque sphériques.

Théorème 0.3. *Soient M_1 et M_2 deux variétés riemanniennes fermées de dimension $n_1 \geq 2$ et $n_2 \geq 2$ respectivement. Suppose que $\forall i$, (M_i, g_i) satisfait (5). Il existe une constante strictement positive $\varepsilon_0 > 0$ qui ne dépend que de n_i pour $i = 1, 2$, telle que, si*

$$\|Riem_i - \frac{1}{2} g_i \owedge g_i\|_{C^2(M_i, g_i)} < \varepsilon_0,$$

alors pour tout $(k, \alpha) \in \mathbb{N} \times (0, 1)$, avec $k \geq 2$, et pour toutes mesures de probabilité à densité strictement positive de classe $C^{k,\alpha}$ sur $M_1 \times M_2$ $\rho_0 dvol$ et $\rho_1 dvol$, le potentiel du transport optimal envoyant $\rho_0 dvol$ vers $\rho_1 dvol$ est de classe $C^{k+2,\alpha}$.

Une conséquence directe est la suivante.

Corollaire 0.3. *Sous les mêmes hypothèses du Théorème 0.3, toutes mesures de probabilité à densité strictement positive de classe C^∞ sur $M_1 \times M_2$ $\rho_0 dvol$ et $\rho_1 dvol$, le potentiel du transport optimal envoyant $\rho_0 dvol$ vers $\rho_1 dvol$ est de classe C^∞.*

Dans [106], on montre également que si la métrique sur $M_1 \times M_2$ n'est pas sous forme de produit, alors l'application du transport optimal n'est pas forcément régulière même si la métrique est proche du produit des sphères en norme C^4. Plus précisement, on a

Théorème 0.4. *On note $g^\times$ le produit des métriques canoniques sur $\mathbb{S}^{n_1} \times \mathbb{S}^{n_2}$ avec $n_1 \geq 2$ et $n_2 \geq 2$. Alors $\forall \varepsilon > 0$, il existe une métrique g sur $\mathbb{S}^{n_1} \times \mathbb{S}^{n_2}$ conforme à $g^\times$ satisfaisant*

$$\|g - g^\times\|_{C^4} < \varepsilon$$

telle que l'on puisse trouver des mesures de probabilité à densité C^∞ strictement positive sur $\mathbb{S}^{n_1} \times \mathbb{S}^{n_2}$ dont l'application du transport optimal correspondante n'est pas lisse.

Ceci provient du fait que l'on n'a pas la stabilité de la condition A3W sur l'espace produit si l'on sort de la classe de produit des métriques.

Chapter 1

Preliminaries

1.1 Basic notations and conventions

In this section, some basic notations from Riemannian geometry will be stated. See [1] [2] [18] [19] [20] [80] as references on Riemannian geometry.

Let (M, g) be a complete connected smooth Riemannian manifold of dimension $n \geq 2$. Let X, Y, Z, W be smooth vector fields on M. The (3,1)-type Riemann curvature tensor of the Riemaniann manifold (M, g) is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

where ∇ denotes the Levi-Civita connection of g.

We set Riem for the associated (4,0)-type Riemann curvature tensor[1], i.e.

$$\mathrm{Riem}(X,Y,Z,W) = \langle R(Z,W)Y, X\rangle.$$

Throughout the thesis, we adopt the Einstein summation convention over repeated indices.

In a local coordinate system$\{x^1, \cdots, x^n\}$, the components of Riemann curvature tensor are given by $R(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k})\frac{\partial}{\partial x^i} = R^l_{ijk}\frac{\partial}{\partial x^l}$ and $R_{ijkl} = g_{ip}R^p_{jkl}$ respectively. The Ricci tensor is obtained by the contraction $\mathrm{Ric}_{ij} = g^{kl}R_{ikjl}$ and the scalar curvature by $\mathrm{Scal} = g^{ij}\mathrm{Ric}_{ij}$.

The Riemannian metric induces norms on all the tensor bundles. Precisely, the squared norm of (r, s)-tensor field T in the coordinate system $x = (x^1, \cdots, x^n)$ is given by

$$|T|^2 = g_{i_1k_1} \cdots g_{i_sk_s} g^{j_1l_1} \cdots g^{j_rl_r} T^{i_1\cdots i_s}_{j_1\cdots j_r} T^{k_1\cdots k_s}_{l_1\cdots l_r},$$

where $T^{i_1\cdots i_s}_{j_1\cdots j_r}$ are components of T in the coordinate system x.

We will need notation for the second covariant derivative of a tensor field, we write

$$\nabla^2_{X,Y} T := (\nabla^2 T)(X, Y, \cdots).$$

It is remarkable to note that $\nabla^2_{X,Y}T = \nabla_X(\nabla_Y T) - \nabla_{\nabla_X Y}T$.

Another fact will be used frequently is that the tensor $g \owedge g$ is parallel, i.e.

$$\nabla(g \owedge g) = 0. \tag{1.1}$$

For later use, the (3,1)-form of $\frac{1}{2} g \owedge g$ is denoted by $\bar{R}$, i.e.

$$\bar{R}(X,Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y.$$

Given a local coordinate system$\{x^1, \cdots, x^n\}$, the components of $\bar{R}$ are given by

$$\bar{R}(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k})\frac{\partial}{\partial x^i} = \bar{R}^l_{ijk}\frac{\partial}{\partial x^l}, \text{ with } \bar{R}^l_{ijk} = \delta^l_j g_{ik} - \delta^l_k g_{ij}. \tag{1.2}$$

[1]We use $g(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$ interchangeably.

Let $K : \mathrm{Gr}_2(M) \to \mathbb{R}$ be the sectional curvature defined on the Grassmann bundle of tangent 2-planes(to see [27]). We suppose that (M, g) be closed manifold of dimension $n \geq 2$ throughout the thesis unless otherwise specified. We also assume the sectional curvature of M satisfies

$$\min_{\mathrm{Gr}_2(M)} K = 1. \tag{1.3}$$

and the Riemann curvature tensor satisfies

$$\|\mathrm{Riem} - \frac{1}{2} g \owedge g\|_{C^2(M,g)} < \varepsilon, \text{with } \varepsilon \text{ small.} \tag{1.4}$$

If (M, g) satisfies (1.3) and (1.4), we say that (M, g) is nearly spherical.

It is readily to see that (1.3) and (1.4) hold on round sphere $\mathbb{S}^n$.

In addition, by (1.4), it follows that

$$\|\mathrm{Ric} - (n-1)g\|_{C^2(M,g)} < \varepsilon,$$
$$\|\mathrm{Scal} - n(n-1)\|_{C^2(M,g)} < \varepsilon.$$

In two dimension, up to a constant, the curvature assumption (1.4) is equivalent to

$$|K - 1|_{C^2(M,g)} < \varepsilon.$$

While in $n \geq 3$ dimension, up to a constant, the curvature assumption (1.4) deduces

$$\|\mathrm{Riem} - \frac{\mathrm{Scal}}{2n(n-1)} g \owedge g\|_{C^2(M,g)} < \varepsilon.$$

Indeed, from the well-known decomposition of *Riem* we obtain the identity

$$|\mathrm{Riem} - \frac{1}{2} g \owedge g|^2 = |\mathrm{Riem} - \frac{\mathrm{Scal}}{2n(n-1)} g \owedge g|^2 + \frac{2}{n(n-1)} (\mathrm{Scal} - n(n-1))^2.$$

Together with the parallel property (1.1) and the definition of Scalar curvature, the result is derived.

The assumption (1.3) and (1.4) contain some geometric information of (M, g). In view of Bonnet myers theorem [20], the normalization (1.3) implies that (M, g) is compact and there is at least one conjugate point along every geodesic. The compactness infers that there is cut point along every geodesic [18].

Let $m \in M$. For $\forall \nu \in T_m M, |\nu|_m = 1$. Set $t_C(m, \nu)$ be the distance from point m to the cut point of m along the geodesic $\exp_m(t\nu)$, *i.e.*

$$t_C(m, \nu) = \sup\{t \geq 0; \exp_m(s\nu)|_{0 \leq s \leq t} \text{ is a minimizing geodesic}\}.$$

The injectivity domain at m is denoted by $I(m)$, i.e.

$$I(m) = \{t\nu; 0 \leq t < t_C(m, \nu), \nu \in T_m M \backslash \{0\}\}.$$

The focal time $t_F(m, \nu)$ is defined by

$$t_F(m, \nu) = \inf\{t \geq 0; \exp_m(t\nu) \text{is conjugate to } m\}.$$

Recall that the cut time is smaller than the focal time. The injectivity domain is an open subset contains the origin in $T_m M$ and star-shaped with respect to the origin. Moreover,$M = \exp_m(I(m)) \bigsqcup \mathrm{Cut}_m$, where $\bigsqcup$ means disjoint union. The exponential map $\exp_m : I(m) \to M \backslash \mathrm{Cut}_m$ is a diffeomorphism.

The geometry of injectivity domain is complicated. But on some special manifolds they have nice geometric properties. For instance, the injectivity domain of round sphere $\mathbb{S}^n$ is the open ball of radius π centered at the origin. If the Riemannian manifold is simply connected, complete and with non-positive curvature, from theorem of Hadamard [18], we know that the injectivity domains are $\mathbb{R}^n$. Figalli-Rifford-Villani [45] established that the injectivity domains are uniformly convex on the Riemannian manifold which is the C^4 metric perturbation of round sphere $\mathbb{S}^n$. Bonnard [6] proved that the injectivity domain of the ellipsoid $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + \frac{z^2}{\mu^2} = 1\}(\mu \in (0, 1])$ is convex if and only if $\mu \geq \frac{1}{\sqrt{3}}$.

The notation $\exp_m^{-1}(y)$ stands for all the velocities $\nu \in T_m M$ such that the geodesic $\exp_m(s\nu)|_{0 \leq s \leq 1}$ is minimizing and $\exp_m \nu = y$. Under the curvature assumption (1.3), by the Bonnet-Myers theorem [20], for each $\nu \in \exp_m^{-1}(y)$, the length of ν is not great than π. Moreover, if $\nu \in I(m)$, by Rauch comparison theorem, the length of ν is strictly less than π. We will use these facts frequently throughout the thesis.

For $y \in M$, we consider the functions $\frac{d_y^2}{2}(\cdot) = \frac{1}{2} d^2(\cdot, y)$ which is smooth in $M \backslash \mathrm{Cut}_y$. For any $m \notin \mathrm{Cut}_y$, the Gauss lemma implies that

$$\operatorname{grad} \frac{d_y^2}{2}(m) = -\exp_m^{-1} y. \tag{1.5}$$

Given a real smooth function u defined on M. The Hessian of u at m is given by the linear operator from $T_m M$ to $T_m M$ defined by the identity

$$\text{for } \forall \xi \in T_m M, \nabla_m^2 u(\xi) := \nabla_\xi(\operatorname{grad} u).$$

It is easy to see that the Hessian is self-adjoint and can be calculated as follows

$$\langle \nabla_m^2 u(\xi), \xi \rangle = \frac{d^2}{ds^2}\bigg|_{s=0} u(\gamma(s)), \tag{1.6}$$

where γ is a geodesic with the initial point m and the initial velocity ξ.

1.2 Jacobi matrix

1.2.1 Initial Jacobi matrices

In this section, we give the definition and some basic facts about the initial Jacobi matrices. The definition is stated as follows. See Chapter 14 in [99] as references on the initial Jacobi matrices and the Jacobi matrix.

Definition 1.1. *Given $m \in M, \nu \in T_m M \backslash \{0\}$. Let $\{E_1, E_2, \cdots, E_n\}$ be an orthonormal basis of $T_m M$ with $E_1 = \nu / |\nu|_m$. Let $\gamma(\cdot)$ be a geodesic with initial point m and initial velocity ν and $\{e_1, e_2, \cdots, e_n\}$ be the parallel transport of $\{E_1, E_2, \cdots, E_n\}$ along γ with $e_i(0) = E_i$. We define the matrices $J_0(m, \nu, t)$ and $J_1(m, \nu, t)$ as the matrix valued solutions of the second order equation*

$$\ddot{J}_a + R J_a = 0, a = 0, 1, \tag{1.7}$$

with the initial condition

$$J_0(m, \nu, 0) = 0, \dot{J}_0(m, \nu, 0) = I_n,$$
$$J_1(m, \nu, 0) = I_n, \dot{J}_1(m, \nu, 0) = 0,$$

where the elements of R are given by

$$R_{ij}(t) = \langle R(e_i(t), \dot{\gamma}(t))\dot{\gamma}(t), e_j(t) \rangle. \tag{1.8}$$

The matrices $J_0(m, \nu, t)$ and $J_1(m, \nu, t)$ are called the initial Jacobi matrices. The matrix (R_{ij}) is called curvature matrix. The equation (1.7) is called the Jacobi equation. A matrix J is called Jacobi matrix if it satisfies the Jacobi equation (1.7).

We give some facts about the curvature matrix. It is clear that the curvature matrix (R_{ij}) has vanishing first row and first column. The curvature matrix (R_{ij}) is symmetric and its the trace gives the Ricci curvature. In addition, the curvature matrix (R_{ij}) is positive semi-definite if the sectional curvature of M is non-negative.

As same as on the sphere, we define $\bar{J}_a(m,\nu,t)$ as the matrix-valued of the second order equation

$$\begin{cases} \ddot{\bar{J}}_a + \bar{R}\bar{J}_a = 0, a = 0,1, \\ \bar{J}_0(m,\nu,0) = 0, \dot{\bar{J}}_0(m,\nu,0) = I_n, \\ \bar{J}_1(m,\nu,0) = I_n, \dot{\bar{J}}_1(m,\nu,0) = 0. \end{cases} \tag{1.9}$$

The elements of $\bar{R}$ are given by

$$\bar{R}_{ij}(m,\nu,t) = \langle \bar{R}(e_i(t),\dot{\gamma}(t))\dot{\gamma}(t), e_j(t)\rangle.$$

It is readily to see that $\bar{J}_0$ and $\bar{J}_1$ are given respectively by

$$\bar{J}_0 = \begin{bmatrix} t & 0 \\ 0 & \frac{\sin(|\nu|t)}{|\nu|} I_{n-1} \end{bmatrix}, \bar{J}_1 = \begin{bmatrix} 1 & 0 \\ 0 & \cos(|\nu|t) I_{n-1} \end{bmatrix}.$$

From the homogeneity of a geodesic(to see [18] p.64), we get the homogeneity of the initial Jacobi matrices,i.e.

$$\begin{aligned} \lambda J_0(m,\lambda\nu,t) &= J_0(m,\nu,\lambda t), \\ J_1(m,\lambda\nu,t) &= J_1(m,\nu,\lambda t), \lambda > 0. \end{aligned}$$

For $t \in [0,1], a = 0,1$, we can extend the initial matrix J_a by continuity at $\nu = 0$ by $J_a(m,0,t) = I_n$. Similarly, we can extend $\dot{J}_0, J_0^{(k)}, k \geq 2$ at $\nu = 0$ by $\dot{J}_0(m,0,t) = I_n, J_0^{(k)}(m,0,t) = 0$ and $J_1^{(k)}, k \geq 1$ at $\nu = 0$ by $J_1^{(k)}(m,0,t) = 0$.

For simplicity, the initial Jacobi matrix $J_a(m,\nu,t)$ is abbreviated to $J_a(t)$ unless otherwise specified.

By definition of the conjugate point, the initial matrix $J_0(t)$ is invertible for every $t \in (0, t_F(m,\nu))$. Moreover, in view of Proposition 14.30 in [99], the continuity derives that $\det J_0(t) > 0$ for every $t \in (0, t_F(m,\nu))$.

We now present the Hessian of the squared distance in terms of the initial Jacobi matrix and a representation formula of the inhomogeneous Jacobi equation.

Proposition 1.1. *Under the hypothesis of Definition 1.1, we have*

(a) Let $J(t)$ be the Jacobi field along the geodesic $\exp_m(t\nu)$ defined by the conditions $J(0) = \xi, J(1) = 0$. Then $J(t) = -J_0(t)J_0^{-1}(1)J_1(1)(\xi) + J_1(t)(\xi)$;

(b) For $t \in [0, t_F(m,\nu))$. Let $\mathcal{S}(m,\nu,t)$ be the linear operator from T_mM to T_mM whose matrix in the orthonormal basis $\{E_1, E_2, \cdots, E_n\}$ is given by $tJ_0(t)^{-1}J_1(t)$. Then the linear operator $\mathcal{S}(m,\nu,t) : T_mM \to T_mM$ is self adjoint. Moreover, if $\nu \in I(m)$, then for $\forall \xi \in T_mM$,

$$\langle \nabla_m^2 c(\cdot, \exp_m \nu)(\xi), \xi\rangle = \langle \mathcal{S}(m,\nu,1)(\xi), \xi\rangle.$$

(c) (Representation formula)The solution of the matrix valued inhomogeneous Jacobi equation

$$\ddot{J}(t) + R(t)J(t) = B(t)$$

is given by the formula

$$J(t) = J_0(t)\dot{J}(0) + J_1(t)J(0) + J_0(t)\int_0^t J_1^* B ds - J_1(t)\int_0^t J_0^* B ds,$$

where $*$ means transpose of matrix.

Remark 1.1. *1.(Homogeneity)From the homogeneity of the initial Jacobi matrices, we infer that*

$$\mathcal{S}(m, \lambda\nu, t) = \mathcal{S}(m, \nu, \lambda t), \lambda > 0.$$

Thus we can extended $\mathcal{S}$ by continuity at $\nu = 0$ by $\mathcal{S}(m, 0, t) = Id$;
2.The linear operator $\mathcal{S}(m, \nu, t)$ has explicit formula on space forms [68], for instance, on the round sphere $\mathbb{S}^n$,

$$\bar{\mathcal{S}}(m, \nu, t)(\xi) = \xi - (1 - t|\nu| \cot(t|\nu|))(\xi - \langle \xi, \frac{\nu}{|\nu|}\rangle \frac{\nu}{|\nu|}).$$

Equivalently, the associated covariant symmetric 2-tensor field is given by $g - (1 - t|\nu| \cot(t|\nu|))(g - \frac{\nu}{|\nu|} \otimes \frac{\nu}{|\nu|})$.
3.We obtain the representation formula for the scalar function,i.e.

$$f(t) = f(0)\cos t + \dot{f}(0)\sin t + \sin t \int_0^t \phi(s)\cos s ds - \cos t \int_0^t \phi(s)\sin s ds, \tag{1.10}$$

where $\phi = \ddot{f} + f$.

Proof. (a) is direct result of the uniqueness of the second order ordinary differential equation.
(b)The self-adjoint property of $\mathcal{S}$ refers to Proposition 14.30 in [99]. We prove the second assertion here. Let ξ be a tangent vector based at m.
If $\nu = 0$, $tJ_0(t)^{-1}$ can be extended by continuity at $t = 0$ by I_n. So

$$\langle \nabla^2 c(\cdot, m)(\xi), \xi\rangle = |\xi|^2 = \langle \mathcal{S}(m, \nu, 1)(\xi), \xi\rangle.$$

If $\nu \in I(m)\backslash\{0\}$, then the curve $\sigma(t) = \exp_m(t\nu)|_{t\in[0,1]}$ is the unique minimizing geodesic from m to $\exp_m \nu$.

Let $\gamma(\cdot)$ be the geodesic with the initial point m and the initial velocity ξ. Consider the family of the geodesics $\sigma(t, s) = \exp_{\gamma(s)}(t \exp^{-1}_{\gamma(s)}(\exp_m \nu))$, so that $\sigma(t, 0) = \sigma(t)$.

By the definition of the Jacobi field, it follows that $J(t) = \frac{\partial}{\partial s}|_{s=0}\sigma$ is a Jacobi field along the geodesic $\sigma(t)$ with $J(0) = \xi$ and $J(1) = 0$. Moreover,

$$\dot{J}(0) = \frac{\partial^2}{\partial s \partial t}|_{s=t=0}\sigma = \frac{d}{ds}|_{s=0} \exp^{-1}_{\gamma(s)}(\exp_m \nu) = -\nabla^2_m c(\cdot, \exp_m \nu)(\xi).$$

where the last equality follows from (1.5) and the definition of Hessian.
Hence

$$\langle \nabla^2_m c(\cdot, \exp_m \nu)(\xi), \xi\rangle = -\langle \dot{J}(0), J(0)\rangle.$$

From (a) the term $J(t)$ is equal to $-J_0(t)J_0^{-1}(1)J_1(1)(\xi) + J_1(t)(\xi)$. So

$$\dot{J}(0) = -\mathcal{S}(m, \nu, 1)(\xi).$$

Therefore

$$\langle \nabla^2_m c(\cdot, \exp_m \nu)(\xi), \xi\rangle = \langle \mathcal{S}(m, \nu, 1)(\xi), \xi\rangle.$$

(c) The result follows from a direct calculation(the details refer to Lemma 3.2 in [45]). This finishes the proof of Proposition 1.1. □

1.2.2 The approximation of initial Jacobi matrices

In this subsection, we present the approximation of the initial Jacobi matrices.

We first give a basic fact from the theory of second order differential equations [29].

Lemma 1.1. *Let (M, g) be a closed manifold of dimension $n \geq 2$. Suppose that M satisfies the curvature assumptions (1.3) and (1.4). Then there exists a positive constant C_1 depending only on n such that for every $m \in M, \nu \in T_mM$ and every $t \in [0, 1]$, the g norms of the initial Jacobi matrices:*

$$|J_a(m, \nu, t)|, |\dot{J}_a(m, \nu, t)|, a = 0, 1.$$

are all bounded above by C_1.

Keeping in mind the assumption (1.4), the initial Jacobi matrices can be estimated as follows [29].

Lemma 1.2. *Let (M,g) be a closed manifold of dimension $n \geq 2$. Suppose that M satisfies the curvature assumptions (1.3) and (1.4). Then there exists a positive constant C_2 depending only on n such that for every $m \in M, \nu \in T_mM$ and every $t \in [0,1]$, the following estimates hold:*

$$\begin{aligned} |J_a(m,\nu,t) - \bar{J}_a(m,\nu,t)| &\leq C_2\varepsilon, \\ |\dot{J}_a(m,\nu,t) - \dot{\bar{J}}_a(m,\nu,t)| &\leq C_2\varepsilon, a = 0,1. \end{aligned}$$

Remark 1.2. *For later use, regarding $|J_0 - \bar{J}_0|$, the constant C_2 can be taken value $2\sqrt{n-1}$.(to see Remark 5 of [29])*

Proof. It is obvious for $\nu = 0$. Without generality, we assume that $\nu \neq 0$. The length of the tangent vector ν is denoted by τ.

Note that the Jacobi equation (1.7) can be rewritten as follows:

$$\ddot{J}_a + \bar{R}J_a = (\bar{R} - R)J_a.$$

Applying a representation formula in [29] to the elements of the matrix J_a, we have

$$J_a = \bar{J}_a + \tau^2 \sin(\tau t) \int_0^t \frac{ds}{\sin^2(\tau s)} \int_0^s \sin(\tau\theta)(\bar{R} - R)J_a d\theta.$$

Using (1.3) and the Cauchy-Schwarz inequality, we derive

$$|J_a - \bar{J}_a| \leq \varepsilon \max_{t\in[0,1]} |J_a^{\perp}|\tau^2 \sin(\tau t)\left| \int_0^t \frac{ds}{\sin^2(|\nu_0|s)} \int_0^s \sin(\tau\theta)d\theta \right|.$$

Together with Lemma 1.1 and the expression

$$1 - \cos(\tau t) = \tau^2 \sin(\tau t) \int_0^t \frac{ds}{\sin^2(\tau s)} \int_0^s \sin(\tau\theta)d\theta,$$

we obtain that there exists a positive constant C depending only n such that

$$|J_a(m,\nu,t) - \bar{J}_a(m,\nu,t)| \quad \leq \quad C\varepsilon, a = 0,1. \tag{1.11}$$

Note that Remark 1.2 follows from the Rauch comparison theorem [18].

For the second inequality, from the initial condition $\dot{J}_a(0) = \dot{\bar{J}}_a(0)$, we get

$$\begin{aligned} \dot{J}_a - \dot{\bar{J}}_a &= \int_0^t \left(\ddot{J}_a - \ddot{\bar{J}}_a\right) ds \\ &= \int_0^t \left(\bar{R}\bar{J}_a - RJ_a\right) ds \\ &= \int_0^t \bar{R}(\bar{J}_a - J_a) + (\bar{R} - R)J_a ds. \end{aligned}$$

Together with the curvature assumption (1.4), Lemma 1.1 and (1.11), we imply that there exist a positive constant $\tilde{C}$ such that

$$|\dot{J}_a(m,\nu,t) - \dot{\bar{J}}_a(m,\nu,t)| \leq \tilde{C}\varepsilon.$$

This finishes the proof of Lemma 1.2 by choosing $C_2 = \max\{C, \tilde{C}\}$. □

1.3 Fermi coordinate system

In this subsection, we state the definition of Fermi coordinate system.

Definition 1.2. *(Fermi coordinate system) Let (M, g) be an n-dimensional Riemannian manifold. Given a compact interval $I \subset \mathbb{R}$ which contains 0. Let $\gamma : I \to M$ be a geodesic with $|\dot{\gamma}| = 1$ and $\{e_1(t), e_2(t), \cdots, e_n(t)\}$ be a parallel orthonormal moving frame of vector fields along the geodesic γ with $e_1(t) = \dot{\gamma}(t)$. The Fermi coordinate system $\{x^1, \cdots, x^n\}$ are defined by*

$$x^1(\exp_{\gamma(t)}(\sum_{\beta=2}^{n} \lambda^\beta e_\beta(t))) = t,$$

$$x^\alpha(\exp_{\gamma(t)}(\sum_{\beta=2}^{n} \lambda^\beta e_\beta(t))) = \lambda^\alpha, 2 \le \alpha \le n, t \in I.$$

where λ^β are sufficiently small so that the exponential maps are defined.

In the Fermi coordinate system $\{x^1, \cdots, x^n\}$, γ is called the axis.

Observe that the differential of the map $(x^1, \cdots, x^n) \to \exp_{\gamma(x^1)}(\sum_{\beta=2}^n x^\beta e_\beta(t))$ on the axis is equal to the identity. Thanks to the inverse function theorem, the Fermi coordinate system makes sense.

The Fermi coordinate system is generalization of the normal coordinate system. To see this, along the axis we have

$$\forall i, j, k \in \{1, 2, \cdots, n\}, g_{ij}(x_1, 0) = \delta_{ij}, \partial_k g_{ij}(x_1, 0) = 0. \tag{1.12}$$

We will require higher order derivatives of the metric and Christoffel symbols on the axis. In the following, the Latin indices run over $1, \cdots, n$ and the Greek indices run over $2, \cdots, n$.

On the axis, we have the following expressions.

Lemma 1.3. *The following identities hold on the axis:*

$$\partial^2_{ij} g_{11} = -2R_{1i1j}, \partial^2_{\alpha\beta} g_{1\mu} = -\frac{2}{3}(R_{\alpha 1\beta\mu} + R_{\alpha\mu\beta 1}), \tag{1.13}$$

$$\partial^2_{\alpha\beta} g_{\rho\mu} = -\frac{1}{3}(R_{\alpha\rho\beta\mu} + R_{\alpha\mu\beta\rho}), \tag{1.14}$$

$$\partial_k \Gamma^i_{1j} = R^i_{jk1}, \partial_\alpha \Gamma^i_{\beta\mu} = \frac{1}{3}(R^i_{\beta\alpha\mu} + R^i_{\mu\alpha\beta}), \tag{1.15}$$

$$\partial^2_{\alpha\beta} \Gamma^i_{11} = \nabla_1 R^i_{\beta\alpha 1} + \nabla_\alpha R^i_{1\beta 1}, \tag{1.16}$$

$$\partial^2_{\alpha\beta} \Gamma^1_{1\mu} = \frac{1}{3}(\nabla_1 R^1_{\beta\alpha\mu} - \nabla_1 R^1_{\mu\beta\alpha}) - \nabla_\alpha R^1_{\mu 1\beta}, \tag{1.17}$$

$$\partial^2_{\alpha\beta} \Gamma^\rho_{1\mu} = \frac{1}{2}(\nabla_\alpha R^\rho_{\mu\beta 1} + \nabla_\beta R^\rho_{\mu\alpha 1}) + \frac{1}{6}(\nabla_1 R^\rho_{\alpha\beta\mu} + \nabla_1 R^\rho_{\beta\alpha\mu}). \tag{1.18}$$

Furthermore, applying p times $\frac{\partial}{\partial x^1}$ (axis-derivative) to any of the preceding left side quantities, yields on the axis the p-th covariant derivative ∇_1^p of the corresponding intrinsic right side quantity. For instance:

$$\partial_1(\partial_k \Gamma^i_{1j}) = \nabla_1 R^i_{jk1}.$$

Proof. The results are given in [29] except (1.14). We only prove (1.14) here. By definition of the Riemann curvature tensor, on the axis, we have

$$R_{\alpha\rho\beta\mu} = \frac{1}{2}(\partial^2_{\beta\rho} g_{\alpha\mu} + \partial^2_{\alpha\mu} g_{\beta\rho} - \partial^2_{\alpha\beta} g_{\rho\mu} - \partial^2_{\rho\mu} g_{\alpha\beta}).$$

Thanks to the identity:

$$\partial^2_{\rho\mu} g_{\alpha\beta} = \partial^2_{\rho\mu} g_{\alpha\beta},$$

We derive

$$R_{\alpha\rho\beta\mu} = \partial^2_{\alpha\mu} g_{\beta\rho} - \partial^2_{\alpha\beta} g_{\rho\mu}.$$

Thus

$$\begin{aligned} R_{\alpha\rho\beta\mu} + R_{\alpha\mu\beta\rho} &= \partial^2_{\alpha\mu} g_{\beta\rho} + \partial^2_{\alpha\rho} g_{\beta\mu} - 2\partial^2_{\alpha\beta} g_{\rho\mu} \\ &= -3\partial^2_{\alpha\beta} g_{\rho\mu}. \end{aligned}$$

where the last inequality follows the well known identity

$$\partial^2_{\mu\alpha} g_{\beta\rho} + \partial^2_{\mu\beta} g_{\alpha\rho} + \partial^2_{\mu\rho} g_{\beta\alpha} = 0.$$

This finishes the proof of Lemma 1.3. □

Chapter 2

The behaviour of the curvature matrix

In this chapter, we study the behaviour of the curvature matrix (R_{ij}). We start by giving some notations. Let (M, g) be an n-dimensional Riemannian manifold satisfying (1.3) and (1.4) for some small positive number ε. Fix $m_0 \in M$ and $\nu_0 \in I(m_0)\backslash\{0\}$. The length of the tangent vector ν_0 is denoted by τ.

Taking the orthonormal basis $\{E_1, E_2, \cdots, E_n\}$ of $T_{m_0}M$. Let the curve $\gamma_\theta(t)$ be the geodesic with the initial point m_0 and the initial velocity $\cos\theta E_1 + \sin\theta E_2$. For $|\theta|$ sufficiently small, let $\{e_1(\theta,t), e_2(\theta,t), \cdots, e_n(\theta,t)\}$ be the parallel transport along the geodesic $\gamma_\theta(t)$ with $e_1(\theta,0) = \cos\theta E_1 + \sin\theta E_2$, $e_2(\theta,0) = -\sin\theta E_1 + \cos\theta E_2$, $e_i(\theta,0) = E_i$ for $i \geq 3$. Then $\{e_1(0,t), e_2(0,t), \cdots, e_n(0,t)\}$ is the parallel orthonormal moving frame along the geodesic $\gamma_0(t)$ with $e_1(0,t) = \dot\gamma_0(t)$.

Let $x = (x^1, x^2, \cdots, x^n)$ be the Fermi coordinate system along the geodesic $\gamma_0(t)$ and $v = (v^1, v^2, \cdots, v^n)$ be the fiber coordinates of $TM \to M$ naturally associated to x. We abbreviate the partial derivatives as follows:

$$\partial_i = \frac{\partial}{\partial x^i}, \partial^2_{ij} = \frac{\partial^2}{\partial x^i \partial x^j}, D_i = \frac{\partial}{\partial v^i}, D^2_{ij} = \frac{\partial^2}{\partial v^i \partial v^j}.$$

For each $m \in M, \nu \in I(m)$ with m in the domain of the Fermi coordinate system x, we set:

$$X = X(x,v,t) = (X^1(x,v,t), X^2(x,v,t), \cdots, X^n(x,v,t)) = x(\exp_m(t\nu)),$$

where $x = x(m)$ and $\nu = v^i\partial_i$.

As $t \mapsto \exp_m(t\nu)$ is a geodesic, thus the n-tuple $X = X(x,v,t)$ is the solution of the following Cauchy problem:

$$\begin{cases} \ddot X^i + \Gamma^i_{jk}(X)\dot X^j \dot X^k = 0, \\ X^i(x,v,0) = x^i, \dot X^i(x,v,0) = v^i. \end{cases} \tag{2.1}$$

In the sequel, the dot will stand for the derivative with respect to t and the prime for the derivative with respect to θ. We will say that a constant is under control whenever it depends only on the dimension n. Given two real function $f(t)$ and $h(t)$, we write $f(t) = \mathcal{B}(h(t))$ if there exists a positive constant C under control such that $|f(t)| \leq C|h(t)|$ for all t in a given range. The third derivative of $f(t)$ with respect to t will be denoted by $\dddot f(t)$.

2.1 The geodesic motion

In this subsection, we examine the geodesic motion on the axis. Let $X(\theta,t) = (X^1(\theta,t), X^2(\theta,t), \cdots, X^n(\theta,t))$ denotes the coordinate of the geodesic $\gamma_\theta(t)$ in the

Fermi coordinate system x, i.e.

$$X^i(\theta,t) = X^i(0,(\cos\theta,\sin\theta,0,\cdots,0),t).$$

The geodesic motion is stated as follows.

Lemma 2.1. *Let (M,g) be a closed Riemannian manifold of dimension $n\geq 2$. Assume that (M,g) satisfies (1.3) and (1.4). Fix $m_0\in M$ and $\nu_0\in I(m_0)\backslash\{0\}$. In the Fermi coordinate system x, for any $t\in[0,\tau]$, the geodesic motion X on the axis satisfies:*

1) $X'(0,t) = (0,\sin t+\mathcal{B}(\varepsilon t^3),\mathcal{B}(\varepsilon t^3),\cdots,\mathcal{B}(\varepsilon t^3))^T,$
$\dot{X}'(0,t) = (0,\cos t+\mathcal{B}(\varepsilon t^2),\mathcal{B}(\varepsilon t^2),\cdots,\mathcal{B}(\varepsilon t^2))^T,$
$\ddot{X}'(0,t) = (0,-\sin t+\mathcal{B}(\varepsilon t),\mathcal{B}(\varepsilon t),\cdots,\mathcal{B}(\varepsilon t))^T;$

2) $X''(0,t) = (-\sin t\cos t+\mathcal{B}(\varepsilon t^3),\mathcal{B}(\varepsilon t^3),\cdots,\mathcal{B}(\varepsilon t^3))^T,$
$\dot{X}''(0,t) = (-\cos(2t)+\mathcal{B}(\varepsilon t^2),\mathcal{B}(\varepsilon t^2),\cdots,\mathcal{B}(\varepsilon t^2))^T,$
$\ddot{X}''(0,t) = (4\sin t\cos t+\mathcal{B}(\varepsilon t),\mathcal{B}(\varepsilon t),\cdots,\mathcal{B}(\varepsilon t))^T;$

3) $\dddot{X}'(0,0) = (0,-R^2_{121}(0),\cdots,-R^n_{121}(0))^T,$
$\dddot{X}''(0,0) = (-4R^1_{221}(0),0,-4R^3_{221}(0),\cdots,-4R^n_{221}(0))^T.$

Proof. Under the above assumptions, from (2.1), we know that $X(\theta,t)$ is the solution of the following Cauchy problem:

$$\begin{cases}\ddot{X}^i+\Gamma^i_{jk}(X)\dot{X}^j\dot{X}^k=0,\\ X(\theta,0)=0,\dot{X}(\theta,0)=(\cos\theta,\sin\theta,0,\cdots,0)^T.\end{cases} \tag{2.2}$$

On the axis, since the Christoffel symbols vanish, we have

$$X(0,t)=(t,0,\cdots,0)^T. \tag{2.3}$$

We will settle Lemma 2.1 from 1) to 3) term by term.
1). We first handle the term $\ddot{X}'$. Differentiating (2.2) with respect to θ, it follows that

$$\ddot{X}^{i'}+\partial_p\Gamma^i_{jk}(X)\dot{X}^j\dot{X}^kX^{p'}+2\Gamma^i_{jk}(X)\dot{X}^{j'}\dot{X}^k=0,$$

with the initial condition

$$X^{i'}(0,0)=0,\dot{X}^{i'}(0,0)=\delta^i_2.$$

Evaluated on the axis, from (1.15),(2.3) and (1.12), we get the following equations

$$\begin{cases}\ddot{X}^{i'}+R^i_{1\alpha1}(X)X^{\alpha'}=0,\\ X'(0,0)=0,\dot{X}'(0,0)=(0,1,0,\cdots,0)^T.\end{cases} \tag{2.4}$$

It is clear that $X^{1'}(0,t)\equiv 0$.
For $i>1$, we first establish the following standard estimate.

Claim 2.1. *For any $t\in[0,\tau]\subset[0,\pi]$,*

$$\max\{|X'|,|\dot{X}'|\}\leq e^{\frac{\pi}{2}}.$$

Proof of Claim 2.1. Let $f_1=|X'|^2+|\dot{X}'|^2$ with $f_1(0)=1$. The derivative of f_1 takes the form

$$\dot{f}_1=2(\bar{R}^\beta_{1\alpha1}(X)-R^\beta_{1\alpha1}(X))X^{\alpha'}\dot{X}^{\beta'},$$

where the term $\bar{R}^\beta_{1\alpha1}$ is defined by (1.2).
Suppose that $\varepsilon\leq 1$, using the Cauchy-Schwarz inequality, we derive that

$$\dot{f}_1\leq f_1.$$

Finally

$$f_1 \le e^t \le e^\pi.$$

This completes the proof of Claim 2.1.

Before proceeding, we define $\overline{X}'(t) = (\overline{X}^{1'}(t), \cdots, \overline{X}^{n'}(t))$ as follows

$$\begin{cases} \ddot{\overline{X}}^{i'} + \bar{R}^i_{1\alpha 1}(\overline{X})\overline{X}^{\alpha'} = 0, \\ \overline{X}'(0,0) = 0, \dot{\overline{X}}'(0,0) = (0,1,0,\cdots,0)^T. \end{cases}$$

Let $\mathcal{E}^i$ be the difference $X^{i'} - \overline{X}^{i'}$. Let us rewrite the equation (2.4) in the perturbative form:

$$\ddot{\mathcal{E}}^i + \mathcal{E}^i = (\bar{R}^i_{1\alpha 1} - R^i_{1\alpha 1})X^{\alpha'},$$

with the null initial conditions:

$$\mathcal{E}^i(0) = \dot{\mathcal{E}}^i(0) = 0.$$

Applying the representation formula of scalar function (1.10) to $\mathcal{E}^i$, it follows that

$$\mathcal{E}^i = \sin t \int_0^t (\bar{R}^i_{1\alpha 1} - R^i_{1\alpha 1})X^{\alpha'} \cos s ds - \cos t \int_0^t (\bar{R}^i_{1\alpha 1} - R^i_{1\alpha 1})X^{\alpha'} \sin s ds.$$

From the inequality $\sin t \le t$ in $[0,\tau)$, the Cauchy-Schwarz inequality and Claim 2.1, we infer that $|\mathcal{E}^i| \le \frac{3}{2}e^{\frac{\pi}{2}}\varepsilon t^2$. Precisely,

$$|X^{2'} - \sin t| \le \frac{3}{2}\varepsilon t^2 e^{\frac{\pi}{2}} \le \frac{3}{2}t^2 e^{\frac{\pi}{2}},$$
$$|X^{j'}| \le \frac{3}{2}\varepsilon t^2 e^{\frac{\pi}{2}} \le \frac{3}{2}t^2 e^{\frac{\pi}{2}}, 3 \le j \le n.$$

Keep in mind that $\ddot{\mathcal{E}}^i = -\mathcal{E}^i + (\bar{R}^i_{1\alpha 1} - R^i_{1\alpha 1})X^{\alpha'}$, by the Cauchy-Schwarz inequality and the inequality $\sin t \le t$ in $[0,\tau)$ again, we conclude

$$|\ddot{\mathcal{E}}^i| \le |\mathcal{E}^i| + \varepsilon|X'| \le \left(\sqrt{\frac{9}{4}\pi^2 e^\pi (n-1) + 3\pi e^{\frac{\pi}{2}} + 1} + \frac{3}{2}\pi e^{\frac{\pi}{2}}\right)\varepsilon t,$$

as desired.

By integrating $\mathcal{E}^i$ with respect to t, we derive the expression for X' and $\dot{X}'$,i.e.

$$\dot{\mathcal{E}}(t) = \int_0^t \ddot{\mathcal{E}}(s)ds = \mathcal{B}(\varepsilon t^2),$$
$$\mathcal{E}(t) = \int_0^t \dot{\mathcal{E}}(s)ds = \mathcal{B}(\varepsilon t^3).$$

2). As similar as 1), we first deal with $\ddot{X}''$. Differentiating the equation (2.2) twice with respect to θ :

$$\ddot{X}^{i''} + \partial_p\Gamma^i_{jk}\dot{X}^j\dot{X}^k X^{p''} + \partial^2_{pq}\Gamma^i_{jk}\dot{X}^j\dot{X}^k X^{p'}X^{q'} +$$
$$4\partial_p\Gamma^i_{jk}\dot{X}^{j'}\dot{X}^k X^{p'} + 2\Gamma^i_{jk}(\dot{X}^{j''}\dot{X}^k + \dot{X}^{j'}\dot{X}^{k'}) = 0,$$

with the initial condition

$$X^{i''}(0,0) = 0, \dot{X}^{i''}(0,0) = -\delta^i_1.$$

Evaluating on the axis, from (1.15),(2.3),(1.16) and (1.12), we get the following equations

$$\begin{cases} \ddot{X}^{i''} + R^i_{1\alpha 1}X^{\alpha''} + (\nabla_\alpha R^i_{1\beta 1} + \nabla_1 R^i_{\beta\alpha 1})X^{\alpha'}X^{\beta'} + 4R^i_{\beta\alpha 1}X^{\alpha'}\dot{X}^{\beta'} = 0, \\ X''(0,0) = 0, \dot{X}''(0,0) = (-1,0,\cdots,0)^T. \end{cases}$$

Similarly, we need the following Claim.

Claim 2.2. *There exists a positive constant C under control such that, for any $t \in [0,\tau] \subset [0,\pi]$,*

$$\max\{|X''|, |\dot{X}''|\} \leq 2e^{\frac{\pi}{2}C}.$$

Proof of Claim 2.2. Let $f_2(t) = |X''|^2 + |\dot{X}''|^2$. Then

$$\begin{aligned}
\dot{f}_2(t) &= 2X^{i''}\dot{X}^{i''} + 2\dot{X}^{i''}\ddot{X}^{i''} \\
&= 2X^{i''}\dot{X}^{i''} - 2R^i_{1\alpha1}X^{\alpha''}\dot{X}^{i''} - 2[(\nabla_\alpha R^i_{1\beta1} + \\
&\quad \nabla_1 R^i_{\beta\alpha1})X^{\alpha'}X^{\beta'} + 4R^i_{\beta\alpha1}X^{\alpha'}\dot{X}^{\beta'}]\dot{X}^{i''} \\
&\leq Cf_2(t) + C.
\end{aligned}$$

We thus conclude that

$$f_2(t) \leq 2e^{Ct} \leq 2e^{C\pi}.$$

This ends the proof of Claim 2.2.

We go back to the proof of Lemma 2.1.
If $i = 1$, let $f_3 = X^{1''} + \sin t\cos t$, then

$$\begin{aligned}
\ddot{f}_3 &= -\nabla_1 R^1_{\beta\alpha1}X^{\alpha'}X^{\beta'} + 4(\bar{R}^1_{\beta\alpha1} - R^1_{\beta\alpha1})X^{\alpha'}\dot{X}^{\beta'} - \\
&\quad 4\bar{R}^1_{\beta\alpha1}X^{\alpha'}\dot{X}^{\beta'} - 4\sin t\cos t \\
&= 4X^{\alpha'}\dot{X}^{\alpha'} - 4\sin t\cos t + \mathcal{B}(\varepsilon t) \\
&= 4X^{2'}\dot{X}^{2'} - 4\sin t\cos t + \mathcal{B}(\varepsilon t) \\
&= 4(X^{2'} - \sin t)\dot{X}^{2'} + 4(\dot{X}^{2'} - \cos t)\sin t + \mathcal{B}(\varepsilon t) \\
&= \mathcal{B}(\varepsilon t).
\end{aligned}$$

If $i > 1$, the term $X^{i''}$ satisfies the following equation

$$\begin{aligned}
\ddot{X}^{i''} + X^{i''} &= (\bar{R}^i_{1\alpha1} - R^i_{1\alpha1})X^{\alpha''} - (\nabla_\alpha R^i_{1\beta1} + \\
&\quad \nabla_1 R^i_{\beta\alpha1})X^{\alpha'}X^{\beta'} - 4R^i_{\beta\alpha1}X^{\alpha'}\dot{X}^{\beta'},
\end{aligned}$$

with the homogenuous initial condition

$$X^{i''}(0,0) = \dot{X}^{i''}(0,0) = 0. \tag{2.5}$$

Using the representation formula of scalar function (1.10) to $X^{i''}$, we see that

$$\begin{aligned}
X^{i''} &= \sin t\int_0^t [(\bar{R}^i_{1\alpha1} - R^i_{1\alpha1})X^{\alpha''} - (\nabla_\alpha R^i_{1\beta1} + \nabla_1 R^i_{\beta\alpha1})X^{\alpha'}X^{\beta'} - \\
&\quad 4R^i_{\beta\alpha1}X^{\alpha'}\dot{X}^{\beta'}]\cos s ds - \cos t\int_0^t [(\bar{R}^i_{1\alpha1} - R^i_{1\alpha1})X^{\alpha''} - \\
&\quad (\nabla_\alpha R^i_{1\beta1} + \nabla_1 R^i_{\beta\alpha1})X^{\alpha'}X^{\beta'} - 4R^i_{\beta\alpha1}X^{\alpha'}\dot{X}^{\beta'}]\sin s ds.
\end{aligned}$$

By $\bar{R}^i_{1\alpha1} - R^i_{1\alpha1} = \mathcal{B}(\varepsilon), \nabla_\alpha R^i_{1\beta1} = \mathcal{B}(\varepsilon), \nabla_1 R^i_{\beta\alpha1} = \mathcal{B}(\varepsilon), R^i_{\beta\alpha1} = \mathcal{B}(\varepsilon)$, Claim 2.1 and Claim 2.2, we have

$$X^{i''} = \mathcal{B}(\varepsilon t).$$

Arguing as above, we have

$$\ddot{X}''(0,t) - (4\sin t\cos t, 0, \cdots, 0)^T = \mathcal{B}(\varepsilon t).$$

By integrating, together with the initial condition (2.5), we obtain

$$\begin{aligned}
\dot{X}''(0,t) - (-\cos(2t), 0, \cdots, 0)^T &= \mathcal{B}(\varepsilon t^2), \\
X''(0,t) - (-\sin t\cos t, 0, \cdots, 0)^T &= \mathcal{B}(\varepsilon t^3).
\end{aligned}$$

3). The results are derived by tedious computation. We include the details in the following. Differentiating (2.2) once with respect to θ and t respectively:

$$\dddot{X}^{i'} + \partial^2_{pq}\Gamma^i_{jk}\dot{X}^j\dot{X}^k\dot{X}^p X^{q'} + \partial_p\Gamma^i_{jk}(\ddot{X}^j\dot{X}^k X^{p'} +$$
$$\dot{X}^j\ddot{X}^k X^{p'} + \dot{X}^j\dot{X}^k\dot{X}^{p'}) + 2\partial_p\Gamma^i_{jk}\dot{X}^{j'}\dot{X}^k\dot{X}^p +$$
$$2\Gamma^i_{jk}(\ddot{X}^{j'}\dot{X}^k + \dot{X}^{j'}\ddot{X}^k) = 0.$$

Evaluating at the origin, and combining with the following relations $X'(0,0) = 0, \dot{X}(0,0) = (1,0,\cdots,0)^T$, $\dot{X}'(0,0) = (0,1,0,\cdots,0)^T$, (1.15) and $\partial_1\Gamma^i_{jk}(0) = \Gamma^i_{jk}(0) = 0$, we conclude

$$\dddot{X}^{i'}(0,0) = -R^i_{121}(0).$$

Differentiating the equation (2.2) twice with respect to θ and once with respect to t respectively, we have

$$\dddot{X}^{i''} + \partial^2_{pq}\Gamma^i_{jk}\dot{X}^j\dot{X}^k\dot{X}^p X^{q''} +$$
$$\partial_p\Gamma^i_{jk}(2\ddot{X}^j\dot{X}^k X^{p''} + \dot{X}^j\dot{X}^k\dot{X}^{p''}) +$$
$$\partial^3_{pql}\Gamma^i_{jk}\dot{X}^j\dot{X}^k X^{p'}X^{q'}\dot{X}^l +$$
$$2\partial^2_{pq}\Gamma^i_{jk}(\ddot{X}^j\dot{X}^k X^{p'}X^{q'} + \dot{X}^j\dot{X}^k\dot{X}^{p'}X^{q'}) +$$
$$4\partial^2_{pq}\Gamma^i_{jk}\dot{X}^{j'}\dot{X}^k\dot{X}^p X^{q'} + 4\partial_p\Gamma^i_{jk}(\ddot{X}^{j'}\dot{X}^k X^{p'} +$$
$$\dot{X}^{j'}\ddot{X}^k X^{p'} + \dot{X}^{j'}\dot{X}^k\dot{X}^{p'}) +$$
$$2\partial_p\Gamma^i_{jk}(\dot{X}^{j''}\dot{X}^k + \dot{X}^{j'}\dot{X}^{k'})\dot{X}^p +$$
$$2\Gamma^i_{jk}(\ddot{X}^{j''}\dot{X}^k + \dot{X}^{j''}\ddot{X}^k + 2\ddot{X}^{j'}\dot{X}^{k'}) = 0.$$

Evaluating at the origin, and combining with the following relations $X''(0,0) = 0, \dot{X}''(0,0) = (-1,0\cdots,0)^T, \partial_1\Gamma^i_{jk}(0) = 0, X'(0,0) = 0, \dot{X}'(0,0) = (0,1,0\cdots,0)^T, \dot{X}(0,0) = (1,0,\cdots,0)^T$, (1.15) and $\Gamma^i_{jk}(0) = 0$, we have

$$\dddot{X}^{i'}(0,0) = -4R^i_{221}(0).$$

The Lemma 2.1 is proved. □

2.2 The orthonormal chart motion

To proceed, we study the orthonormal chart motion.

Along the geodesic $\gamma_\theta(t)$, there are two charts:the natural chart $\{\frac{\partial}{\partial x^i}\}$ and the orthonormal chart $\{e_1,\cdots,e_2\}$. To differentiate the curvature matrix (1.8), we need the coordinate of $\{e_1,\cdots,e_2\}$. Set $(Y^j_i(\theta,t))$ for the coordinates of the orthonormal chart $\{e_1,\cdots,e_2\}$, i.e.

$$e_i(\theta,t) = Y^j_i(\theta,t)\partial_j.$$

It is clear that $Y^i_1(\theta,t) = \dot{X}^i(\theta,t)$.

Since the orthonormal moving chart $\{e_1,\cdots,e_n\}$ is parallel, we have the equation:

$$\dot{Y}^i_j + \Gamma^i_{kl}(X)\dot{X}^l Y^k_j = 0, \tag{2.6}$$

with the initial condition

$$Y(\theta,0) = \begin{bmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & I_{n-2} \end{bmatrix}.$$

It is useful to note that $Y^i_1 = \dot{X}^i$. Moreover, it is obvious that $Y(0,t) = I_n$.

The orthonormal chart motion is presented as follows.

Lemma 2.2. *Let (M,g) be a closed Riemannian manifold of dimension $n \geq 2$. Assume that (M,g) satisfies (1.3) and (1.4). Fix $m_0 \in M$ and $\nu_0 \in I(m_0)\backslash\{0\}$. In the Fermi coordinate system x, for any $t \in [0,\tau]$, the derivatives of the orthonormal chart motion along the axis satisfy:*

1) $Y_j^{i'}(0,t) = -(\delta_1^i\delta_j^2 - \delta_2^i\delta_j^1)\cos t + \mathcal{B}(\varepsilon t^2)$,
$\dot{Y}_j^{i'}(0,t) = (\delta_1^i\delta_j^2 - \delta_2^i\delta_j^1)\sin t + \mathcal{B}(\varepsilon t)$;

2) $Y_j^{i''}(0,t) = -\delta_1^i\delta_j^1\cos(2t) - \delta_2^i\delta_j^2\cos^2 t + \frac{1}{3}(\delta_j^i - \delta_1^i\delta_j^1 - \delta_2^i\delta_j^2)\sin^2 t + \mathcal{B}(\varepsilon t^2)$,
$\dot{Y}_j^{i''}(0,t) = 4\delta_1^i\delta_j^1\sin t\cos t + 2\delta_2^i\delta_j^2\sin t\cos t + \frac{2}{3}(\delta_j^i - \delta_1^i\delta_j^1 - \delta_2^i\delta_j^2)\sin t\cos t + \mathcal{B}(\varepsilon t)$;

3) $\ddot{Y}_j^{i'}(0,0) = -R_{j21}^i(0)$,
$\ddot{Y}_j^{i''}(0,0) = 2\delta_j^2 R_{121}^i(0) - \frac{2}{3}(1+5\delta_j^1)R_{22j}^i(0)$.

Proof. We will settle Lemma 2.2 from 1) to 3) term by term.
1). Similar to the proof of Lemma 2.1, we first consider $\dot{Y}'$. Differentiating the equation (2.6) with respect to θ :

$$\dot{Y}_j^{i'} + \partial_p\Gamma_{kl}^i\dot{X}^l X^{p'} Y_j^k + \Gamma_{kl}^i(\dot{X}^{l'}Y_j^k + \dot{X}^l Y_j^{k'}) = 0,$$

Evaluating on the axis, and replacing $\dot{X}, Y, \Gamma_{kl}^i, \partial_p\Gamma_{k1}^i$ by the related values, we get

$$\dot{Y}_j^{i'} + R_{j\alpha 1}^i X^{\alpha'} = 0,$$

with initial condition

$$Y_j^{i'}(0,0) = \delta_2^i\delta_j^1 - \delta_1^i\delta_j^2. \tag{2.7}$$

Then

$$\begin{aligned}\dot{Y}_j^{i'}(0,t) &= -R_{j\alpha 1}^i X^{\alpha'} \\ &= (\bar{R}_{j\alpha 1}^i - R_{j\alpha 1}^i)X^{\alpha'} - \bar{R}_{j\alpha 1}^i X^{\alpha'}.\end{aligned}$$

Lemma 2.1 yields,

$$\begin{aligned}\dot{Y}_j^{i'}(0,t) &= -\bar{R}_{j\alpha 1}^i X^{\alpha'} + \mathcal{B}(\varepsilon t) \\ &= -\bar{R}_{j21}^i X^{2'} + \mathcal{B}(\varepsilon t) \\ &= (\delta_1^i\delta_j^2 - \delta_2^i\delta_j^1)\sin t + \mathcal{B}(\varepsilon t).\end{aligned}$$

Integrating with respect to t, together with the initial condition (2.7), we derive

$$\begin{aligned}Y_j^{i'}(0,t) &= Y_j^{i'}(0,0) + \int_0^t \dot{Y}_j^{i'}(0,s)ds \\ &= \delta_2^i\delta_j^1 - \delta_1^i\delta_j^2 + (\delta_1^i\delta_j^2 - \delta_2^i\delta_j^1)\int_0^t \sin s ds + \mathcal{B}(\varepsilon t^2) \\ &= (\delta_2^i\delta_j^1 - \delta_1^i\delta_j^2)\cos t + \mathcal{B}(\varepsilon t^2).\end{aligned}$$

2). Similarly, we take account of $\dot{Y}''$. Differentiating equation (2.6) twice with respect to the parameter θ :

$$\begin{aligned}&\dot{Y}_j^{i''} + \partial_{pq}^2\Gamma_{kl}^i\dot{X}^l X^{p'}X^{q'}Y_j^k + 2\partial_p\Gamma_{kl}^i\dot{X}^{l'}X^{p'}Y_j^k + \\ &\partial_p\Gamma_{kl}^i\dot{X}^l X^{p''}Y_j^k + 2\partial_p\Gamma_{kl}^i\dot{X}^l X^{p'}Y_j^{k'} + \\ &\Gamma_{kl}^i(\dot{X}^{l''}Y_j^k + 2\dot{X}^{l'}Y_j^{k'} + \dot{X}^l Y_j^{k''}) = 0,\end{aligned}$$

with the initial condition

$$Y_j^{i''}(0,0) = -\delta_1^i\delta_j^1 - \delta_2^i\delta_j^2. \tag{2.8}$$

On the axis, replacing $\dot{X}, Y, \Gamma_{kl}^i, \partial_p\Gamma_{k1}^i$ by the corresponding values, the above equation reduces to

$$\begin{aligned}&\dot{Y}_j^{i''} + \partial_{\alpha\beta}^2\Gamma_{1j}^i X^{\alpha'}X^{\beta'} + 2\partial_\alpha\Gamma_{\beta j}^i X^{\alpha'}\dot{X}^{\beta'} + \\ &R_{j\alpha 1}^i X^{\alpha''} + 2R_{k\alpha 1}^i X^{\alpha'}Y_j^{k'} = 0,\end{aligned}$$

As a result, the term $\dot{Y}_j^{i''}$ has the form

$$\begin{aligned}\dot{Y}_j^{i''} &= -\partial_{\alpha\beta}^2\Gamma_{1j}^i X^{\alpha'}X^{\beta'} - 2\partial_\alpha\Gamma_{\beta j}^i X^{\alpha'}\dot{X}^{\beta'} - \\ &\quad R_{j\alpha 1}^i X^{\alpha''} - 2R_{k\alpha 1}^i X^{\alpha'}Y_j^{k'}.\end{aligned}$$

Since the result follows from Lemma 2.1 if j is equal to 1, thus it is sufficient to assume that $j > 1$.

From (1.17), Lemma 2.1 and (1.15), it follows that

$$\begin{aligned}\dot{Y}_j^{i''} &= -\frac{2}{3}(R_{\beta\alpha j}^i + R_{j\alpha\beta}^i)X^{\alpha'}\dot{X}^{\beta'} - 2R_{k\alpha 1}^i X^{\alpha'}Y_j^{k'} + \mathcal{B}(\varepsilon t) \\ &= -\frac{2}{3}(\bar{R}_{\beta\alpha j}^i + \bar{R}_{j\alpha\beta}^i)X^{\alpha'}\dot{X}^{\beta'} - 2\bar{R}_{k\alpha 1}^i X^{\alpha'}Y_j^{k'} + \mathcal{B}(\varepsilon t) \\ &= -\frac{2}{3}(\bar{R}_{22j}^i + \bar{R}_{j22}^i)X^{2'}\dot{X}^{2'} - 2\delta_j^2\bar{R}_{121}^i X^{2'}Y_2^{1'} + \mathcal{B}(\varepsilon t) \\ &= -\frac{2}{3}\bar{R}_{22j}^i \sin t\cos t + 2\delta_j^2\bar{R}_{121}^i \sin t\cos t + \mathcal{B}(\varepsilon t) \\ &= \frac{2}{3}(\delta_j^i - \delta_2^i\delta_j^2)\sin t\cos t + 2\delta_2^i\delta_j^2 \sin t\cos t + \mathcal{B}(\varepsilon t).\end{aligned}$$

Integrating with respect to t, together with the initial condition (2.8), we have

$$\begin{aligned}Y_j^{i''}(0,t) &= Y_j^{i''}(0,0) - \int_0^t \dot{Y}_j^{i''}(0,s)ds \\ &= -\delta_2^i\delta_j^2 + 2\delta_2^i\delta_j^2\int_0^t \sin s\cos s ds + \\ &\quad \frac{2}{3}(\delta_j^i - \delta_2^i\delta_j^2)\int_0^t \sin s\cos s ds + \mathcal{B}(\varepsilon t^2) \\ &= -\delta_2^i\delta_j^2 + \delta_2^i\delta_j^2\sin^2 t + \frac{1}{3}(\delta_j^i - \delta_2^i\delta_j^2)\sin^2 t + \mathcal{B}(\varepsilon t^2) \\ &= -\delta_2^i\delta_j^2\cos^2 t + \frac{1}{3}(\delta_j^i - \delta_2^i\delta_j^2)\sin^2 t + \mathcal{B}(\varepsilon t^2).\end{aligned}$$

3). Differentiating equation (2.6) one with respect to t and θ respectively:

$$\begin{aligned}&\ddot{Y}_j^{i'} + \partial_{pq}^2\Gamma_{kl}^i\dot{X}^l\dot{X}^p X^{q'}Y_j^k + \partial_p\Gamma_{kl}^i(\ddot{X}^l X^{p'}Y_j^k + \\ &\dot{X}^l\dot{X}^{p'}Y_j^k + \dot{X}^l X^{p'}\dot{Y}_j^k) + \partial_p\Gamma_{kl}^i(\dot{X}^{l'}Y_j^k + \\ &\dot{X}^l Y_j^{k'})\dot{X}^p + \Gamma_{kl}^i(\ddot{X}^{l'}Y_j^k + \dot{X}^{l'}\dot{Y}_j^k + \\ &\ddot{X}^l Y_j^{k'} + \dot{X}^l\dot{Y}_j^{k'}) = 0.\end{aligned}$$

Evaluating at the origin, together with the following relations $X'(0,0) = 0, \dot{X}(0,0) = (1,0\cdots,0)^T$, $\dot{X}'(0,0) = (0,1,0\cdots,0)^T, Y(0,0) = I_n$, (1.15) and $\Gamma_{kl}^i(0) = \partial_1\Gamma_{kl}^i(0) = 0$, we see that

$$\ddot{Y}_j^{i'}(0,0) = -R_{j21}^i(0).$$

Differentiating equation (2.6) twice with respect to θ and once with respect to t respectively:

$$\begin{aligned}&\ddot{Y}_j^{i''}+2\partial^2_{pq}\Gamma^i_{kl}\dot{X}^l\dot{X}^pX^{q'}Y_j^{k'}+2\partial_p\Gamma^i_{kl}(\ddot{X}^lX^{p'}Y_j^{k'}+\dot{X}^l\dot{X}^{p'}Y_j^{k'}+\\&\dot{X}^lX^{p'}\dot{Y}_j^{k'})+\partial^2_{pq}\Gamma^i_{kl}\dot{X}^l\dot{X}^pX^{q''}Y^k_j+\partial_p\Gamma^i_{kl}(\ddot{X}^lX^{p''}Y^k_j+\dot{X}^l\dot{X}^{p''}Y^k_j+\\&\dot{X}^lX^{p''}\dot{Y}^k_j)+2\partial^2_{pq}\Gamma^i_{kl}\dot{X}^{l'}\dot{X}^pX^{q'}Y^k_j+2\partial_p\Gamma^i_{kl}(\ddot{X}^{l'}X^{p'}Y^k_j+\dot{X}^{l'}\dot{X}^{p'}Y^k_j+\\&\dot{X}^{l'}X^{p'}\dot{Y}^k_j)+\partial^3_{apq}\Gamma^i_{kl}\dot{X}^l\dot{X}^aX^{p'}X^{q'}Y^k_j+\partial^2_{pq}\Gamma^i_{kl}(\ddot{X}^lX^{p'}X^{q'}Y^k_j+\\&\dot{X}^l\dot{X}^{p'}X^{q'}Y^k_j+\dot{X}^lX^{p'}\dot{X}^{q'}Y^k_j+\dot{X}^lX^{p'}X^{q'}\dot{Y}^k_j)+\partial_p\Gamma^i_{kl}(\dot{X}^{l''}Y^k_j+\\&2\dot{X}^{l'}Y_j^{k'}+\dot{X}^lY_j^{k''})\dot{X}^p+\Gamma^i_{kl}(\ddot{X}^{l''}Y^k_j+\dot{X}^{l''}\dot{Y}^k_j+2\ddot{X}^{l'}Y_j^{k'}+\\&2\dot{X}^{l'}\dot{Y}_j^{k'}+\ddot{X}^lY_j^{k''}+\dot{X}^l\dot{Y}_j^{k''})=0,\end{aligned}$$

Evaluating at the origin, combining (20) with the relations $\dot{X}(0,0)=(1,0\cdots,0)^T$, $X'(0,0)=X''(0)=0,\dot{X}'(0,0)=(0,1,0\cdots,0)^T,\dot{X}''(0,0)=(-1,0\cdots,0)^T$, $Y(0,0)=I_n$ and $\Gamma^k_{ij}(0)=\partial_1\Gamma^k_{ij}(0)=0$, we get

$$\begin{aligned}\ddot{Y}_j^{i''}(0,0)&=-2R^i_{k21}(0)Y_j^{k'}(0)-2\partial_2\Gamma^i_{2j}(0)\\&=-2\delta^1_jR^i_{221}(0)+2\delta^2_jR^i_{121}(0)-\\&\quad 2\delta^1_jR^i_{221}(0)-\frac{2}{3}(1-\delta^1_j)R^i_{22j}(0)\\&=-\frac{10}{3}\delta^1_jR^i_{221}+2\delta^2_jR^i_{121}(0)-\frac{2}{3}R^i_{22j}(0)\\&=2\delta^2_jR^i_{121}(0)-\frac{2}{3}(1+5\delta^1_j)R^i_{22j}(0).\end{aligned}$$

This completes the proof of Lemma 2.2. □

2.3 The behaviour of the curvature matrix

In this section, we take account of the behaviour of the curvature matrix. Combining (1.8) with $\gamma_\theta(t)=e_1(\theta,t)$, by the anti-symmetry of the Riemann curvature tensor, we see that $R_{ij}(\theta,t)=0$, if $i=1$ or $j=1$. Thus all order partial derivatives of R_{1j} and R_{i1} with respect to θ or t vanish identically. Without loss of generality, we assume that $i,j>1$ in this section. In the following, the Riemann curvature tensor are evaluated at $\gamma_0(t)$, otherwise specifically.

The behaviour of curvature matrix is illustrated as follows.

Proposition 2.1. *Let (M,g) be a closed Riemannian manifold of dimension $n\geq 2$. Assume that (M,g) satisfies the curvature assumptions (1.3) and (1.4). Fix $m_0\in M$ and $\nu_0\in I(m_0)\backslash\{0\}$. In the Fermi coordinate system, for any $t\in[0,\tau]$ and $i,j>1$, we have*

a) $\dot{R}_{ij}(0,t)=\nabla_1R_{1i1j}$,
$R'_{ij}(0,t)=(R_{1i2j}+R_{1j2i})\cos t+\nabla_2R_{1i1j}\sin t+\mathcal{B}(\varepsilon t^2)$;

b) $\ddot{R}_{ij}(0,t)=\nabla^2_{11}R_{1i1j}$,
$\dot{R}'_{ij}(0,t)=(\nabla_1R_{1i2j}+\nabla_1R_{1j2i}+\nabla_2R_{1i1j})\cos t+\mathcal{B}(\varepsilon t)$,
$R''_{ij}(0,t)=[2R_{2i2j}+(\delta_{2i}+\delta_{2j}-2)R_{1i1j}]\cos^2 t+$
$(-\nabla_1R_{1i1j}+2\nabla_2R_{1i2j}+2\nabla_2R_{1j2i})\sin t\cos t+\mathcal{B}(\varepsilon t^2)$;

c) $\dot{R}''_{ij}(0,0)=2\nabla_1R_{2i2j}(0)+(\delta_{2i}+\delta_{2j}-3)\nabla_1R_{1i1j}(0)+$
$2(\nabla_2R_{1i2j}(0)+\nabla_2R_{1j2i}(0))$,
$\ddot{R}'_{ij}(0,0)=\mathcal{B}(\varepsilon)$,
$\ddot{R}''_{ij}(0,0)=\mathcal{B}(\varepsilon)$.

Remark 2.1. *(1).We will need the initial value of $\dot{R}, R', \dot{R}'$ and R''. Based on a) and b), we know that*

$$\dot{R}(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & \nabla_1 R_{1i1j}(0) \end{bmatrix},$$

$$R'(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & R_{1i2j}(0) + R_{1j2i}(0) \end{bmatrix},$$

$$\dot{R}'(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & \nabla_1 R_{1i2j}(0) + \nabla_1 R_{1j2i}(0) + \nabla_2 R_{1i1j}(0) \end{bmatrix},$$

$$R''(0,0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -R_{121j}(0) \\ 0 & -R_{1i12}(0) & 2R_{2i2j}(0) - 2R_{1i1j}(0) \end{bmatrix};$$

(2).From a) and b), we know that $\dot{R}, R', \ddot{R}, \dot{R}'$ and R'' are all globally small enough,i.e. there exists a positive constant C under control such that

$$\forall t \in [0,\pi), \ \ \max\{|\dot{R}|, |R'|, |\ddot{R}|, |\dot{R}'|, |R''|\} \leq C\varepsilon.$$

Proof. Fix $i, j > 1$. To differentiate R_{ij}, we need to rewrite the expression (1.8) in the Fermi coordinate system. Recall that

$$e_1(\theta,t) = \dot{X}^j(\theta,t)\partial_j, e_i(\theta,t) = Y_i^j(\theta,t)\partial_j.$$

From (1.8), we can write

$$\begin{aligned} R_{ij}(\theta,t) &= \langle R(e_i(\theta,t), e_1(\theta,t))e_1(\theta,t), e_j(\theta,t)\rangle \\ &= R_{dcba}(X)\dot{X}^a Y_i^b \dot{X}^c Y_j^d \\ &= R_{abcd}(X)\dot{X}^a Y_i^b \dot{X}^c Y_j^d, \end{aligned} \tag{2.9}$$

where the last equality holds with respect to the symmetry of the Riemann curvature tensor.

a). Differentiating (2.9) with respect to t :

$$\begin{aligned} \dot{R}_{ij}(\theta,t) &= \partial_p R_{abcd}\dot{X}^a Y_i^b \dot{X}^c Y_j^d \dot{X}^p + R_{abcd}\partial_t(\dot{X}^a Y_i^b \dot{X}^c Y_j^d) \\ &= \partial_p R_{abcd}\dot{X}^a Y_i^b \dot{X}^c Y_j^d \dot{X}^p + R_{abcd}(\ddot{X}^a Y_i^b \dot{X}^c Y_j^d + \\ &\quad \dot{X}^a \dot{Y}_i^b \dot{X}^c Y_j^d + \dot{X}^a Y_i^b \ddot{X}^c Y_j^d + \dot{X}^a Y_i^b \dot{X}^c \dot{Y}_j^d). \end{aligned} \tag{2.10}$$

The partial derivative in the first term can be written in terms of the covariant derivative. The definition of the first covariant derivative gives:

$$\begin{aligned} \partial_p R_{abcd} &= \nabla_p R_{abcd} + \Gamma_{pa}^h R_{hbcd} + \Gamma_{pb}^h R_{ahcd} + \\ &\quad \Gamma_{pc}^h R_{abhd} + \Gamma_{pd}^h R_{abch}. \end{aligned}$$

Since the Christoffel symbols vanish identically on the axis, it follows that

$$\partial_p R_{abcd} = \nabla_p R_{abcd}. \tag{2.11}$$

Evaluating on the axis, and substituting (2.11) and $X(0,t) = (t, 0, \cdots, 0)^T, Y(0,t) = I_n$ into (2.10), the result is

$$\dot{R}_{ij}(0,t) = \nabla_1 R_{1i1j}.$$

To finish the proof of a), it remains to estimate R'_{ij}. Differentiating (2.9) with respect to θ :

$$\begin{aligned} R'_{ij}(\theta,t) &= \partial_p R_{abcd}\dot{X}^a Y_i^b \dot{X}^c Y_j^d X^{p'} + R_{abcd}\partial_\theta(\dot{X}^a Y_i^b \dot{X}^c Y_j^d) \\ &= \partial_p R_{abcd}\dot{X}^a Y_i^b \dot{X}^c Y_j^d X^{p'} + R_{abcd}(\dot{X}^{a'} Y_i^b \dot{X}^c Y_j^d + \\ &\quad \dot{X}^a Y_i^{b'} \dot{X}^c Y_j^d + \dot{X}^a Y_i^b \dot{X}^{c'} Y_j^d + \dot{X}^a Y_i^b \dot{X}^c Y_j^{d'}). \end{aligned}$$

Evaluating on the axis, (2.11) and the relations $X(0,t) = (t, 0, \cdots, 0)^T, X^{1'}(0,t) = 0, Y(0,t) = I_n$ infer

$$\begin{aligned} R'_{ij}(0,t) &= \nabla_\alpha R_{1i1j} X^{\alpha'} + (R_{\alpha i1j} + R_{1i\alpha j})\dot{X}^{\alpha'} + \\ &\quad R_{1\alpha 1j} Y_i^{\alpha'} + R_{1i1\alpha} Y_j^{\alpha'}. \end{aligned}$$

Using Lemma 2.1 and Lemma 2.2, we obtain

$$\begin{aligned} R'_{ij}(0,t) &= \nabla_2 R_{1i1j} X^{2'} + (R_{2i1j} + R_{1i2j})\dot{X}^{2'} + \mathcal{B}(\varepsilon t^2) \\ &= \nabla_2 R_{1i1j} \sin t + (R_{1i2j} + R_{1j2i}) \cos t + \mathcal{B}(\varepsilon t^2). \end{aligned}$$

b) Differentiating (2.9) twice with respect to t, we have

$$\begin{aligned} \ddot{R}_{ij}(\theta,t) &= \partial^2_{pq} R_{abcd} \dot{X}^a Y_i^b \dot{X}^c Y_j^d \dot{X}^p \dot{X}^q + \partial_p R_{abcd} \dot{X}^a Y_i^b \dot{X}^c Y_j^d \ddot{X}^p + \\ &\quad 2\partial_p R_{abcd} \partial_t (\dot{X}^a Y_i^b \dot{X}^c Y_j^d) \dot{X}^p + R_{abcd} \partial_t^2 (\dot{X}^a Y_i^b \dot{X}^c Y_j^d) \\ &= \partial^2_{pq} R_{abcd} \dot{X}^a Y_i^b \dot{X}^c Y_j^d \dot{X}^p \dot{X}^q + \partial_p R_{abcd} \dot{X}^a Y_i^b \dot{X}^c Y_j^d \ddot{X}^p + \\ &\quad 2\partial_p R_{abcd} (\ddot{X}^a Y_i^b \dot{X}^c Y_j^d + \dot{X}^a \dot{Y}_i^b \dot{X}^c Y_j^d + \dot{X}^a Y_i^b \ddot{X}^c Y_j^d + \\ &\quad \dot{X}^a Y_i^b \dot{X}^c \dot{Y}_j^d) \dot{X}^p + R_{abcd} (\dddot{X}^a Y_i^b \dot{X}^c Y_j^d + 2\ddot{X}^a \dot{Y}_i^b \dot{X}^c Y_j^d + \\ &\quad 2\ddot{X}^a Y_i^b \ddot{X}^c Y_j^d + 2\ddot{X}^a Y_i^b \dot{X}^c \dot{Y}_j^d + \dot{X}^a \ddot{Y}_i^b \dot{X}^c Y_j^d + \\ &\quad 2\dot{X}^a \dot{Y}_i^b \ddot{X}^c Y_j^d + 2\dot{X}^a \dot{Y}_i^b \dot{X}^c \dot{Y}_j^d + \dot{X}^a Y_i^b \dddot{X}^c Y_j^d + \\ &\quad 2\dot{X}^a Y_i^b \ddot{X}^c \dot{Y}_j^d + \dot{X}^a Y_i^b \dot{X}^c \ddot{Y}_j^d). \end{aligned} \tag{2.12}$$

We write the second partial derivatives in above expression in terms of the related covariant derivatives. By the definition of the second covariant derivatives, we see that

$$\begin{aligned} \partial^2_{pq} R_{abcd} &= \nabla^2_{pq} R_{abcd} + \partial_p \Gamma^h_{qa} R_{hbcd} + \partial_p \Gamma^h_{qb} R_{ahcd} + \partial_p \Gamma^h_{qc} R_{abhd} + \\ &\quad \partial_p \Gamma^h_{qd} R_{abch} + \Gamma^h_{qa} \partial_p R_{hbcd} + \Gamma^h_{qb} \partial_p R_{ahcd} + \Gamma^h_{qc} \partial_p R_{abhd} + \\ &\quad \Gamma^h_{qd} \partial_p R_{abch} + \Gamma^h_{pq} \nabla_h R_{abcd} + \Gamma^h_{pa} \nabla_q R_{hbcd} + \Gamma^h_{pb} \nabla_q R_{ahcd} + \\ &\quad \Gamma^h_{pc} \nabla_q R_{abhd} + \Gamma^h_{pd} \nabla_q R_{abch}. \end{aligned}$$

Evaluating on the axis, since the the Christoffel symbols vanish identically, we have

$$\begin{aligned} \partial^2_{pq} R_{abcd} &= \nabla^2_{pq} R_{abcd} + \partial_p \Gamma^h_{qa} R_{hbcd} + \partial_p \Gamma^h_{qb} R_{ahcd} + \\ &\quad \partial_p \Gamma^h_{qc} R_{abhd} + \partial_p \Gamma^h_{qd} R_{abch}. \end{aligned} \tag{2.13}$$

Recall that $\partial_1 \Gamma^i_{jk} = 0$ on the axis. Thus, we get

$$\partial^2_{1q} R_{1i1j} = \nabla^2_{1q} R_{1i1j}. \tag{2.14}$$

Substituting (2.14) and $X(0,t) = (t, 0, \cdots, 0)^T, Y(0,t) = I_n$ into (2.12), the result is

$$\ddot{R}_{ij}(0,t) = \nabla^2_{11} R_{1i1j}.$$

Next, we begin to estimate $\dot{R}'_{ij}$. Differentiating (2.9) with respect to θ and t respectively:

$$
\begin{aligned}
\dot{R}'_{ij}(\theta,t) &= \partial^2_{pq}R_{abcd}\dot{X}^aY^b_i\dot{X}^cY^d_j\dot{X}^qX^{p'} + \partial_pR_{abcd}\dot{X}^aY^b_i\dot{X}^cY^d_j\dot{X}^{p'} + \\
&\quad \partial_pR_{abcd}\partial_t(\dot{X}^aY^b_i\dot{X}^cY^d_j)X^{p'} + \partial_pR_{abcd}\partial_\theta(\dot{X}^aY^b_i\dot{X}^cY^d_j)\dot{X}^p + \\
&\quad R_{abcd}\partial^2_{t\theta}(\dot{X}^aY^b_i\dot{X}^cY^d_j) \\
&= \partial^2_{pq}R_{abcd}\dot{X}^aY^b_i\dot{X}^cY^d_j\dot{X}^qX^{p'} + \partial_pR_{abcd}\dot{X}^aY^b_i\dot{X}^cY^d_j\dot{X}^{p'} + \\
&\quad \partial_pR_{abcd}(\ddot{X}^aY^b_i\dot{X}^cY^d_j + \dot{X}^a\dot{Y}^b_i\dot{X}^cY^d_j + \dot{X}^aY^b_i\ddot{X}^cY^d_j + \\
&\quad \dot{X}^aY^b_i\dot{X}^c\dot{Y}^d_j)X^{p'} + \partial_pR_{abcd}(\dot{X}^{a'}Y^b_i\dot{X}^cY^d_j + \dot{X}^aY^{b'}_i\dot{X}^cY^d_j + \\
&\quad \dot{X}^aY^b_i\dot{X}^{c'}Y^d_j + \dot{X}^aY^b_i\dot{X}^cY^{d'}_j)\dot{X}^p + R_{abcd}(\ddot{X}^{a'}Y^b_i\dot{X}^cY^d_j + \\
&\quad \dot{X}^{a'}\dot{Y}^b_i\dot{X}^cY^d_j + \dot{X}^{a'}Y^b_i\ddot{X}^cY^d_j + \dot{X}^{a'}Y^b_i\dot{X}^c\dot{Y}^d_j + \ddot{X}^aY^{b'}_i\dot{X}^cY^d_j + \\
&\quad \dot{X}^a\dot{Y}^{b'}_i\dot{X}^cY^d_j + \dot{X}^aY^{b'}_i\ddot{X}^cY^d_j + \dot{X}^aY^{b'}_i\dot{X}^c\dot{Y}^d_j + \ddot{X}^aY^b_i\dot{X}^{c'}Y^d_j + \\
&\quad \dot{X}^a\dot{Y}^b_i\dot{X}^{c'}Y^d_j + \dot{X}^a\dot{Y}^b_i\ddot{X}^{c'}Y^d_j + \dot{X}^aY^b_i\dot{X}^{c'}\dot{Y}^d_j + \ddot{X}^aY^b_i\dot{X}^cY^{d'}_j + \\
&\quad \dot{X}^a\dot{Y}^b_i\dot{X}^cY^{d'}_j + \dot{X}^aY^b_i\ddot{X}^cY^{d'}_j + \dot{X}^aY^b_i\dot{X}^c\dot{Y}^{d'}_j).
\end{aligned}
$$

Evaluating on the axis, the relations $X(0,t) = (t, 0, \cdots, 0)^T$, $X^{1'}(0,t) = 0$, and $Y(0,t) = I_n$ infer

$$
\begin{aligned}
\dot{R}'_{ij}(0,t) &= \partial^2_{1\alpha}R_{1i1j}X^{\alpha'} + \partial_\alpha R_{1i1j}\dot{X}^{\alpha'} + \partial_1R_{\alpha i1j}\dot{X}^{\alpha'} + \\
&\quad \partial_1R_{1\alpha1j}Y^{\alpha'}_i + \partial_1R_{1i\alpha j}\dot{X}^{\alpha'} + \partial_1R_{1i1\alpha}Y^{\alpha'}_j + \\
&\quad R_{\alpha i1j}\ddot{X}^{\alpha'} + R_{1\alpha1j}\dot{Y}^{\alpha'}_i + R_{1i\alpha j}\ddot{X}^{\alpha'} + R_{1i1\alpha}\dot{Y}^{\alpha'}_j \\
&= \partial^2_{1\alpha}R_{1i1j}X^{\alpha'} + (\partial_\alpha R_{1i1j} + \partial_1R_{\alpha i1j} + \partial_1R_{1i\alpha j})\dot{X}^{\alpha'} + \\
&\quad (R_{\alpha i1j} + R_{1i\alpha j})\ddot{X}^{\alpha'} + \partial_1R_{1\alpha1j}Y^{\alpha'}_i + \partial_1R_{1i1\alpha}Y^{\alpha'}_j + \\
&\quad + R_{1\alpha1j}\dot{Y}^{\alpha'}_i + R_{1i1\alpha}\dot{Y}^{\alpha'}_j.
\end{aligned}
$$

From (2.11), (2.14), Lemma 2.1 and Lemma 2.2, we get

$$
\begin{aligned}
\dot{R}'_{ij}(0,t) &= \nabla^2_{1\alpha}R_{1i1j}X^{\alpha'} + (\nabla_\alpha R_{1i1j} + \nabla_1R_{\alpha i1j} + \nabla_1R_{1i\alpha j})\dot{X}^{\alpha'} + \\
&\quad (R_{\alpha i1j} + R_{1i\alpha j})\ddot{X}^{\alpha'} + \nabla_1R_{1\alpha1j}Y^{\alpha'}_i + \nabla_1R_{1i1\alpha}Y^{\alpha'}_j + \\
&\quad + R_{1\alpha1j}\dot{Y}^{\alpha'}_i + R_{1i1\alpha}\dot{Y}^{\alpha'}_j \\
&= \nabla^2_{12}R_{1i1j}X^{2'} + (\nabla_2R_{1i1j} + \nabla_1R_{2i1j} + \nabla_1R_{1i2j})\dot{X}^{2'} + \\
&\quad (R_{2i1j} + R_{1i2j})\ddot{X}^{2'} + \mathcal{B}(\varepsilon t) \\
&= \nabla^2_{12}R_{1i1j}\sin t + (\nabla_2R_{1i1j} + \nabla_1R_{2i1j} + \nabla_1R_{1i2j})\cos t - \\
&\quad (R_{2i1j} + R_{1i2j})\sin t + \mathcal{B}(\varepsilon t) \\
&= (\nabla_1R_{1i2j} + \nabla_1R_{1j2i} + \nabla_2R_{1i1j})\cos t + \mathcal{B}(\varepsilon t),
\end{aligned}
$$

where in the last equality the fact was used that $R_{1i2j} = \mathcal{B}(\varepsilon)$.

To finish the proof of b), it remains to estimate R''_{ij}. Differentiating (2.9) twice with respect to θ, we infer

$$\begin{aligned}R''_{ij}(\theta,t) &= \partial^2_{pq}R_{abcd}\dot{X}^aY^b_i\dot{X}^cY^d_jX^{p'}X^{q'}+\partial_pR_{abcd}\dot{X}^aY^b_i\dot{X}^cY^d_jX^{p''}+\\ &\quad 2\partial_pR_{abcd}\partial_\theta(\dot{X}^aY^b_i\dot{X}^cY^d_j)X^{p'}+R_{abcd}\partial^2_{\theta\theta}(\dot{X}^aY^b_i\dot{X}^cY^d_j)\\ &= \partial^2_{pq}R_{abcd}\dot{X}^aY^b_i\dot{X}^cY^d_jX^{p'}X^{q'}+\partial_pR_{abcd}\dot{X}^aY^b_i\dot{X}^cY^d_jX^{p''}+\\ &\quad 2\partial_pR_{abcd}(\dot{X}^{a'}Y^b_i\dot{X}^cY^d_j+\dot{X}^aY^{b'}_i\dot{X}^cY^d_j+\dot{X}^aY^b_i\dot{X}^{c'}Y^d_j+\\ &\quad \dot{X}^aY^b_i\dot{X}^cY^{d'}_j)X^{p'}+R_{abcd}(\dot{X}^{a''}Y^b_i\dot{X}^cY^d_j+2\dot{X}^{a'}Y^{b'}_i\dot{X}^cY^d_j+\\ &\quad 2\dot{X}^{a'}Y^b_i\dot{X}^{c'}Y^d_j+2\dot{X}^{a'}Y^b_i\dot{X}^cY^{d'}_j+\dot{X}^aY^{b''}_i\dot{X}^cY^d_j+\\ &\quad 2\dot{X}^aY^{b'}_i\dot{X}^{c'}Y^d_j+2\dot{X}^aY^{b'}_i\dot{X}^cY^{d'}_j+\dot{X}^aY^b_i\dot{X}^{c''}Y^d_j+\\ &\quad 2\dot{X}^aY^b_i\dot{X}^{c'}Y^{d'}_j+\dot{X}^aY^b_i\dot{X}^cY^{d''}_j).\end{aligned}$$

Evaluated on the axis, the relations $X(0,t)=(t,0,\cdots,0)^T, X^{1'}(0,t)=0, Y(0,t)=I_n$ imply

$$\begin{aligned}R''_{ij}(0,t) &= \partial^2_{\alpha\beta}R_{1i1j}X^{\alpha'}X^{\beta'}+\partial_pR_{1i1j}X^{p''}+2\partial_\alpha R_{\beta i1j}X^{\alpha'}\dot{X}^{\beta'}+\\ &\quad 2\partial_\alpha R_{1\beta 1j}X^{\alpha'}Y^{\beta'}_i+2\partial_\alpha R_{1i\beta j}X^{\alpha'}\dot{X}^{\beta'}+2\partial_\alpha R_{1i1\beta}X^{\alpha'}Y^{\beta'}_j+\\ &\quad R_{ai1j}\dot{X}^{a''}+2R_{\alpha b1j}\dot{X}^{\alpha'}Y^{b'}_i+2R_{\alpha i\beta j}\dot{X}^{\alpha'}\dot{X}^{\beta'}+2R_{\alpha i1\beta}\dot{X}^{\alpha'}Y^{\beta'}_j+\\ &\quad R_{1\beta 1j}Y^{\beta''}_i+2R_{1\beta\alpha j}\dot{X}^{\alpha'}Y^{\beta'}_i+2R_{1\alpha 1\beta}Y^{\alpha'}_iY^{\beta'}_j+R_{1iaj}\dot{X}^{a''}+\\ &\quad 2R_{1i\alpha b}\dot{X}^{\alpha'}Y^{b'}_j+R_{1i1\beta}Y^{\beta''}_j\\ &= \partial^2_{\alpha\beta}R_{1i1j}X^{\alpha'}X^{\beta'}+\partial_pR_{1i1j}X^{p''}+2(\partial_\alpha R_{\beta i1j}+\\ &\quad \partial_\alpha R_{1i\beta j})X^{\alpha'}\dot{X}^{\beta'}+2(\partial_\alpha R_{1\beta 1j}X^{\alpha'}Y^{\beta'}_i+\partial_\alpha R_{1i1\beta}X^{\alpha'}Y^{\beta'}_j)+\\ &\quad (R_{ai1j}+R_{1iaj})\dot{X}^{a''}+2R_{\alpha i\beta j}\dot{X}^{\alpha'}\dot{X}^{\beta'}+2(R_{\alpha b1j}\dot{X}^{\alpha'}Y^{b'}_i+\\ &\quad R_{1i\alpha b}\dot{X}^{\alpha'}Y^{b'}_j)+2(R_{\alpha i1\beta}\dot{X}^{\alpha'}Y^{\beta'}_j+R_{1\beta\alpha j}\dot{X}^{\alpha'}Y^{\beta'}_i)+\\ &\quad 2R_{1\alpha 1\beta}Y^{\alpha'}_iY^{\beta'}_j+R_{1\beta 1j}Y^{\beta''}_i+R_{1i1\beta}Y^{\beta''}_j.\end{aligned}$$

We are now in the position to calculate $\partial^2_{\alpha\beta}R_{1i1j}$ and ∂_pR_{1i1j}. For the first covariant derivative, from (2.11), we have $\partial_pR_{1i1j}=\nabla_pR_{1i1j}$. For the second covariant derivative, from (2.13), it follows that

$$\begin{aligned}\partial^2_{\alpha\beta}R_{1i1j} &= \nabla^2_{\alpha\beta}R_{1i1j}+\partial_\alpha\Gamma^p_{1\beta}R_{pi1j}+\partial_\alpha\Gamma^p_{i\beta}R_{1p1j}+\\ &\quad \partial_\alpha\Gamma^p_{1\beta}R_{1ipj}+\partial_\alpha\Gamma^p_{j\beta}R_{1i1p}\\ &= \nabla^2_{\alpha\beta}R_{1i1j}+\partial_\alpha\Gamma^p_{1\beta}(R_{pi1j}+R_{1ipj})+\partial_\alpha\Gamma^p_{i\beta}R_{1p1j}+\\ &\quad \partial_\alpha\Gamma^p_{j\beta}R_{1i1p}\\ &= \nabla^2_{\alpha\beta}R_{1i1j}+R^p_{\beta\alpha 1}(R_{1ipj}+R_{1jpi})+\frac{1}{3}(R^p_{i\alpha\beta}+R^p_{\beta\alpha i})R_{1p1j}+ \qquad (2.15)\\ &\quad \frac{1}{3}(R^p_{j\alpha\beta}+R^p_{\beta\alpha j})R_{1i1p}.\end{aligned}$$

where the last equality holding due to (1.15) and (1.16).
Hence $\partial^2_{\alpha\beta}R_{1i1j}$ is uniformly bounded. Therefore

$$\begin{aligned}R''_{ij}(0,t) &= \partial^2_{\alpha\beta}R_{1i1j}X^{\alpha'}X^{\beta'}+\nabla_pR_{1i1j}X^{p''}+2(\nabla_\alpha R_{\beta i1j}+\\ &\quad \nabla_\alpha R_{1i\beta j})X^{\alpha'}\dot{X}^{\beta'}+2(\nabla_\alpha R_{1\beta 1j}X^{\alpha'}Y^{\beta'}_i+\nabla_\alpha R_{1i1\beta}X^{\alpha'}Y^{\beta'}_j)+\\ &\quad (R_{ai1j}+R_{1iaj})\dot{X}^{a''}+2R_{\alpha i\beta j}\dot{X}^{\alpha'}\dot{X}^{\beta'}+2(R_{\alpha b1j}\dot{X}^{\alpha'}Y^{b'}_i+\\ &\quad R_{1i\alpha b}\dot{X}^{\alpha'}Y^{b'}_j)+2(R_{\alpha i1\beta}\dot{X}^{\alpha'}Y^{\beta'}_j+R_{1\beta\alpha j}\dot{X}^{\alpha'}Y^{\beta'}_i)+\\ &\quad 2R_{1\alpha 1\beta}Y^{\alpha'}_iY^{\beta'}_j+R_{1\beta 1j}Y^{\beta''}_i+R_{1i1\beta}Y^{\beta''}_j.\end{aligned}$$

From Lemma 2.1 and Lemma 2.2, we obtain

$$\begin{aligned}
R''_{ij}(0,t) &= \partial^2_{22}R_{1i1j}X^{2'}X^{2'} + \nabla_1 R_{1i1j}X^{1''} + 2(\nabla_2 R_{2i1j} + \\
&\nabla_2 R_{1i2j})X^{2'}\dot{X}^{2'} + 2R_{1i1j}\dot{X}^{1''} + 2R_{2i2j}\dot{X}^{2'}\dot{X}^{2'} + \\
&2(\delta_{2i}R_{211j}\dot{X}^{2'}Y_2^{1'} + R_{1i21}\delta_{2j}\dot{X}^{2'}Y_2^{1'}) + \\
&R_{1i1j}Y_i^{i''} + R_{1i1j}Y_j^{j''} + \mathcal{B}(\varepsilon t^2) \\
&= \partial^2_{22}R_{1i1j}\sin^2 t - \nabla_1 R_{1i1j}\sin t\cos t + 2(\nabla_2 R_{2i1j} + \\
&\nabla_2 R_{1i2j})\sin t\cos t - 2R_{1i1j}\cos(2t) + 2R_{2i2j}\cos^2 t + \\
&2(\delta_{2i}+\delta_{2j})R_{1i1j}\cos^2 t - (\delta_{2i}+\delta_{2j})R_{1i1j}\cos^2 t + \\
&\frac{1}{3}(\delta_{i3}+\delta_{j3}+\cdots+\delta_{in}+\delta_{jn})R_{1i1j}\sin^2 t + \mathcal{B}(\varepsilon t^2) \\
&= \partial^2_{22}R_{1i1j}\sin^2 t + (-\nabla_1 R_{1i1j} + 2\nabla_2 R_{2i1j} + \\
&2\nabla_2 R_{1i2j})\sin t\cos t - 2R_{1i1j}\cos(2t) + 2R_{2i2j}\cos^2 t + \\
&(\delta_{2i}+\delta_{2j})R_{1i1j}\cos^2 t + \\
&\frac{1}{3}(\delta_{i3}+\delta_{j3}+\cdots+\delta_{in}+\delta_{jn})R_{1i1j}\sin^2 t + \mathcal{B}(\varepsilon t^2).
\end{aligned}$$

For the term $\partial^2_{22}R_{1i1j}$, the formula (2.15) gives:

$$\begin{aligned}
\partial^2_{22}R_{1i1j} &= \nabla^2_{22}R_{1i1j} + R^p_{221}(R_{pi1j} + R_{1ipj}) + \\
&\frac{1}{3}R^p_{22i}R_{1p1j} + \frac{1}{3}R^p_{22j}R_{1i1p} \\
&= -\frac{8}{3}R_{1i1j} + \frac{1}{3}(\delta_{2i}+\delta_{2j})R_{1i1j} + \mathcal{B}(\varepsilon),
\end{aligned}$$

where the last equality follows from the assumption (1.4).
Therefore

$$\begin{aligned}
R''_{ij}(0,t) &= -\frac{8}{3}R_{1i1j}\sin^2 t + \frac{1}{3}(\delta_{2i}+\delta_{2j})R_{1i1j}\sin^2 t + \\
&(-\nabla_1 R_{1i1j} + 2\nabla_2 R_{2i1j} + 2\nabla_2 R_{1i2j})\sin t\cos t - \\
&2R_{1i1j}\cos(2t) + 2R_{2i2j}\cos^2 t + (\delta_{2i}+\delta_{2j})R_{1i1j}\cos^2 t + \\
&\frac{1}{3}(\delta_{i3}+\delta_{j3}+\cdots+\delta_{in}+\delta_{jn})R_{1i1j}\sin^2 t + \mathcal{B}(\varepsilon t^2).
\end{aligned}$$

Since $\delta_{i3}+\cdots+\delta_{in} = 1-\delta_{2i}, \delta_{j3}+\cdots+\delta_{jn} = 1-\delta_{2j}$, for $i,j>1$. Then

$$\begin{aligned}
R''_{ij}(0,t) &= -\frac{8}{3}R_{1i1j}\sin^2 t + \frac{1}{3}(\delta_{2i}+\delta_{2j})R_{1i1j}\sin^2 t + \\
&(-\nabla_1 R_{1i1j} + 2\nabla_2 R_{2i1j} + 2\nabla_2 R_{1i2j})\sin t\cos t - \\
&2R_{1i1j}\cos(2t) + 2R_{2i2j}\cos^2 t + (\delta_{2i}+\delta_{2j})R_{1i1j}\cos^2 t + \\
&\frac{1}{3}(2-\delta_{2i}-\delta_{2j})R_{1i1j}\sin^2 t + \mathcal{B}(\varepsilon t^2) \\
&= -2R_{1i1j}\sin^2 t + (-\nabla_1 R_{1i1j} + 2\nabla_2 R_{2i1j} + \\
&2\nabla_2 R_{1i2j})\sin t\cos t - 2R_{1i1j}\cos(2t) + 2R_{2i2j}\cos^2 t + \\
&(\delta_{2i}+\delta_{2j})R_{1i1j}\cos^2 t + \mathcal{B}(\varepsilon t^2) \\
&= 2(R_{2i2j} - R_{1i1j})\cos^2 t + (-\nabla_1 R_{1i1j} + 2\nabla_2 R_{1i2j} + \\
&2\nabla_2 R_{1j2i})\sin t\cos t + (\delta_{2i}+\delta_{2j})R_{1i1j}\cos^2 t + \mathcal{B}(\varepsilon t^2).
\end{aligned}$$

c). Differentiating (2.9) twice with respect to θ and once with respect to t :

$$\begin{aligned}\dot{R}''_{ij}(\theta,t) &= \partial^3_{pqk}R_{abcd}\dot{X}^aY^b_i\dot{X}^cY^d_j\dot{X}^kX^{p'}X^{q'}+\\ &\quad \partial^2_{pq}R_{abcd}\partial_t(\dot{X}^aY^b_i\dot{X}^cY^d_j)X^{p'}X^{q'}+\\ &\quad \partial^2_{pq}R_{abcd}\dot{X}^aY^b_i\dot{X}^cY^d_j(\dot{X}^{p'}X^{q'}+X^{p'}\dot{X}^{q'})+\\ &\quad \partial^2_{pq}R_{abcd}\dot{X}^aY^b_i\dot{X}^cY^d_j\dot{X}^qX^{p''}+\\ &\quad \partial_pR_{abcd}\partial_t(\dot{X}^aY^b_i\dot{X}^cY^d_j)X^{p''}+\\ &\quad \partial_pR_{abcd}\dot{X}^aY^b_i\dot{X}^cY^d_j\dot{X}^{p''}+\\ &\quad 2\partial^2_{pq}R_{abcd}\partial_\theta(\dot{X}^aY^b_i\dot{X}^cY^d_j)\dot{X}^qX^{p'}+\\ &\quad 2\partial_pR_{abcd}\partial^2_{t\theta}(\dot{X}^aY^b_i\dot{X}^cY^d_j)X^{p'}+\\ &\quad 2\partial_pR_{abcd}\partial_\theta(\dot{X}^aY^b_i\dot{X}^cY^d_j)\dot{X}^{p'}+\\ &\quad \partial_pR_{abcd}\partial^2_{\theta\theta}(\dot{X}^aY^b_i\dot{X}^cY^d_j)\dot{X}^p+\\ &\quad R_{abcd}\partial^3_{t\theta\theta}(\dot{X}^aY^b_i\dot{X}^cY^d_j).\end{aligned}$$

Evaluating at the origin, the initial conditions $X'(0,0)=X''(0,0)=0$ infer

$$\begin{aligned}\dot{R}''_{ij}(0,0) &= \partial_pR_{abcd}\dot{X}^aY^b_i\dot{X}^cY^d_j\dot{X}^{p''}+\\ &\quad 2\partial_pR_{abcd}\partial_\theta(\dot{X}^aY^b_i\dot{X}^cY^d_j)\dot{X}^{p'}+\\ &\quad \partial_pR_{abcd}\partial^2_{\theta\theta}(\dot{X}^aY^b_i\dot{X}^cY^d_j)\dot{X}^p+\\ &\quad R_{abcd}\partial^3_{t\theta\theta}(\dot{X}^aY^b_i\dot{X}^cY^d_j).\end{aligned}$$

The term $2\partial_pR_{abcd}\partial_\theta(\dot{X}^aY^b_i\dot{X}^cY^d_j)\dot{X}^{p'}$ can be handled as follows

$$\begin{aligned}&\quad 2\partial_pR_{abcd}\partial_\theta(\dot{X}^aY^b_i\dot{X}^cY^d_j)\dot{X}^{p'}\\ &= 2\partial_pR_{abcd}(\dot{X}^{a'}Y^b_i\dot{X}^cY^d_j+\dot{X}^aY^{b'}_i\dot{X}^cY^d_j+\\ &\quad \dot{X}^aY^b_i\dot{X}^{c'}Y^d_j+\dot{X}^aY^b_i\dot{X}^cY^{d'}_j)\dot{X}^{p'}\\ &= 2\partial_\alpha R_{\beta i1j}\dot{X}^{\alpha'}\dot{X}^{\beta'}+2\partial_\alpha R_{1\beta 1j}\dot{X}^{\alpha'}Y^{\beta'}_i+\\ &\quad 2\partial_\alpha R_{1i\beta j}\dot{X}^{\alpha'}\dot{X}^{\beta'}+2\partial_\alpha R_{1i1\beta}\dot{X}^{\alpha'}Y^{\beta'}_j\\ &= 2(\nabla_2R_{1i2j}+\nabla_2R_{1j2i}).\end{aligned}$$

where the last equality follows from $\dot{X}'(0,0)=(0,1,0,\cdots,0)^T$ and $Y^{i'}_j(0,0)=-\delta^i_1\delta^2_j+\delta^i_2\delta^1_j$.

The term $\partial_pR_{abcd}\partial^2_{\theta\theta}(\dot{X}^aY^b_i\dot{X}^cY^d_j)\dot{X}^p$.

$$\begin{aligned}&\quad \partial_pR_{abcd}\partial^2_{\theta\theta}(\dot{X}^aY^b_i\dot{X}^cY^d_j)\dot{X}^p\\ &= \partial_pR_{abcd}(\dot{X}^{a''}Y^b_i\dot{X}^cY^d_j+2\dot{X}^{a'}Y^{b'}_i\dot{X}^cY^d_j+2\dot{X}^{a'}Y^b_i\dot{X}^{c'}Y^d_j+\\ &\quad 2\dot{X}^{a'}Y^b_i\dot{X}^cY^{d'}_j+\dot{X}^aY^{b''}_i\dot{X}^cY^d_j+2\dot{X}^aY^{b'}_i\dot{X}^{c'}Y^d_j+\\ &\quad 2\dot{X}^aY^{b'}_i\dot{X}^cY^{d'}_j+\dot{X}^aY^b_i\dot{X}^{c''}Y^d_j+2\dot{X}^aY^b_i\dot{X}^{c'}Y^{d'}_j+\\ &\quad \dot{X}^aY^b_i\dot{X}^cY^{d''}_j)\dot{X}^p\\ &= \partial_1R_{ai1j}\dot{X}^{a''}+2\partial_1R_{\alpha b1j}\dot{X}^{\alpha'}Y^{b'}_i+2\partial_1R_{\alpha i\beta j}\dot{X}^{\alpha'}\dot{X}^{\beta'}+2\partial_1R_{\alpha i1\beta}\dot{X}^{\alpha'}Y^{\beta'}_j+\\ &\quad \partial_1R_{1\beta 1j}Y^{\beta''}_i+2\partial_1R_{1\beta\alpha j}\dot{X}^{\alpha'}Y^{\beta'}_i+2\partial_1R_{1\alpha 1\beta}Y^{\alpha'}_iY^{\beta'}_j+\partial_1R_{1iaj}\dot{X}^{a''}+\\ &\quad 2\partial_1R_{1i\alpha b}\dot{X}^{\alpha'}Y^{b'}_j+\partial_1R_{1i1\beta}Y^{\beta''}_j\\ &= -2\nabla_1R_{1i1j}+(\delta_{2i}+\delta_{2j})\nabla_1R_{1i1j}+2\nabla_1R_{2i2j},\end{aligned}$$

where the last equality follows from the following relations $\dot{X}'(0,0) = (0,1,0,\cdots,0)^T$, $\ddot{X}'(0,0) = (0,-1,0,\cdots,0)^T$, $\dot{X}''(0,0) = (-1,0,\cdots,0)^T$ and $Y_j^{i'}(0,0) = -\delta_1^i\delta_j^2 + \delta_2^i\delta_j^1$.
The term $\partial^3_{t\theta\theta}(\dot{X}^aY_i^b\dot{X}^cY_j^d)$.

$$
\begin{aligned}
& \partial^3_{t\theta\theta}(\dot{X}^aY_i^b\dot{X}^cY_j^d) \\
= \; & \ddot{X}^{a''}Y_i^b\dot{X}^cY_j^d + \dot{X}^{a''}\dot{Y}_i^b\dot{X}^cY_j^d + \dot{X}^{a''}Y_i^b\ddot{X}^cY_j^d + \dot{X}^{a''}Y_i^b\dot{X}^c\dot{Y}_j^d + \\
& 2\ddot{X}^{a'}Y_i^{b'}\dot{X}^cY_j^d + 2\dot{X}^{a'}\dot{Y}_i^{b'}\dot{X}^cY_j^d + 2\dot{X}^{a'}Y_i^{b'}\ddot{X}^cY_j^d + 2\dot{X}^{a'}Y_i^{b'}\dot{X}^c\dot{Y}_j^d + \\
& 2\ddot{X}^{a'}Y_i^b\dot{X}^{c'}Y_j^d + 2\dot{X}^{a'}\dot{Y}_i^b\dot{X}^{c'}Y_j^d + 2\dot{X}^{a'}Y_i^b\ddot{X}^{c'}Y_j^d + 2\dot{X}^{a'}Y_i^b\dot{X}^{c'}\dot{Y}_j^d + \\
& 2\ddot{X}^{a'}Y_i^b\dot{X}^cY_j^{d'} + 2\dot{X}^{a'}\dot{Y}_i^b\dot{X}^cY_j^{d'} + 2\dot{X}^{a'}Y_i^b\ddot{X}^cY_j^{d'} + 2\dot{X}^{a'}Y_i^b\dot{X}^c\dot{Y}_j^{d'} + \\
& \ddot{X}^aY_i^{b''}\dot{X}^cY_j^d + \dot{X}^a\dot{Y}_i^{b''}\dot{X}^cY_j^d + \dot{X}^aY_i^{b''}\ddot{X}^cY_j^d + \dot{X}^aY_i^{b''}\dot{X}^c\dot{Y}_j^d + \\
& 2\ddot{X}^aY_i^{b'}\dot{X}^{c'}Y_j^d + 2\dot{X}^a\dot{Y}_i^{b'}\dot{X}^{c'}Y_j^d + 2\dot{X}^aY_i^{b'}\ddot{X}^{c'}Y_j^d + 2\dot{X}^aY_i^{b'}\dot{X}^{c'}\dot{Y}_j^d + \\
& 2\ddot{X}^aY_i^{b'}\dot{X}^cY_j^{d'} + 2\dot{X}^a\dot{Y}_i^{b'}\dot{X}^cY_j^{d'} + 2\dot{X}^aY_i^{b'}\ddot{X}^cY_j^{d'} + 2\dot{X}^aY_i^{b'}\dot{X}^c\dot{Y}_j^{d'} + \\
& \ddot{X}^aY_i^b\dot{X}^{c''}Y_j^d + \dot{X}^a\dot{Y}_i^b\dot{X}^{c''}Y_j^d + \dot{X}^aY_i^b\ddot{X}^{c''}Y_j^d + \dot{X}^aY_i^b\dot{X}^{c''}\dot{Y}_j^d + \\
& 2\ddot{X}^aY_i^b\dot{X}^{c'}Y_j^{d'} + 2\dot{X}^a\dot{Y}_i^b\dot{X}^{c'}Y_j^{d'} + 2\dot{X}^aY_i^b\ddot{X}^{c'}Y_j^{d'} + 2\dot{X}^aY_i^b\dot{X}^{c'}\dot{Y}_j^{d'} + \\
& \ddot{X}^aY_i^b\dot{X}^cY_j^{d''} + \dot{X}^a\dot{Y}_i^b\dot{X}^cY_j^{d''} + \dot{X}^aY_i^b\ddot{X}^cY_j^{d''} + \dot{X}^aY_i^b\dot{X}^c\dot{Y}_j^{d''} \\
= \; & 0,
\end{aligned}
$$

where the last equality holds because of $\ddot{X}(0,0) = \ddot{X}'(0,0) = \ddot{X}''(0,0) = 0$ and $\dot{Y}(0,0) = \dot{Y}'(0,0) = \dot{Y}''(0,0) = 0$.

Therefore

$$
\begin{aligned}
\dot{R}''_{ij}(0,0) \; = \; & 2\nabla_1 R_{2i2j} + (\delta_{2i} + \delta_{2j} - 3)\nabla_1 R_{1i1j} + \\
& 2(\nabla_2 R_{1i2j} + \nabla_2 R_{1j2i}).
\end{aligned}
$$

By a lengthy computation, one can prove that $\ddot{R}'_{ij}(0,0) = \mathcal{B}(\varepsilon)$ and $\ddot{R}''_{ij}(0,0) = \mathcal{B}(\varepsilon)$. This ends the proof of Proposition 2.1. □

Chapter 3

A reinforced MTW condition

This chapter contains the various subtle estimates of the *MTW tensor*. The *MTW tensor* plays an important role in the regularity of the optimal transport map. Thus it is useful to understand the behaviour of the *MTWtensor*. Firstly, we exploit the formula in [45] to calculate the *MTWtensor*. Secondly, we recast the approximation of the *MTWtensor* by the *MTWtensor* on the round sphere originated from Theorem 2 in [29]. In the end, we prove that the *MTWtensor* on the nearly spherical manifold satisfies a reinforced *MTWcondition*.

3.1 The calculation of MTW tensor

In this section, we calculate the *MTWtensor*. As a starting point, we give some notations. Let (M, g) be an n-dimensional Riemannian manifold. Henceforth, we fix $m \in M, \nu \in I(m)\backslash\{0\}$ and $(\xi, \eta) \in T_mM \times T_mM$. Since the *MTWtensor* is homogeneous with degree 2 in both ξ and η, it suffices to assume that $|\xi|_m = |\eta|_m = 1$.

Taking the orthonormal basis $\{E_1, E_2, \cdots, E_n\}$ of the tangent space T_mM so that $\nu = |\nu|_m E_1, \xi = \xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3, \eta = \eta_1 E_1 + \eta_2 E_2$ and identify the tangent vectors at m with their coordinates in this basis. Then the metric g_m is given by the canonical scalar product of $\mathbb{R}^n$. It will be implicitly understood throughout the calculations that the inner product and the Riemann curvature tensor are evaluated at the point m.

Recalling the definition of *MTWtensor* (0.2), it follows that

$$\begin{aligned} \mathcal{C}(m,\nu)(\xi,\eta) &= -\frac{3}{2}\frac{d^2}{ds^2}\Big|_{s=0} \langle \nabla_m^2 c(., \exp_m(\nu + s\eta))(\xi), \xi\rangle_m \\ &= -\frac{3}{2}\frac{d^2}{ds^2}\Big|_{s=0} \langle \mathcal{S}(m, \nu + s\eta, 1)(\xi), \xi\rangle_m, \end{aligned}$$

where the second equality follows from Proposition 1.1(b).

For any $s \in \mathbb{R}$ small enough, we can write

$$\nu + s\eta = t(s)(\cos\theta E_1 + \sin\theta E_2),$$

where

$$t(s) = |\nu + s\eta|_m, \theta(s) = \tan^{-1}\left(\frac{s\eta_2}{\tau + s\eta_1}\right).$$

From the Remark 1.1(1), it follows that

$$\mathcal{C}(m,\nu)(\xi,\eta) = -\frac{3}{2}\frac{d^2}{ds^2}|_{s=0}\langle \mathcal{S}(m, \cos\theta(s)E_1 + \sin\theta(s)E_2, t(s))(\xi), \xi\rangle_m.$$

To proceed, we give some more notations. Let $\gamma_\theta(t)$ be the geodesic with initial point m and initial velocity $\cos\theta E_1 + \sin\theta E_2$. For $|\theta|$ sufficiently small, let the orthonormal frame $\{e_1(\theta,t), e_2(\theta,t), \cdots, e_n(\theta,t)\}$ be the parallel transport along the geodesic

$\gamma_\theta(t)$ with the initial conditions $e_1(\theta,0) = \cos\theta E_1 + \sin\theta E_2$, $e_2(\theta,0) = -\sin\theta E_1 + \cos\theta E_2$, $e_i(\theta,0) = E_i$ for $i \geq 3$. Let $J_0(\theta,t)$ and $J_1(\theta,t)$ be the solutions of the Jacobi equation (1.7) and $R(\theta,t)$ be defined by (1.8) along the geodesic $\gamma_\theta(t)$. Recall that the matrix of the linear operator $\mathcal{S}(m, e_1(\theta,0), t)$ in the orthonormal basis $\{e_1(\theta,0), e_2(\theta,0), \cdots, e_n(\theta,0)\}$ is $tJ_0(\theta,t)^{-1}J_1(\theta,t)$.

Then the matrix of the linear operator $\mathcal{S}(m, e_1(\theta,0), t)$ in the orthonormal basis $\{E_1, E_2, \cdots, E_n\}$ is $Q(\theta)^T\mathcal{S}(\theta,t)Q(\theta)$, i.e.

$$\langle \mathcal{S}(m, e_1(\theta,0), t)(\xi), \xi\rangle = \langle \mathcal{S}(\theta,t)Q(\theta)\xi, Q(\theta)\xi\rangle. \tag{3.1}$$

where

$$Q(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & \\ -\sin\theta & \cos\theta & \\ & & I_{n-2} \end{bmatrix}.$$

In the sequel, the dot will stand for the derivative with respect to t and the prime for the derivative with respect to θ.

Differentiating (3.1) once and twice with respect to s, respectively:

$$\begin{aligned} &\frac{d}{ds}\langle \mathcal{S}(m, e_1(\theta,0), t)(\xi), \xi\rangle = [\langle \mathcal{S}'Q\xi, Q\xi\rangle + 2\langle \mathcal{S}Q\xi, Q'\xi\rangle]\frac{d\theta}{ds} + \\ &\langle \dot{\mathcal{S}}Q\xi, Q\xi\rangle\frac{dt}{ds}, \\ &\frac{d^2}{ds^2}\langle \mathcal{S}(m, e_1(\theta,0), t)(\xi), \xi\rangle = \\ &[\langle \mathcal{S}''Q\xi, Q\xi\rangle + 4\langle \mathcal{S}'Q\xi, Q'\xi\rangle + 2\langle \mathcal{S}Q'\xi, Q'\xi\rangle + 2\langle \mathcal{S}Q\xi, Q''\xi\rangle](\frac{d\theta}{ds})^2 + \\ &[2\langle \dot{\mathcal{S}}'Q\xi, Q\xi\rangle + 4\langle \dot{\mathcal{S}}Q\xi, Q'\xi\rangle]\frac{dt}{ds}\frac{d\theta}{ds} + \langle \ddot{\mathcal{S}}Q\xi, Q\xi\rangle(\frac{dt}{ds})^2 + \\ &[\langle \mathcal{S}'Q\xi, Q\xi\rangle + 2\langle \mathcal{S}Q\xi, Q'\xi\rangle]\frac{d^2\theta}{ds^2} + \langle \dot{\mathcal{S}}Q\xi, Q\xi\rangle\frac{d^2t}{ds^2}. \end{aligned} \tag{3.2}$$

By a straitforward computation, we have

$$\begin{aligned} \frac{dt}{ds} &= \frac{s(\eta_1^2+\eta_2^2)+\tau\eta_1}{|\nu+s\eta|}, \frac{d^2t}{ds^2} = \frac{\tau^2\eta_2^2}{|\nu+s\eta|^3}, \\ \frac{d\theta}{ds} &= \frac{\tau\eta_2}{s^2(\eta_1^2+\eta_2^2)+2s\tau\eta_1+\tau^2}, \frac{d^2\theta}{ds^2} = -\frac{2\tau\eta_2[s(\eta_1^2+\eta_2^2)+\tau\eta_1]}{[s^2(\eta_1^2+\eta_2^2)+2s\tau\eta_1+\tau^2]^2}, \end{aligned}$$

where $\tau = |\nu|_m$.
Evaluating at $s=0$, we have

$$t = \tau, \frac{dt}{ds} = \eta_1, \frac{d^2t}{ds^2} = \frac{\eta_2^2}{\tau}, \theta = 0, \frac{d\theta}{ds} = \frac{\eta_2}{\tau}, \frac{d^2\theta}{ds^2} = -\frac{2\eta_1\eta_2}{\tau^2}. \tag{3.3}$$

We write

$$P\xi = (\xi_1, \xi_2, 0, \cdots, 0)^T, P^\perp\xi = (\xi_2, -\xi_1, 0, \cdots, 0)^T.$$

Then at $s=0$, we have

$$Q\xi = \xi, Q'\xi = P^\perp\xi, Q''\xi = -P\xi. \tag{3.4}$$

At $s=0$, plugging (3.3),(3.4) into (3.2), we obtain

$$\begin{aligned}
& \frac{d^2}{ds^2}|_{s=0}\langle \mathcal{S}(m, e_1(\theta,0),t)(\xi),\xi\rangle \\
= \; & [\langle \mathcal{S}''\xi,\xi\rangle + 4\langle \mathcal{S}'\xi, P^{\perp}\xi\rangle + 2\langle \mathcal{S}P^{\perp}\xi, P^{\perp}\xi\rangle - 2\langle \mathcal{S}\xi, P\xi\rangle]\frac{\eta_2^2}{\tau^2} + \\
& [2\langle \dot{\mathcal{S}}'\xi,\xi\rangle + 4\langle \dot{\mathcal{S}}\xi, P^{\perp}\xi\rangle]\frac{\eta_1\eta_2}{\tau} + \langle \ddot{\mathcal{S}}\xi,\xi\rangle\eta_1^2 + \\
& [\langle \mathcal{S}'\xi,\xi\rangle + 2\langle \mathcal{S}\xi, P^{\perp}\xi\rangle](-\frac{2\eta_1\eta_2}{\tau^2}) + \langle \dot{\mathcal{S}}\xi,\xi\rangle\frac{\eta_2^2}{\tau} \\
= \; & \langle \ddot{\mathcal{S}}\xi,\xi\rangle\eta_1^2 + [\frac{2}{\tau}\langle \dot{\mathcal{S}}'\xi,\xi\rangle + \frac{4}{\tau}\langle \dot{\mathcal{S}}\xi, P^{\perp}\xi\rangle - \frac{2}{\tau^2}\langle \mathcal{S}'\xi,\xi\rangle - \\
& \frac{4}{\tau^2}\langle \mathcal{S}\xi, P^{\perp}\xi\rangle]\eta_1\eta_2 + [\frac{1}{\tau^2}\langle \mathcal{S}''\xi,\xi\rangle + \frac{1}{\tau}\langle \dot{\mathcal{S}}\xi,\xi\rangle + \\
& \frac{4}{\tau^2}\langle \mathcal{S}'\xi, P^{\perp}\xi\rangle + \frac{2}{\tau^2}\langle \mathcal{S}P^{\perp}\xi, P^{\perp}\xi\rangle - \frac{2}{\tau^2}\langle \mathcal{S}\xi, P\xi\rangle]\eta_2^2.
\end{aligned}$$

Finally, we get

$$\begin{aligned}
& \mathcal{C}(m,\nu)(\xi,\eta) \\
& = -\frac{3}{2}\langle \ddot{\mathcal{S}}\xi,\xi\rangle\eta_1^2 + 3[-\frac{1}{\tau}\langle \dot{\mathcal{S}}'\xi,\xi\rangle - \frac{2}{\tau}\langle \dot{\mathcal{S}}\xi, P^{\perp}\xi\rangle + \\
& \frac{1}{\tau^2}\langle \mathcal{S}'\xi,\xi\rangle + \frac{2}{\tau^2}\langle \mathcal{S}\xi, P^{\perp}\xi\rangle]\eta_1\eta_2 + \frac{3}{2}[-\frac{1}{\tau^2}\langle \mathcal{S}''\xi,\xi\rangle - \\
& \frac{1}{\tau}\langle \dot{\mathcal{S}}\xi,\xi\rangle - \frac{4}{\tau^2}\langle \mathcal{S}'\xi, P^{\perp}\xi\rangle - \frac{2}{\tau^2}\langle \mathcal{S}P^{\perp}\xi, P^{\perp}\xi\rangle + \frac{2}{\tau^2}\langle \mathcal{S}\xi, P\xi\rangle]\eta_2^2 \\
& = a_{11}(m,\nu,\xi)\eta_1^2 + a_{12}(m,\nu,\xi)\eta_1\eta_2 + a_{22}(m,\nu,\xi)\eta_2^2.
\end{aligned} \tag{3.5}$$

We note that the term ξ_1^2 does not appear in the coefficients a_{11} and a_{12}. The details will be discussed in section 3.3.

3.2 The approximation of MTW tensor

In this section, we recast Theorem 2 in [29] as follows.

Theorem 3.1. *Let (M,g) be a closed n-dimensional Riemannian manifold satisfying (1.3) and (1.4) for some positive constant ε. Given $m \in M$ and $\nu \in I(m)$. Assume ε is small enough such that*

$$\frac{|\nu|}{\sin|\nu|}\varepsilon \leq \frac{1}{4\sqrt{n-1}}.$$

Then there exists a positive constant $C \geq 1$ (independent of (m,ν,ε)) such that, for any unit tangent vectors $\xi,\eta \in T_mM$, the following inequality holds:

$$|\mathcal{C}(m,\nu)(\xi,\eta) - \bar{\mathcal{C}}(m,\nu)(\xi,\eta)| \leq C\varepsilon(\frac{|\nu|}{\sin|\nu|})^4(|\xi^{\perp}|^2 + |\eta^{\perp}|^2).$$

We give some comments about Theorem 3.1. Theorem 3.1 provides a qualified control of $MTW tensor$ by the $MTW tensor$ on the sphere. The control depends on ε and $|\nu|$ which will be important for the blow up rates when $|\nu|$ is close to π. It will be used to show that, under the hypothesis of (1.3) and (1.4), the $MTW tensor$ satisfies a reinforced $MTW condition$ in section 3.3.

Remark 3.1. *Theorem 3.1 is obvious when ν is equal to 0, because of (4) and (9). Therefore, we will consider only the case $\nu \in I(m)\backslash\{0\}$.*

Proof of Theorem 3.1. Recall that we can calculate the *MTWtensor* by formula (3.5). Replacing the term $\mathcal{C}$ by $\bar{\mathcal{C}}$, we get the corresponding formula for MTW tensor on the sphere $\bar{\mathcal{C}}$. The difference between $\mathcal{C}$ and $\bar{\mathcal{C}}$ can be estimated as follows:

$$\begin{aligned} & \mathcal{C}(m,\nu)(\xi,\eta) - \bar{\mathcal{C}}(m,\nu)(\xi,\eta) \\ = \; & -\frac{3}{2}\langle(\ddot{\mathcal{S}} - \ddot{\bar{\mathcal{S}}})\xi,\xi\rangle\eta_1^2 + 3[-\frac{1}{\tau}\langle\dot{\mathcal{S}}'\xi,\xi\rangle + \frac{1}{\tau^2}\langle\mathcal{S}'\xi,\xi\rangle - \frac{2}{\tau}\langle(\dot{\mathcal{S}} - \\ & \dot{\bar{\mathcal{S}}})\xi, P^{\perp}\xi\rangle + \frac{2}{\tau^2}\langle(\mathcal{S} - \bar{\mathcal{S}})\xi, P^{\perp}\xi\rangle]\eta_1\eta_2 + \frac{3}{2}[-\frac{1}{\tau^2}\langle\mathcal{S}''\xi,\xi\rangle - \\ & \frac{1}{\tau}\langle(\dot{\mathcal{S}} - \dot{\bar{\mathcal{S}}})\xi,\xi\rangle - \frac{4}{\tau^2}\langle\mathcal{S}'P^{\perp}\xi,\xi\rangle - \frac{2}{\tau^2}\langle(\mathcal{S} - \bar{\mathcal{S}})P^{\perp}\xi, P^{\perp}\xi\rangle + \\ & \frac{2}{\tau^2}\langle(\mathcal{S} - \bar{\mathcal{S}})\xi, P\xi\rangle]\eta_2^2. \end{aligned} \tag{3.6}$$

Observe we calculate the above expression on the right hand side at the point (m, E_1, τ) where $\tau = |\nu|$.

The proof is divided into four steps. From now on, we will compute at the point (m, E_1, t) where $t \in (0, \tau]$.

Step 1.In first step, we will prove that there exist a positive constant C such that,$\forall t \in (0, \tau]$,

$$|\mathcal{S} - \bar{\mathcal{S}}| \leq C\varepsilon\frac{t^6}{\sin^4 t}, \tag{3.7}$$

$$|\dot{\mathcal{S}} - \dot{\bar{\mathcal{S}}}| \leq C\varepsilon\frac{t^5}{\sin^4 t}, \tag{3.8}$$

$$|\ddot{\mathcal{S}} - \ddot{\bar{\mathcal{S}}}| \leq C\varepsilon\frac{t^4}{\sin^4 t}. \tag{3.9}$$

By integrating with respect to t, it suffices to show the last inequality (3.9). We start by evaluating the difference $J_0^{-1} - \bar{J}_0^{-1}$. For this purpose, we write

$$J_0 = \bar{J}_0[I_n - \bar{J}_0^{-1}(\bar{J}_0 - J_0)].$$

Then we infer the formal expansion

$$J_0^{-1} = \sum_{k=0}^{\infty}[\bar{J}_0^{-1}(\bar{J}_0 - J_0)]^k\bar{J}_0^{-1}.$$

Recalling that $\bar{J}_0$ is diagonal and first row and first column of the matrix $\bar{J}_0 - J_0$ vanish, we have

$$J_0^{-1} - \bar{J}_0^{-1} = \frac{t}{\sin t}\sum_{k=1}^{\infty}(\frac{t}{\sin t})^k(\bar{J}_0 - J_0)^k.$$

From Remark 1.2, we derive that $|J_0 - \bar{J}_0| \leq 2\varepsilon\sqrt{n-1}$. Together with the assumption $\frac{\tau}{\sin\tau}\varepsilon \leq \frac{1}{4\sqrt{n-1}}$ and the real function $\frac{t}{\sin t}$ is increase in the interval $(0,\pi)$, the latter expansion is uniformly convergent in any compact subset in $(0,\pi)$. Moreover, we have

$$|J_0^{-1} - \bar{J}_0^{-1}| \leq 4\sqrt{n-1}(\frac{t}{\sin t})^2\varepsilon, \tag{3.10}$$

provided ε is sufficiently small. The triangle inequality provides the upper bound

$$|(J_0^{-1})^{\perp}| \leq 2\sqrt{n-1}\frac{t}{\sin t}. \tag{3.11}$$

We are in position to estimate the term $\ddot{\mathcal{S}} - \ddot{\bar{\mathcal{S}}}$. The second derivative of $\mathcal{S}$ with respect to t takes the form

$$\ddot{\mathcal{S}} = 2J_0^{-1}\dot{J}_1 - 2tJ_0^{-1}\dot{J}_0J_0^{-1}\dot{J}_1 + 2tJ_0^{-1}\dot{J}_0J_0^{-1}\dot{J}_0J_0^{-1}J_1 - 2J_0^{-1}\dot{J}_0J_0^{-1}J_1. \tag{3.12}$$

Replacing $\ddot{\mathcal{S}}$ by $\ddot{\bar{\mathcal{S}}}$, one can get the corresponding formula. After using the finite differences trick in a systematic way, we have

$$\begin{aligned}
\ddot{\mathcal{S}} - \ddot{\bar{\mathcal{S}}} \; = \; & 2(J_0^{-1} - \bar{J}_0^{-1})\dot{J}_1 + 2\bar{J}_0^{-1}(\dot{J}_1 - \dot{\bar{J}}_1) - 2t(J_0^{-1} - \bar{J}_0^{-1})\dot{J}_0 J_0^{-1}\dot{J}_1 - \\
& 2t\bar{J}_0^{-1}(\dot{J}_0 - \dot{\bar{J}}_0)J_0^{-1}\dot{J}_1 - 2t\bar{J}_0^{-1}\dot{\bar{J}}_0(J_0^{-1} - \bar{J}_0^{-1})\dot{J}_1 - \\
& 2t\bar{J}_0^{-1}\dot{\bar{J}}_0\bar{J}_0^{-1}(\dot{J}_1 - \dot{\bar{J}}_1) + 2t(J_0^{-1} - \bar{J}_0^{-1})\dot{J}_0 J_0^{-1}\dot{J}_0 J_0^{-1}J_1 + \\
& 2t\bar{J}_0^{-1}(\dot{J}_0 - \dot{\bar{J}}_0)J_0^{-1}\dot{J}_0 J_0^{-1}J_1 + 2t\bar{J}_0^{-1}\dot{\bar{J}}_0(J_0^{-1} - \bar{J}_0^{-1})\dot{J}_0 J_0^{-1}J_1 + \\
& 2t\bar{J}_0^{-1}\dot{\bar{J}}_0\bar{J}_0^{-1}(\dot{J}_0 - \dot{\bar{J}}_0)J_0^{-1}J_1 + 2t\bar{J}_0^{-1}\dot{\bar{J}}_0\bar{J}_0^{-1}\dot{\bar{J}}_0(J_0^{-1} - \bar{J}_0^{-1})J_1 + \\
& 2t\bar{J}_0^{-1}\dot{\bar{J}}_0\bar{J}_0^{-1}\dot{\bar{J}}_0\bar{J}_0^{-1}(J_1 - \bar{J}_1) - 2t(J_0^{-1} - \bar{J}_0^{-1})\dot{J}_0 J_0^{-1}\dot{J}_1 - \\
& 2t\bar{J}_0^{-1}(\dot{J}_0 - \dot{\bar{J}}_0)J_0^{-1}\dot{J}_1 - 2t\bar{J}_0^{-1}\dot{\bar{J}}_0(J_0^{-1} - \bar{J}_0^{-1})\dot{J}_1 - \\
& 2t\bar{J}_0^{-1}\dot{\bar{J}}_0\bar{J}_0^{-1}(\dot{J}_1 - \dot{\bar{J}}_1) + t(J_0^{-1} - \bar{J}_0^{-1})\ddot{J}_1 + t\bar{J}_1^{-1}(\ddot{J}_1 - \ddot{\bar{J}}_1).
\end{aligned}$$

From (3.10), (3.11), Lemma 1.1 and Lemma 1.2, we infer that there exsits a positive constant C such that

$$|\ddot{\mathcal{S}} - \ddot{\bar{\mathcal{S}}}| \leq C\varepsilon \frac{t^4}{\sin^4 t}, \forall t \in (0, \tau]. \tag{3.13}$$

Step 2.In this step, we will show that there exists a positive constant $C > 0$ such that

$$|\mathcal{S}'| \leq C\varepsilon \frac{t^4}{\sin^2 t}, \tag{3.14}$$

$$|\dot{\mathcal{S}}'| \leq C\varepsilon \frac{t^4}{\sin^3 t}, \forall t \in (0, \tau]. \tag{3.15}$$

We first deal with J_0' and J_1'. By differentiating the equation (1.7) with respect to the variable θ, evaluating at the point $(0, t)$, one can derive that J_0' and J_1' satisfy the following equations

$$\begin{cases} \ddot{J}_a' + RJ_a' = -R'J_a, \\ J_a'(0) = 0 = \dot{J}_a'(0), \; a = 0, 1. \end{cases}$$

By the representation formula (Proposition 1.1 (c)), we obtain

$$J_a' = -J_0 \int_0^t J_1^* R' J_a ds + J_1 \int_0^t J_0^* R' J_a ds, a = 0, 1. \tag{3.16}$$

By virtue of $a)$ in Proposition 2.1 and Lemma 1.1, we derive that there exists a positive constant C such that

$$|J_a'| \leq C\varepsilon t, a = 0, 1. \tag{3.17}$$

Note that

$$\mathcal{S}' = -tJ_0^{-1}J_0'J_0^{-1}J_1 + tJ_0^{-1}J_1'.$$

From (3.11), (3.17) and Lemma 1.1, we infer that there exists a positive constant C such that

$$|\mathcal{S}'| \leq C\varepsilon \frac{t^4}{\sin^2 t}.$$

We are in position to estimate (3.15). We first handle $\dot{J}_0'$ and $\dot{J}_1'$.

By differentiating the equation (1.7) with respect to θ and t once respectively, evaluated at the point $(0, t)$, one can derive that $\dot{J}_0'$ and $\dot{J}_1'$ satisfy the following equations

$$\begin{cases} \ddot{\dot{J}}_a' + R\dot{J}_a' = -\dot{R}'J_a - R'\dot{J}_a - \dot{R}J_a', \; a = 0, 1. \\ \dot{J}_0'(0) = 0 = \ddot{J}_0'(0), \\ \dot{J}_1'(0) = 0, \ddot{J}_1'(0) = -R'(0). \end{cases}$$

From the representation formula (Proposition 1.1 (c)), we have

$$\begin{aligned}
\dot{J}_0' &= -J_0\int_0^t J_1^*(\dot{R}'J_0 + R'\dot{J}_0 + \dot{R}J_0')ds + \\
&\quad J_1\int_0^t J_0^*(\dot{R}'J_0 + R'\dot{J}_0 + \dot{R}J_0')ds, \qquad (3.18)\\
\dot{J}_1' &= -J_0R'(0) - J_0\int_0^t J_1^*(\dot{R}'J_1 + R'\dot{J}_1 + \dot{R}J_1')ds + \\
&\quad J_1\int_0^t J_0^*(\dot{R}'J_1 + R'\dot{J}_1 + \dot{R}J_1')ds. \qquad (3.19)
\end{aligned}$$

By virtue of $a), b)$ in Proposition 2.1 and Lemma 1.1, we derive that there exists a positive constant C such that

$$|\dot{J}_0'| \le C\varepsilon t, |\dot{J}_1'| \le C\varepsilon.$$

Note that

$$\begin{aligned}
\dot{\mathcal{S}}' &= -J_0^{-1}J_0'J_0^{-1}J_1 + tJ_0^{-1}\dot{J}_0J_0^{-1}J_0'J_0^{-1}J_1 - \qquad (3.20)\\
&\quad tJ_0^{-1}\dot{J}_0'J_0^{-1}J_1 + tJ_0^{-1}J_0'J_0^{-1}\dot{J}_0J_0^{-1}J_1 - \\
&\quad tJ_0^{-1}J_0'J_0^{-1}\dot{J}_1 + J_0^{-1}J_1' - tJ_0^{-1}\dot{J}_0J_0^{-1}J_1' + \\
&\quad tJ_0^{-1}\dot{J}_1'.
\end{aligned}$$

Together with (3.17), (3.11) and Lemma 1.1, we infer that there exists a positive constant C such that

$$|\dot{\mathcal{S}}'| \le C\varepsilon\frac{t^4}{\sin^3 t}.$$

Step 3.In this step, we will verify that there exists a positive constant C such that

$$\forall t \in (0,\tau],\ |\mathcal{S}''| \le C\varepsilon\frac{t^5}{\sin^3 t}. \qquad (3.21)$$

We first deal with J_0'' and J_1''. Differentiating the equation (1.7) with respect to θ twice, evaluating at the point $(0,t)$, one can derive that J_0'' and J_1'' satisfy the following equations

$$\begin{cases} \ddot{J}_a'' + RJ_a'' = -R''J_a - 2R'J_a', \quad a = 0,1, \\ J_a''(0) = 0 = \dot{J}_a''(0). \end{cases}$$

Making use of the representation formula (Proposition 1.1 (c)), we see that

$$J_a'' = -J_0\int_0^t J_1^*(R''J_a + 2R'J_a')ds + J_1\int_0^t (R''J_a + 2R'J_a')ds, a = 0,1. \qquad (3.22)$$

By virtue of $a), b)$ in Proposition 2.1 and Lemma 1.1, we derive that there exists a positive constant C such that

$$|J_0''| \le C\varepsilon t, |J_1''| \le C\varepsilon t.$$

Note that

$$\begin{aligned}
\mathcal{S}'' &= 2tJ_0^{-1}J_0'J_0^{-1}J_0'J_0^{-1}J_1 - tJ_0^{-1}J_0''J_0^{-1}J_1 - \qquad (3.23)\\
&\quad 2tJ_0^{-1}J_0'J_0^{-1}J_1' + tJ_0^{-1}J_1''.
\end{aligned}$$

Together with (3.17), (3.11) and Lemma 1.1, we infer that there exists a positive constant C such that

$$|\mathcal{S}''| \le C\varepsilon\frac{t^5}{\sin^3 t}, \forall t \in (0,\tau].$$

Step 4. Notice that the first row and the first column of the matrices $\ddot{\mathcal{S}} - \ddot{\bar{\mathcal{S}}}, \dot{\mathcal{S}}', \mathcal{S}', \dot{\mathcal{S}} - \dot{\bar{\mathcal{S}}}, \mathcal{S} - \bar{\mathcal{S}}, \mathcal{S}''$ all vanish. Using (3.6), (3.7), (3.8), (3.9), (3.14), (3.15), (3.21) and the fact that $|\xi_2^2\eta_1\eta_2|, |\xi_2\xi_3\eta_1\eta_2|, |\xi_3^2\eta_1\eta_2|, |\xi_1\xi_2\eta_1\eta_2|, |\xi_1\xi_3\eta_1\eta_2|$ are all controlled by $|\xi^\perp|^2 + |\eta^\perp|^2$, we get the desired result. This completes the proof of Theorem 3.1. □

3.3 A reinforced MTW condition

This section is devoted to showing that the *MTW tensor* on nearly spherical manifold has a reinforced lower bound. The result is formulated as follows.

Theorem 3.2. *Let (M, g) be a closed n-dimensional Riemannian manifold satisfying the curvature assumption (1.3). Then there exist some positive constants ε_0 and κ_0 such that if*

$$\|Riem - \frac{1}{2} g \owedge g\|_{C^2(M,g)} < \varepsilon_0.$$

Then for any $m \in M, \nu \in I(m)$ and any tangent vectors ξ, η in T_mM,

$$\mathcal{C}(m,\nu)(\xi,\eta) \geq \kappa_0(|\xi \wedge \eta|_m^2 + |\xi|_m^2|\eta \wedge \nu|_m^2 + |\xi \wedge \nu|_m^2|\eta|_m^2), \tag{3.24}$$

where $|\xi \wedge \eta|_m^2, |\eta \wedge \nu|_m^2$ and $|\xi \wedge \nu|_m^2$ stand for the squared areas of the parallelograms defined in T_mM,i.e.: $|\xi \wedge \eta|_m^2 = |\xi|_m^2|\eta|_m^2 - g_m(\xi,\eta)^2, |\eta \wedge \nu|_m^2 = |\eta|_m^2|\nu|_m^2 - g_m(\eta,\nu)^2, |\xi \wedge \nu|_m^2 = |\xi|_m^2|\nu|_m^2 - g_m(\xi,\nu)^2$.

Notice that Theorem 3.2 was established for $n = 2$ in [28]. Moreover, the round sphere $\mathbb{S}^n$ satisfies the curvature assumptions in Theorem 3.2, the associated result was established in [31]. The similar result was obtained under the C^4 perturbation of the round spheres in [45]. It is easy to see that Theorem 3.2 implies that the closed Riemannian manifold under the above hypotheses satisfies the $A3S$ condition.

As a direct consequence of Theorem 3.2, the $A3W$ condition holds on Riemannian product of nearly spherical manifolds.

Corallary 3.1. *Let M_1 and M_2 be two closed Riemannian manifolds of dimension $n_1 \geq 2$ and $n_2 \geq 2$ respectively. If M_1 and M_2 both satisfy the curvature assumptions as in Theorem 3.2, then the A3W condition holds on the Riemannian product manifold $M_1 \times M_2$. Moreover, the associated MTW tensor is non-negative.*

Proof of Theorem 3.2. Remark the tangent vector ν takes value in $I(m)$. To prove Theorem 3.2, we discuss three cases for MTW tensor: when the tangent vector ν is close to the origin, close to the focalization and rest cases. In the following, the length of the tangent vector ν is denoted by τ.
Step 1.We consider the behaviour of *MTW tensor* when ν is away from the zero and from the focalization.

Assume that $0 < \delta_1 \leq \tau \leq \delta_2 < t_F(m,\nu) \leq \pi$ and $0 < \varepsilon < \frac{\sin\delta_2}{4\delta_2\sqrt{n-1}}$. Since the function $\frac{t}{\sin t}$ is non-decreasing in the interval $[0,\pi)$, thus the condition of Theorem 3.1 is satisfied. Then

$$\mathcal{C}(m,\nu)(\xi,\eta) \quad \geq \quad \bar{\mathcal{C}}(m,\nu)(\xi,\eta) - C(\frac{\delta_2}{\sin\delta_2})^4\varepsilon(\xi_2^2 + \xi_3^2 + \eta_2^2).$$

As it was known that, on the round sphere, there exists a positive constant [31] such that

$$\begin{aligned} \bar{\mathcal{C}}(m,\nu)(\xi,\eta) \quad &\geq \quad \bar{\kappa}_0\left[(|\xi|^2|\eta|^2 - \langle\xi,\eta\rangle^2) + \tau^2(\xi_2^2 + \xi_3^2)|\eta|^2 + \tau^2|\xi|^2\eta_2^2\right] \\ &= \quad \bar{\kappa}_0\left[(|\xi|^2|\eta|^2 - \langle\xi,\eta\rangle^2) + \tau^2(\xi_2^2 + \xi_3^2 + \eta_2^2)\right]. \end{aligned}$$

As a direct consequence, we have

$$\begin{aligned}\mathcal{C}(m,\nu)(\xi,\eta) &\geq \bar{\kappa}_0(|\xi|^2|\eta|^2-\langle\xi,\eta\rangle^2)+(\bar{\kappa}_0\tau^2-\frac{C\delta_2^4}{\sin^4\delta_2}\varepsilon)(\xi_2^2+\xi_3^2+\eta_2^2)\\ &\geq \bar{\kappa}_0(|\xi|^2|\eta|^2-\langle\xi,\eta\rangle^2)+(\bar{\kappa}_0-\frac{C\delta_2^4}{\delta_1^2\sin^4\delta_2}\varepsilon)\tau^2(\xi_2^2+\xi_3^2+\eta_2^2)\\ &\geq \frac{\bar{\kappa}_0}{2}[|\xi|^2|\eta|^2-\langle\xi,\eta\rangle^2+\tau^2(\xi_2^2+\xi_3^2+\eta_2^2)].\end{aligned}$$

where the last inequality follows from the assumption $\varepsilon<\frac{\bar{\kappa}_0\delta_1^2\sin^4\delta_2}{2C\delta_2^4}$.

Then for $\delta_1\leq\tau\leq\delta_2$, we infer the existence of constant $\varepsilon_1=\frac{\bar{\kappa}_0\delta_1^2\sin^4\delta_2}{2C\delta_2^4}$ and $\kappa_1=\frac{\bar{\kappa}_0}{2}$ such that

$$\mathcal{C}(m,\nu)(\xi,\eta) \geq \kappa_1\left[(|\xi|^2|\eta|^2-\langle\xi,\eta\rangle^2)+\tau^2(\xi_2^2+\xi_3^2)|\eta|^2+\tau^2|\xi|^2\eta_2^2\right].$$

Step 2. We investigate the behaviour of $MTWtensor$ near the focalization.

Under the curvature assumption (1.3) and (1.4)(ε is small enough), one can derive by the method in [45] near the focalization that there exist positive numbers $\varepsilon_2,\kappa_2,\bar{\delta}_2(\frac{3\pi}{4}\leq\bar{\delta}_2<\pi)$ such that the $MTWtensor$ has the following estimate:

$$\varepsilon<\varepsilon_2,\tau\geq\bar{\delta}_2,\mathcal{C}(m,\nu)(\xi,\eta)\geq\kappa_2(|S^{\perp}\xi|^2|\eta|^2+\xi_1^2\eta_2^2), \tag{3.25}$$

where $S^{\perp}$ denotes the orthogonal projection of S on the orthogonal subspace $\nu^{\perp}$.

Recalling the curvature assumption (1.3), then the Hessian comparison theorem [19] infers

$$-\mathcal{S}^{\perp}\geq-\frac{\tau\cos\tau}{\sin\tau}I_{n-1}.$$

Hence the term $|\mathcal{S}^{\perp}\xi|^2$ controls $\xi_2^2+\xi_3^2$ if $\tau\geq\frac{3\pi}{4}$, i.e. $|\mathcal{S}^{\perp}\xi|^2\geq2(\xi_2^2+\xi_3^2)$. In addition, by Cauchy-Schwarz inequality, the term $|\xi_1\xi_2\eta_1\eta_2|$ is bounded by $\xi_1^2\eta_2^2+\frac{1}{4}\xi_2^2\eta_1^2$. From (3.25), up to the constant κ_2, we see that

$$\begin{aligned}\mathcal{C}(m,\nu)(\xi,\eta) &\geq \kappa_2\left[\frac{1}{2}\xi_1^2\eta_2^2+\frac{1}{4}\xi_2^2\eta_1^2-\frac{1}{2}\xi_1\xi_2\eta_1\eta_2+(\xi_2^2+\xi_3^2)|\eta|^2\right]\\ &= \kappa_2\left[\frac{1}{4}(|\xi|^2|\eta|^2-\langle\xi,\eta\rangle^2)+\frac{1}{4}\xi_1^2\eta_2^2+\xi_2^2|\eta|^2+\frac{3}{4}\xi_3^2|\eta|^2\right]\\ &\geq \kappa_2\left[(|\xi|^2|\eta|^2-\langle\xi,\eta\rangle^2)+(\xi_2^2+\xi_3^2)|\eta|^2+|\xi|^2\eta_2^2\right]\\ &\geq \kappa_2\left[(|\xi|^2|\eta|^2-\langle\xi,\eta\rangle^2)+\tau^2(\xi_2^2+\xi_3^2)|\eta|^2+\tau^2|\xi|^2\eta_2^2\right],\end{aligned}$$

where the last inequality holding due to $\tau<\pi$.

We derive the existence of constant ε_2,κ_2 and $\bar{\delta}_2$ such that for $0<\varepsilon<\varepsilon_2,\bar{\delta}_2\leq\tau<t_F(m,\nu)$

$$\mathcal{C}(m,\nu)(\xi,\eta) \geq \kappa_2\left[(|\xi|^2|\eta|^2-\langle\xi,\eta\rangle^2)+\tau^2(\xi_2^2+\xi_3^2)|\eta|^2+\tau^2|\xi|^2\eta_2^2\right].$$

Step 3.In the last step, we examine the asymptotic behaviour of MTW tensor when ν is near the origin. From (3.5), we know that it suffices to study the expansion of the coefficients a_{11},a_{12} and a_{22}. By definition, the coefficients a_{11},a_{12} and a_{22} contain $\mathcal{S},\dot{\mathcal{S}},\mathcal{S}',\dot{\mathcal{S}}',\ddot{\mathcal{S}},\mathcal{S}''$. As a result, we only need to deal with the expansion of $\mathcal{S},\dot{\mathcal{S}},\mathcal{S}',\dot{\mathcal{S}}',\ddot{\mathcal{S}},\mathcal{S}''$. It is clear that $\mathcal{S}_j^1(\theta,t)=\mathcal{S}_1^j(\theta,t)=\delta_j^1$. Without loss of generality, we address the expansion of $\mathcal{S}_j^i,\dot{\mathcal{S}}_j^i,\mathcal{S}_j^{i'},\dot{\mathcal{S}}_j^{i'},\ddot{\mathcal{S}}_j^i,\mathcal{S}_j^{i''}$ with $i,j\geq2$.

In the following, given a real function f, we write $f=O(\varepsilon t^p+t^q)(p,q>0)$ if there exists two positive constant μ_0,C which are both independent of ε such that $\forall t\in[0,\mu_0),|f(t)|\leq C(\varepsilon t^p+t^q)$. The notation $f=O(\varepsilon t^p)(p>0)$ or $f=O(t^q)(q>0)$ can be defined in the similar way.

We first take account of the asymptotic behavior of the coefficient a_{11} which involves $\ddot{\mathcal{S}}$.

Lemma 3.1. *Under the curvature assumptions (1.3) and (1.4), on the axis, we have*

$$\mathcal{S}(0,t) = I_n - \frac{t^2}{3}R(0,0) - \frac{t^3}{12}\dot{R}(0,0) - \frac{t^4}{45}A + O(\varepsilon t^4 + t^6), \tag{3.26}$$

$$\dot{\mathcal{S}}(0,t) = -\frac{2}{3}tR(0,0) - \frac{t^2}{4}\dot{R}(0,0) - \frac{4}{45}t^3A + O(\varepsilon t^3 + t^5), \tag{3.27}$$

$$\ddot{\mathcal{S}}(0,t) = -\frac{2}{3}R(0,0) - \frac{t}{2}\dot{R}(0,0) - \frac{4}{15}t^2A + O(\varepsilon t^2 + t^4). \tag{3.28}$$

where $A = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-1} \end{bmatrix}$.

Remark 3.2. *(1).The formula (3.26) can also recover the expression of the MTW tensor in the special case $\nu = 0$. Using a Riemannian normal coordinate system at m, it follows from the definition of the MTW tensor (0.2) that*

$$\begin{aligned} \mathcal{C}(m,0)(\xi,\eta) &= -\frac{3}{2}\frac{d^2}{ds^2}|_{s=0}\langle \mathcal{S}(m,\eta,s)(\xi),\xi\rangle \\ &= \langle R(0,0)\xi,\xi\rangle = R_m(\xi,\eta,\xi,\eta). \end{aligned}$$

Before showing Lemma 3.1, we give some facts about the coefficient a_{11}. Since the first row and first column of $\ddot{\mathcal{S}}$ vanish, the coefficient a_{11} is independent of ξ_1^2.

As a consequence of (3.28), one has the expansion of a_{11}

$$\begin{aligned} a_{11}(m,\nu,\xi) &= \langle R(0,0)\xi,\xi\rangle + \frac{3}{4}\tau\langle \dot{R}(0,0)\xi,\xi\rangle + \frac{2}{5}\tau^2\langle A\xi,\xi\rangle + \\ &\quad O(\varepsilon\tau^2 + \tau^4)(\xi_2^2 + \xi_3^2) \\ &= R_{1212}\xi_2^2 + 2R_{1213}\xi_2\xi_3 + R_{1313}\xi_3^2 + \\ &\quad \frac{3}{4}\tau(\nabla_1 R_{1212}\xi_2^2 + 2\nabla_1 R_{1213}\xi_2\xi_3 + \nabla_1 R_{1313}\xi_3^2) + \\ &\quad \frac{2}{5}\tau^2(\xi_2^2+\xi_3^2) + O(\varepsilon\tau^2+\tau^4)(\xi_2^2+\xi_3^2) \\ &= R(\xi,E_1,\xi,E_1) + \frac{3}{4}\tau(\nabla_1 R_{1212}\xi_2^2 + 2\nabla_1 R_{1213}\xi_2\xi_3 + \\ &\quad \nabla_1 R_{1313}\xi_3^2) + \frac{2}{5}\tau^2(\xi_2^2+\xi_3^2) + O(\varepsilon\tau^2+\tau^4)(\xi_2^2+\xi_3^2). \end{aligned} \tag{3.29}$$

Proof of Lemma 3.1. First, we examine the expansion of $\mathcal{S}$. Let $\mathbb{J}_0$ and $\mathbb{J}_1$ be the solutions of the following second order differential equations:

$$\ddot{\mathbb{J}}_a(t) + R(0,0)\mathbb{J}_a(t) = 0, a = 0,1.$$

with the initial conditions

$$\mathbb{J}_0(0) = 0, \dot{\mathbb{J}}_0(0) = I_n,$$

$$\mathbb{J}_1(0) = I_n, \dot{\mathbb{J}}_1(0) = 0.$$

Obviously, the matrices J_0 and J_1 satisfy the equations

$$\ddot{J}_a(t) + R(0,0)J_a(t) = (R(0,0) - R(0,t))J_a(t), a = 0,1.$$

Using the representation formula (Proposition 1.1 (c)), we derive

$$J_0 = \mathbb{J}_0 + \mathbb{J}_0\int_0^t \mathbb{J}_1^*[R(0,0) - R(0,s)]J_0 ds - \mathbb{J}_1\int_0^t \mathbb{J}_0^*[R(0,0) - R(0,s)]J_0 ds,$$

$$J_1 = \mathbb{J}_1 + \mathbb{J}_0\int_0^t \mathbb{J}_1^*[R(0,0) - R(0,s)]J_1 ds - \mathbb{J}_1\int_0^t \mathbb{J}_0^*[R(0,0) - R(0,s)]J_1 ds.$$

Notice that the matrices $\mathbb{J}_0$ and $\mathbb{J}_1$ have the following expansions

$$\begin{aligned} \mathbb{J}_0(t) &= tI_n - \frac{t^3}{6}R(0,0) + \frac{t^5}{120}R^2(0,0) + O(t^7), \\ \mathbb{J}_1(t) &= I_n - \frac{t^2}{2}R(0,0) + \frac{t^4}{24}R^2(0,0) + O(t^6). \end{aligned}$$

In addition, from the Taylor formula and b) in Proposition 2.1, we see that

$$\begin{aligned} R(0,t) &= R(0,0) + t\dot{R}(0,0) + \int_0^t (t-s)\ddot{R}(0,s)ds, \\ &= R(0,0) + t\dot{R}(0,0) + \int_0^t (t-s)\nabla_{11}^2 R(0,s)ds, \\ &= R(0,0) + t\dot{R}(0,0) + O(\varepsilon t^2). \end{aligned}$$

Thus

$$\begin{aligned} J_0(0,t) &= \mathbb{J}_0(t) - \frac{t^4}{12}\dot{R}(0,0) + O(\varepsilon t^5) \\ &= tI_n - \frac{t^3}{6}R(0,0) - \frac{t^4}{12}\dot{R}(0,0) + \frac{t^5}{120}R^2(0,0) + O(\varepsilon t^5 + t^7) \\ &= tI_n - \frac{t^3}{6}R(0,0) - \frac{t^4}{12}\dot{R}(0,0) + \frac{t^5}{120}A + O(\varepsilon t^5 + t^7), \qquad (3.30) \\ J_1(0,t) &= \mathbb{J}_1(t) - \frac{t^3}{6}\dot{R}(0,0) + O(\varepsilon t^4) \\ &= tI_n - \frac{t^2}{2}R(0,0) - \frac{t^3}{6}\dot{R}(0,0) + \frac{t^4}{24}R^2(0,0) + O(\varepsilon t^4 + t^6) \\ &= I_n - \frac{t^2}{2}R(0,0) - \frac{t^3}{6}\dot{R}(0,0) + \frac{t^4}{24}A + O(\varepsilon t^4 + t^6). \qquad (3.31) \end{aligned}$$

Then

$$\begin{aligned} \mathcal{S}(0,t) &= tJ_0(0,t)^{-1}J_1(0,t) \\ &= t[tI_n - \frac{t^3}{6}R(0,0) - \frac{t^4}{12}\dot{R}(0,0)) + \frac{t^5}{120}A + O(\varepsilon t^5 + t^7)]^{-1}[I_n - \\ &\quad \frac{t^2}{2}R(0,0) - \frac{t^3}{6}\dot{R}(0,0) + \frac{t^4}{24}A + O(\varepsilon t^4 + t^6)] \\ &= [I_n + \frac{t^2}{6}R(0,0) + \frac{t^3}{12}\dot{R}(0,0) + \frac{7}{360}t^4A + O(\varepsilon t^4 + t^6)][I_n - \\ &\quad \frac{t^2}{2}R(0,0) - \frac{t^3}{6}\dot{R}(0,0) + \frac{t^4}{24}A + O(\varepsilon t^4 + t^6)] \\ &= I_n - \frac{t^2}{3}R(0,0) - \frac{t^3}{12}\dot{R}(0,0) - \frac{t^4}{45}A + O(\varepsilon t^4 + t^6). \end{aligned}$$

We are in position to deal with the expansion of $\dot{\mathcal{S}}$. It is easy to see that $\dot{\mathcal{S}} = (I_n - tJ_0^{-1}\dot{J}_0)J_0^{-1}J_1 + tJ_0^{-1}\dot{J}_1$ which involves $J_0^{-1}\dot{J}_0$ and $J_0^{-1}\dot{J}_1$. We shall consider the expansions of $J_0^{-1}\dot{J}_0$ and $J_0^{-1}\dot{J}_1$.
Differentiating (1.7) with respect to t,

$$\begin{cases} \ddot{\dot{J}}_a + R\dot{J}_a = -\dot{R}J_a, \ a = 0,1 \\ \dot{J}_0(0) = I_n, \ddot{J}_0(0) = 0, \\ \dot{J}_1(0) = 0, \ddot{J}_1(0) = -R(0,0). \end{cases}$$

Making use of the representation formula (Proposition 1.1 (c)) again,

$$\begin{aligned} \dot{J}_0 &= J_1 - J_0\int_0^t J_1^*\dot{R}J_0 ds + J_1\int_0^t J_0^*\dot{R}J_0 ds, \\ \dot{J}_1 &= -J_0R(0,0) - J_0\int_0^t J_1^*\dot{R}J_1 ds + J_1\int_0^t J_0^*\dot{R}J_1 ds. \end{aligned}$$

Then

$$\begin{aligned} tJ_0^{-1}\dot{J}_0 &= tJ_0^{-1}J_1 - t\int_0^t J_1^*\dot{R}J_0 ds + tJ_0^{-1}J_1\int_0^t J_0^*\dot{R}J_0 ds \\ &= \mathcal{S}(0,t) - t\int_0^t J_1^*\dot{R}J_0 ds + \mathcal{S}(0,t)\int_0^t J_0^*\dot{R}J_0 ds, \end{aligned} \tag{3.32}$$

$$\begin{aligned} J_0^{-1}\dot{J}_1 &= -R(0,0) - \int_0^t J_1^*\dot{R}J_1 ds + J_0^{-1}J_1\int_0^t J_0^*\dot{R}J_1 ds \\ &= -R(0,0) - \int_0^t J_1^*\dot{R}J_1 ds + J_0^{-1}J_1\int_0^t J_0^*\dot{R}J_1 ds. \end{aligned} \tag{3.33}$$

By b) in Proposition 2.1, it follows that

$$\begin{aligned} \dot{R}(0,t) &= \dot{R}(0,0) + \int_0^t \ddot{R}(0,s)ds \\ &= \dot{R}(0,0) + \int_0^t \nabla_{11}^2 R(0,s)ds \\ &= \dot{R}(0,0) + O(\varepsilon t). \end{aligned}$$

Then

$$\begin{aligned} tJ_0^{-1}\dot{J}_0(0,t) &= \mathcal{S}(0,t) - \frac{t^3}{2}\dot{R}(0,0) + \mathcal{S}(0,t)\frac{t^3}{3}\dot{R}(0,0) + O(\varepsilon t^4) \\ &= I_n - \frac{t^2}{3}R(0,0) - \frac{t^3}{4}\dot{R}(0,0) - \frac{t^4}{45}A + O(\varepsilon t^4 + t^6), \\ J_0^{-1}\dot{J}_1(0,t) &= -R(0,0) - \frac{t}{2}\dot{R}(0,0) + O(\varepsilon t^2 + t^4). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \dot{\mathcal{S}} &= (I_n - tJ_0^{-1}\dot{J}_0)J_0^{-1}J_1 + tJ_0^{-1}\dot{J}_1 \\ &= [\frac{t^2}{3}R(0,0) + \frac{t^3}{4}\dot{R}(0,0) + \frac{t^4}{45}A + O(\varepsilon t^4 + t^6)][tI_n - \frac{t^3}{6}R(0,0) - \\ &\quad \frac{t^4}{12}\dot{R}(0,0) + \frac{t^5}{120}A + O(\varepsilon t^5 + t^7)]^{-1}[I_n - \frac{t^2}{2}R(0,0) - \frac{t^3}{6}\dot{R}(0,0) + \\ &\quad \frac{t^4}{24}A + O(\varepsilon t^4 + t^6)] + t[-R(0,0) - \frac{t}{2}\dot{R}(0,0) + O(\varepsilon t^2 + t^4)] \\ &= [\frac{t}{3}R(0,0) + \frac{t^2}{4}\dot{R}(0,0) + \frac{t^3}{45}A + O(\varepsilon t^3 + t^5)][I_n + \frac{t^2}{6}R(0,0) + \\ &\quad \frac{t^3}{12}\dot{R}(0,0) + \frac{7t^4}{360}A + O(\varepsilon t^4 + t^6)][I_n - \frac{t^2}{2}R(0,0) - \frac{t^3}{6}\dot{R}(0,0) + \\ &\quad \frac{t^4}{24}A + O(\varepsilon t^4 + t^6)] - tR(0,0) - \frac{t^2}{2}\dot{R}(0,0) + O(\varepsilon t^3 + t^5) \\ &= [\frac{t}{3}R(0,0) + \frac{t^2}{4}\dot{R}(0,0) + \frac{7t^3}{90}A + O(\varepsilon t^3 + t^5)][I_n - \frac{t^2}{2}R(0,0) - \\ &\quad \frac{t^3}{6}\dot{R}(0,0) + \frac{t^4}{24}A + O(\varepsilon t^4 + t^6)] - tR(0,0) - \frac{t^2}{2}\dot{R}(0,0) + O(\varepsilon t^3 + t^5) \\ &= \frac{t}{3}R(0,0) + \frac{t^2}{4}\dot{R}(0,0) - \frac{4t^3}{45}A - tR(0,0) - \frac{t^2}{2}\dot{R}(0,0) + O(\varepsilon t^3 + t^5) \\ &= -\frac{2}{3}tR(0,0) - \frac{t^2}{4}\dot{R}(0,0) - \frac{4}{45}t^3 A + O(\varepsilon t^3 + t^5). \end{aligned}$$

Similarly, from (3.12), we know that

$$\begin{aligned} \ddot{\mathcal{S}} &= 2(I_n - tJ_0^{-1}\dot{J}_0)J_0^{-1}\dot{J}_1 + 2(tJ_0^{-1}\dot{J}_0)\frac{1}{t^2}(tJ_0^{-1}\dot{J}_0 - I_n)\mathcal{S} \\ &= 2[\frac{t^2}{3}R(0,0) + \frac{t^3}{4}\dot{R}(0,0) + \frac{t^4}{45}A + O(\varepsilon t^4 + t^6)][-R(0,0) - \end{aligned}$$

$$
\begin{aligned}
&\frac{t}{2}\dot{R}(0,0) + O(\varepsilon t^2 + t^4)] + 2[I_n - \frac{t^2}{3}R(0,0) - \frac{t^3}{4}\dot{R}(0,0) - \\
&\frac{t^4}{45}A + O(\varepsilon t^4 + t^6)][-\frac{1}{3}R(0,0) - \frac{t}{4}\dot{R}(0,0) - \frac{t^2}{45}A + \\
&O(\varepsilon t^2 + t^4)][I_n - \frac{t^2}{3}R(0,0) - \frac{t^3}{12}\dot{R}(0,0) - \frac{t^4}{45}A + O(\varepsilon t^4 + t^6)] \\
= \; & -\frac{2}{3}t^2R^2(0) + O(\varepsilon t^2 + t^4) + 2[-\frac{1}{3}R(0,0) - \frac{t}{4}\dot{R}(0,0) + \\
&\frac{4t^2}{45}A + O(\varepsilon t^2 + t^4)][I_n - \frac{t^2}{3}R(0,0) - \frac{t^3}{12}\dot{R}(0,0) - \frac{t^4}{45}A + O(\varepsilon t^4 + t^6)] \\
= \; & -\frac{2}{3}t^2R^2(0) - \frac{2}{3}R(0,0) - \frac{t}{2}\dot{R}(0,0) + \frac{2}{5}t^2A + O(\varepsilon t^2 + t^4) \\
= \; & -\frac{2}{3}R(0,0) - \frac{t}{2}\dot{R}(0,0) - \frac{4}{15}t^2A + O(\varepsilon t^2 + t^4).
\end{aligned}
$$

This ends the proof of Lemma 3.1. □

We are in position to consider the expansion of the coefficient a_{12}. It is easy to see that the coefficient a_{12} involves the terms $\mathcal{S}'$ and $\dot{\mathcal{S}}'$. We shall examine the expansions of $\mathcal{S}'$ and $\dot{\mathcal{S}}'$.

Lemma 3.2. *Under the curvature assumptions (1.3) and (1.4), on the axis, we have*

$$
\mathcal{S}'(0,t) = -\frac{t^2}{3}R'(0,0) - \frac{t^3}{12}\dot{R}'(0,0) + O(\varepsilon t^4 + t^6), \tag{3.34}
$$

$$
\dot{\mathcal{S}}'(0,t) = -\frac{2}{3}tR'(0,0) - \frac{t^2}{4}\dot{R}'(0,0) + O(\varepsilon t^3 + t^5). \tag{3.35}
$$

As a consequence of Lemma 3.2, we can derive the expansion of the coefficient a_{12}. Noting that $\langle \xi, P\xi \rangle = 0$, thus the coefficient a_{12} takes the form

$$
\begin{aligned}
a_{12}(m,\nu,\xi) \; = \; & -\frac{3}{\tau}\langle \dot{\mathcal{S}}'\xi, \xi\rangle - \frac{6}{\tau}\langle \dot{\mathcal{S}}\xi, P^{\perp}\xi\rangle + \frac{3}{\tau^2}\langle S'\xi, \xi\rangle + \\
&\frac{6}{\tau^2}\langle (\mathcal{S} - I_n)\xi, P^{\perp}\xi\rangle.
\end{aligned}
$$

Since the first row and first column of $\dot{\mathcal{S}}', \dot{\mathcal{S}}, \mathcal{S}'$ and $\mathcal{S} - I_n$ vanish, the coefficient a_{12} is also independent of ξ_1^2.

Plugging (3.35)(3.27)(3.34) and (3.26)into the above expression, we get

$$
\begin{aligned}
a_{12}(m,\nu,\xi) \; = \; & 2\langle R'(0,0)\xi, \xi\rangle + \frac{3}{4}\tau\langle \dot{R}'(0,0)\xi, \xi\rangle + \\
&4\langle R(0,0)\xi, P^{\perp}\xi\rangle + \frac{3}{2}\tau\langle \dot{R}(0,0)\xi, P^{\perp}\xi\rangle + \\
&\frac{8}{15}\tau^2\langle A\xi, P^{\perp}\xi\rangle - \langle R'(0,0)\xi, \xi\rangle - \frac{\tau}{4}\langle \dot{R}'(0,0)\xi, \xi\rangle - \\
&2\langle R(0,0)\xi, P^{\perp}\xi\rangle - \frac{\tau}{2}\langle \dot{R}(0,0)\xi, P^{\perp}\xi\rangle - \\
&\frac{2}{15}\tau^2\langle A\xi, P^{\perp}\xi\rangle + \\
&O(\varepsilon t^2 + t^4)(\xi_2^2 + \xi_3^2 + \xi_1\xi_2 + \xi_1\xi_3) \\
= \; & \langle R'(0,0)\xi, \xi\rangle + 2\langle R(0,0)\xi, P^{\perp}\xi\rangle + \\
&+\tau(\frac{1}{2}\langle \dot{R}'(0,0)\xi, \xi\rangle + \langle \dot{R}(0,0)\xi, P^{\perp}\xi\rangle) + \\
&\frac{2}{5}\tau^2\langle A\xi, P^{\perp}\xi\rangle + O(\varepsilon t^2 + t^4)(\xi_2^2 + \xi_3^2 + \xi_1\xi_2 + \xi_1\xi_3).
\end{aligned}
$$

Using (1) in Remark 2.1, we get that

$$\begin{aligned} a_{12}(m,\nu,\xi) &= 2R_{1223}\xi_2\xi_3 + 2R_{1323}\xi_3^2 + 2(-R_{1212}\xi_1\xi_2 - R_{1213}\xi_1\xi_3) + \\ &\quad \tau[\frac{1}{2}\nabla_2 R_{1212}\xi_2^2 + (\nabla_1 R_{1223} + \nabla_2 R_{1213})\xi_2\xi_3 + (\nabla_1 R_{1323} + \\ &\quad \frac{1}{2}\nabla_2 R_{1313})\xi_3^2 - \nabla_1 R_{1212}\xi_1\xi_2 - \nabla_1 R_{1213}\xi_1\xi_3] - \\ &\quad \frac{2}{5}\tau^2\xi_1\xi_2 + O(\varepsilon\tau^2+\tau^4)(\xi_2^2+\xi_3^2+\xi_1\xi_2+\xi_1\xi_3) \\ &= 2R(\xi,E_1,\xi,E_2) + \tau[\frac{1}{2}\nabla_2 R_{1212}\xi_2^2 + (\nabla_1 R_{1223} + \\ &\quad \nabla_2 R_{1213})\xi_2\xi_3 + (\nabla_1 R_{1323} + \frac{1}{2}\nabla_2 R_{1313})\xi_3^2 - \\ &\quad \nabla_1 R_{1212}\xi_1\xi_2 - \nabla_1 R_{1213}\xi_1\xi_3] - \frac{2}{5}\tau^2\xi_1\xi_2 + \\ &\quad O(\varepsilon\tau^2+\tau^4)(\xi_2^2+\xi_3^2+\xi_1\xi_2+\xi_1\xi_3). \end{aligned} \tag{3.36}$$

Proof of Lemma 3.2. First, we handle the expansion of $\mathcal{S}'$. One can prove that $\mathcal{S}' = -tJ_0^{-1}J_0'J_0^{-1}J_1 + tJ_0^{-1}J_1'$ which involves the terms $J_0^{-1}J_0'$ and $J_0^{-1}J_1'$, thus we shall investigate the expansion of these two terms.

Recall that we can use the representations (3.16) which contain the term R' for the matrices J_0' and J_1'. We shall address the expansion of R'.

Notice that the second order derivative $\ddot{R}'(0,0)$ of R' can be computed in the following way

$$\ddot{R}'(0,0) = 2\lim_{t\to 0^+}\frac{R'(0,t) - R'(0,0) - t\dot{R}'(0,0)}{t^2}.$$

By the definition of the limit, for any $\mu > 0$, there exists a small positive constant $\delta > 0$ which depends on μ such that

$$\forall t\in[0,\delta), |R'(0,t) - R'(0,0) - t\dot{R}'(0,0)| \le (\frac{1}{2}|\ddot{R}'(0,0)| + \mu)t^2.$$

From *c*) in Proposition 2.1, we derive that there exists a positive constant $C > 0$ such that

$$\forall t\in[0,\delta), |R'(0,t) - R'(0,0) - t\dot{R}'(0,0)| \le (C\varepsilon + \mu)t^2.$$

Assume that $\varepsilon < \mu$,

$$\forall t\in[0,\delta), |R'(0,t) - R'(0,0) - t\dot{R}'(0,0)| \le (C+1)\mu t^2.$$

That is

$$R'(0,t) = R'(0,0) + t\dot{R}'(0,0) + O(\mu t^2).$$

Without the confusion, we write

$$R'(0,t) = R'(0,0) + t\dot{R}'(0,0) + O(\varepsilon t^2). \tag{3.37}$$

It follows that

$$\begin{aligned} J_0^{-1}J_0'(0,t) &= -\int_0^t J_1^* R' J_0 ds + J_0^{-1}J_1\int_0^t J_0^* R' J_0 ds \\ &= -\frac{t^2}{6}R'(0,0) - \frac{t^3}{12}\dot{R}'(0,0) + O(\varepsilon t^4 + t^6), \end{aligned} \tag{3.38}$$

$$\begin{aligned} J_0^{-1}J_1'(0,t) &= -\int_0^t J_1^* R' J_1 ds + J_0^{-1}J_1\int_0^t J_0^* R' J_1 ds \\ &= -\frac{t}{2}R'(0,0) - \frac{t^2}{6}\dot{R}'(0,0) + O(\varepsilon t^3 + t^5). \end{aligned} \tag{3.39}$$

As a consequence,

$$
\begin{aligned}
\mathcal{S}' &= -tJ_0^{-1}J_0'J_0^{-1}J_1 + tJ_0^{-1}J_1' \\
&= -J_0^{-1}J_0'\mathcal{S} + tJ_0^{-1}J_1' \\
&= -[-\frac{t^2}{6}R'(0,0) - \frac{t^3}{12}\dot{R}'(0,0) + O(\varepsilon t^4 + t^6)][I_n - \frac{t^2}{3}R(0,0) - \\
&\quad \frac{t^3}{12}\dot{R}(0,0) - \frac{t^4}{45}A + O(\varepsilon t^4 + t^6)] - \frac{t^2}{2}R'(0,0) - \frac{t^3}{6}\dot{R}'(0,0) + O(\varepsilon t^4 + t^6) \\
&= \frac{t^2}{6}R'(0,0) + \frac{t^3}{12}\dot{R}'(0,0) - \frac{t^2}{2}R'(0) - \frac{t^3}{6}\dot{R}'(0,0) + O(\varepsilon t^4 + t^6) \\
&= -\frac{t^2}{3}R'(0,0) - \frac{t^3}{12}\dot{R}'(0,0) + O(\varepsilon t^4 + t^6).
\end{aligned}
$$

Secondly, we deal with the expansion of $\dot{\mathcal{S}}'$. Since $\dot{\mathcal{S}}'$ involves $J_0^{-1}\dot{J}_0'$ and $J_0^{-1}\dot{J}_1'$, thus we shall investigate the expansions of these two terms.

We use again the representations (3.18) and (3.19) for $\dot{J}_0'$ and $\dot{J}_1'$. Therefore, from $c)$ in Proposition 2.1, the first order derivative formula $\ddot{R}'(0,0) = \lim_{t\to 0^+} \frac{\dot{R}'(0,t) - \dot{R}'(0,0)}{t}$ of $\dot{R}'$ and by the definition of the limit, we get

$$
\dot{R}'(0,t) = \dot{R}'(0,0) + O(\varepsilon t). \tag{3.40}
$$

Combining (3.30) (3.38) and (3.39), we know that

$$
\begin{aligned}
J_0'(0,t) &= -\frac{t^3}{6}R'(0,0) - \frac{t^4}{12}\dot{R}'(0,0) + O(\varepsilon t^5 + t^7), \\
J_1'(0,t) &= -\frac{t^2}{2}R'(0,0) - \frac{t^3}{6}\dot{R}'(0,0) + O(\varepsilon t^4 + t^6).
\end{aligned}
$$

Together with (3.30)(3.31) (3.37)(3.40) and $\dot{R}(0,t) = \dot{R}(0,0) + O(\varepsilon t)$, it follows that

$$
\begin{aligned}
J_0^{-1}\dot{J}_0' &= -\int_0^t J_1^*(\dot{R}'J_0 + R'\dot{J}_0 + \dot{R}J_0')ds + \\
&\quad J_0^{-1}J_1\int_0^t J_0^*(\dot{R}'J_0 + R'\dot{J}_0 + \dot{R}J_0')ds \\
&= -\frac{t}{2}R'(0,0) - \frac{t^2}{3}\dot{R}'(0,0) + O(\varepsilon t^3 + t^5), \qquad (3.41) \\
J_0^{-1}\dot{J}_1' &= -R'(0,0) - \int_0^t J_1^*(\dot{R}'J_1 + R'\dot{J}_1 + \dot{R}J_1')ds + \\
&\quad J_0^{-1}J_1\int_0^t J_0^*(\dot{R}'J_1 + R'\dot{J}_1 + \dot{R}J_1')ds \\
&= -R'(0,0) - \frac{t}{2}\dot{R}'(0,0) + O(\varepsilon t^2 + t^4). \qquad (3.42)
\end{aligned}
$$

From(3.20), we obtain

$$
\begin{aligned}
\dot{\mathcal{S}}' &= -J_0^{-1}J_0'J_0^{-1}J_1 + tJ_0^{-1}\dot{J}_0J_0^{-1}J_0'J_0^{-1}J_1 - \\
&\quad tJ_0^{-1}\dot{J}_0'J_0^{-1}J_1 + tJ_0^{-1}J_0'J_0^{-1}\dot{J}_0J_0^{-1}J_1 - \\
&\quad tJ_0^{-1}J_0'J_0^{-1}\dot{J}_1 + J_0^{-1}J_1' - tJ_0^{-1}\dot{J}_0J_0^{-1}J_1' + \\
&\quad tJ_0^{-1}\dot{J}_1' \\
&= (tJ_0^{-1}\dot{J}_0 - I_n)J_0^{-1}J_0'J_0^{-1}J_1 - J_0^{-1}\dot{J}_0'\mathcal{S} + \\
&\quad J_0^{-1}J_0'J_0^{-1}\dot{J}_0\mathcal{S} - tJ_0^{-1}J_0'J_0^{-1}\dot{J}_1 + \\
&\quad (I_n - tJ_0^{-1}\dot{J}_0)J_0^{-1}J_1' + tJ_0^{-1}\dot{J}_1'.
\end{aligned}
$$

Plugging (3.32)(3.38)(3.26)(3.41)(3.33)(3.39)(3.33) into the above expression, we see that

$$\begin{aligned}
\dot{\mathcal{S}}'(0,t) &= [-\frac{t^2}{3}R(0,0)-\frac{t^3}{4}\dot{R}(0,0)-\frac{t^4}{45}A+O(\varepsilon t^4+t^6)][-\frac{t}{6}R'(0,0)- \\
&\quad \frac{t^2}{12}\dot{R}'(0,0)+O(\varepsilon t^3+t^5)][I_n-\frac{t^2}{3}R(0,0)-\frac{t^3}{12}\dot{R}(0,0)-\frac{t^4}{45}A+ \\
&\quad O(\varepsilon t^4+t^6)]-[-\frac{t}{2}R'(0,0)-\frac{t^2}{3}\dot{R}'(0,0)+O(\varepsilon t^3+t^5)][I_n- \\
&\quad \frac{t^2}{3}R(0,0)-\frac{t^3}{12}\dot{R}(0,0)-\frac{t^4}{45}A+O(\varepsilon t^4+t^6)]+[-\frac{t}{6}R'(0,0)- \\
&\quad \frac{t^2}{12}\dot{R}'(0,0)+O(\varepsilon t^3+t^5)][I_n-\frac{t^2}{3}R(0,0)-\frac{t^3}{4}\dot{R}(0,0)-\frac{t^4}{45}A+ \\
&\quad O(\varepsilon t^4+t^6)][I_n-\frac{t^2}{3}R(0,0)-\frac{t^3}{12}\dot{R}(0,0)-\frac{t^4}{45}A+O(\varepsilon t^4+t^6)]- \\
&\quad t[-\frac{t^2}{6}R'(0,0)-\frac{t^3}{12}\dot{R}'(0,0)+O(\varepsilon t^4+t^6)][-R(0,0)-\frac{t}{2}\dot{R}(0,0)+ \\
&\quad O(\varepsilon t^2+t^4)]+[\frac{t^2}{3}R(0,0)+\frac{t^3}{4}\dot{R}(0,0)+\frac{t^4}{45}A+O(\varepsilon t^4+t^6)] \\
&\quad [-\frac{t}{2}R'(0,0)-\frac{t^2}{6}\dot{R}'(0,0)+O(\varepsilon t^3+t^5)]-tR'(0,0)-\frac{t^2}{2}\dot{R}'(0,0)+ \\
&\quad O(\varepsilon t^3+t^5) \\
&= \frac{t}{2}R'(0,0)+\frac{t^2}{3}\dot{R}'(0,0)+[-\frac{t}{6}R'(0,0)-\frac{t^2}{12}\dot{R}'(0,0)+O(\varepsilon t^3+t^5)] \\
&\quad [I_n-\frac{t^2}{3}R(0,0)-\frac{t^3}{12}\dot{R}(0,0)-\frac{t^4}{45}A+O(\varepsilon t^4+t^6)]-tR'(0,0)- \\
&\quad \frac{t^2}{2}\dot{R}'(0,0)+O(\varepsilon t^3+t^5) \\
&= -\frac{t}{6}R'(0,0)-\frac{t^2}{12}\dot{R}'(0,0)-\frac{t}{2}R'(0,0)-\frac{t^2}{6}\dot{R}'(0,0)+O(\varepsilon t^3+t^5) \\
&= -\frac{2}{3}tR'(0,0)-\frac{t^2}{4}\dot{R}'(0,0)+O(\varepsilon t^3+t^5).
\end{aligned}$$

This ends the proof of Lemma 3.2. □

We now consider the expansion of the coefficient a_{22}. The coefficient a_{22} involves the term $\mathcal{S}''$. We shall examine the expansion of $\mathcal{S}''$.

Lemma 3.3. *Under the curvature assumption (1.3) and (1.4), on the axis, we have*

$$\mathcal{S}''(0,t) = -\frac{t^2}{3}R''(0,0)-\frac{t^3}{12}\dot{R}''(0,0)+O(\varepsilon t^4+t^6). \tag{3.43}$$

As a consequence of Lemma 3.3, we can derive the expansion of the coefficient a_{22}. Noting that $\langle P^{\perp}\xi, P^{\perp}\xi\rangle = \langle \xi, P\xi\rangle$, the coefficient a_{22} takes the form

$$\begin{aligned}
a_{22}(m,\nu,\xi) &= -\frac{3}{2\tau^2}\langle \mathcal{S}''\xi,\xi\rangle-\frac{3}{2\tau}\langle\dot{\mathcal{S}}\xi,\xi\rangle-\frac{6}{\tau^2}\langle\mathcal{S}'\xi,P^{\perp}\xi\rangle- \\
&\quad \frac{3}{\tau^2}\langle(\mathcal{S}-I_n)P^{\perp}\xi,P^{\perp}\xi\rangle+\frac{3}{\tau^2}\langle(\mathcal{S}-I_n)\xi,P\xi\rangle]\eta_2^2.
\end{aligned}$$

Note that, through the first row and first column of $\mathcal{S}'', \dot{\mathcal{S}}, \dot{\mathcal{S}}', \mathcal{S}-I_n$ and $\mathcal{S}-I_n$ vanish, the coefficient a_{22} depends on ξ_1^2.

Plugging (3.43) (3.27)(3.34) and (3.26)into the above expression, we get the following expansion:

$$\begin{aligned}
a_{22}(m,\nu,\xi) &= \frac{1}{2}\langle R''(0,0)\xi,\xi\rangle + \frac{\tau}{8}\langle \dot{R}''(0,0)\xi,\xi\rangle + \\
&\langle R(0,0)\xi,\xi\rangle + \frac{3\tau}{8}\langle \dot{R}(0,0)\xi,\xi\rangle + \frac{2}{15}\tau^2\langle A\xi,\xi\rangle + \\
&2\langle R'(0,0)\xi,P^{\perp}\xi\rangle + \frac{\tau}{2}\langle \dot{R}'(0,0)\xi,P^{\perp}\xi\rangle + \\
&\langle R(0,0)P^{\perp}\xi,P^{\perp}\xi\rangle + \frac{\tau}{4}\langle \dot{R}(0,0)P^{\perp}\xi,P^{\perp}\xi\rangle + \\
&\frac{\tau^2}{15}\langle AP^{\perp}\xi,P^{\perp}\xi\rangle - \langle R(0,0)\xi,P\xi\rangle - \frac{\tau}{4}\langle \dot{R}(0,0)\xi,P\xi\rangle - \\
&\frac{\tau^2}{15}\langle A\xi,P\xi\rangle + O(\varepsilon\tau^2+\tau^4) \\
&= \frac{1}{2}\langle R''(0,0)\xi,\xi\rangle + \langle R(0,0)\xi,\xi\rangle + 2\langle R'(0,0)\xi,P^{\perp}\xi\rangle + \\
&\langle R(0,0)P^{\perp}\xi,P^{\perp}\xi\rangle - \langle R(0,0)\xi,P\xi\rangle + \\
&\tau[\frac{1}{8}\langle \dot{R}''(0,0)\xi,\xi\rangle + \frac{3}{8}\langle \dot{R}(0,0)\xi,\xi\rangle + \frac{1}{2}\langle \dot{R}'(0,0)\xi,P^{\perp}\xi\rangle + \\
&\frac{1}{4}\langle \dot{R}(0,0)P^{\perp}\xi,P^{\perp}\xi\rangle - \frac{1}{4}\langle \dot{R}(0,0)\xi,P\xi\rangle] + \\
&\frac{\tau^2}{15}[2\langle A\xi,\xi\rangle + \langle AP^{\perp}\xi,P^{\perp}\xi\rangle - \langle A\xi,P\xi\rangle] + O(\varepsilon\tau^2+\tau^4).
\end{aligned}$$

Applying $c)$ in Proposition 2.1 and (1) in Remark 2.1, we know that

$$\begin{aligned}
a_{22}(m,\nu,\xi) &= -R_{1213}\xi_2\xi_3 + (-R_{1313}+R_{2323})\xi_3^2 + \\
&R_{1212}\xi_2^2 + 2R_{1213}\xi_2\xi_3 + R_{1313}\xi_3^2 - 2R_{1223}\xi_1\xi_3 + \\
&R_{1212}\xi_1^2 - (R_{1212}\xi_2^2 + R_{1213}\xi_2\xi_3) + \\
&\tau\{\frac{1}{8}[-\nabla_1 R_{1212}\xi_2^2 + 4(-\nabla_1 R_{1213} + \nabla_2 R_{1223})\xi_2\xi_3 + \\
&(2\nabla_1 R_{2323} - 3\nabla_1 R_{1313} + 4\nabla_2 R_{1323})\xi_3^2] + \\
&\frac{3}{8}(\nabla_1 R_{1212}\xi_2^2 + 2\nabla_1 R_{1213}\xi_2\xi_3 + \nabla_1 R_{1313}\xi_3^2) - \\
&\frac{1}{2}[\nabla_2 R_{1212}\xi_1\xi_2 + (\nabla_1 R_{1223} + \nabla_2 R_{1213})\xi_1\xi_3] + \\
&\frac{1}{4}\nabla_1 R_{1212}\xi_1^2 - \frac{1}{4}(\nabla_1 R_{1212}\xi_2^2 + \nabla_1 R_{1213}\xi_2\xi_3)\} \\
&\frac{\tau^2}{15}(2\xi_2^2 + 2\xi_3^2 + \xi_1^2 - \xi_2^2) + O(\varepsilon\tau^2+\tau^4).
\end{aligned}$$

After combing the similar terms, we have

$$\begin{aligned}
a_{22}(m,\nu,\xi) &= R_{1212}\xi_1^2 - 2R_{1223}\xi_1\xi_3 + R_{2323}\xi_3^2 + \\
&\tau[\frac{1}{4}\nabla_1 R_{1212}\xi_1^2 - \frac{1}{2}\nabla_2 R_{1212}\xi_1\xi_2 - \frac{1}{2}(\nabla_1 R_{1223} + \\
&\nabla_2 R_{1213})\xi_1\xi_3 + \frac{1}{2}\nabla_2 R_{1223}\xi_2\xi_3 + (\frac{1}{4}\nabla_1 R_{2323} + \\
&\frac{1}{2}\nabla_2 R_{1323})\xi_3^2] + \frac{\tau^2}{15}(\xi_1^2 + \xi_2^2 + 2\xi_3^2) + O(\varepsilon\tau^2+\tau^4) \\
&= R(\xi,E_2,\xi,E_2) + \tau[\frac{1}{4}\nabla_1 R_{1212}\xi_1^2 - \frac{1}{2}\nabla_2 R_{1212}\xi_1\xi_2 - \\
&\frac{1}{2}(\nabla_1 R_{1223} + \nabla_2 R_{1213})\xi_1\xi_3 + \frac{1}{2}\nabla_2 R_{1223}\xi_2\xi_3 + \\
&(\frac{1}{4}\nabla_1 R_{2323} + \frac{1}{2}\nabla_2 R_{1323})\xi_3^2] + \frac{\tau^2}{15}(\xi_1^2 + \xi_2^2 + 2\xi_3^2) + \\
&O(\varepsilon\tau^2+\tau^4).
\end{aligned} \tag{3.44}$$

Proof of Lemma 3.3. In view of (3.23), we know that $\mathcal{S}''$ involves the term $J_0^{-1}J_0''$ and $J_0^{-1}J_1''$. We shall investigate the expansions of these two terms.

From the representations (3.22) for J_0'' and J_1'', together with the Remark 2.1 and the formula $\ddot{R}''(0,0) = 2\lim_{t\to 0^+}\frac{R''(0,t)-R''(0,0)-t\dot{R}''(0,0)}{t^2}$, we get

$$R''(0,t) = R''(0,0) + t\dot{R}''(0,0) + O(\varepsilon t^2).$$

Then

$$\begin{aligned} J_0^{-1}J_0''(0,t) &= -\int_0^t J_1^*(R''J_0 + 2R'J_0')ds + J_0^{-1}J_1\int_0^t (R''J_0 + 2R'J_0')ds \\ &= -\frac{t^2}{6}R''(0,0) - \frac{t^3}{12}\dot{R}''(0,0) + O(\varepsilon t^4 + t^6), \qquad (3.45) \\ J_0^{-1}J_1''(0,t) &= -\int_0^t J_1^*(R''J_1 + 2R'J_1')ds + J_0^{-1}J_1\int_0^t (R''J_1 + 2R'J_1')ds \\ &= -\frac{t}{2}R''(0,0) - \frac{t^2}{6}\dot{R}''(0,0) + O(\varepsilon t^3 + t^5). \qquad (3.46) \end{aligned}$$

From (3.23), we have

$$\begin{aligned} \mathcal{S}'' &= 2J_0^{-1}J_0'J_0^{-1}J_0'\mathcal{S} - J_0^{-1}J_0''\mathcal{S} - \\ & \quad 2tJ_0^{-1}J_0'J_0^{-1}J_1' + tJ_0^{-1}J_1''. \end{aligned}$$

Plugging (3.38)(3.26)(3.45)(3.38)(3.46) into the above expression, we get

$$\begin{aligned} \mathcal{S}'' &= 2[-\frac{t^2}{6}R'(0,0) - \frac{t^3}{12}\dot{R}'(0,0) + O(\varepsilon t^4 + t^6)][-\frac{t^2}{6}R'(0,0) - \\ & \quad \frac{t^3}{12}\dot{R}'(0,0) + O(\varepsilon t^4 + t^6)][I_n - \frac{t^2}{3}R(0,0) - \frac{t^3}{12}\dot{R}(0,0) - \\ & \quad \frac{t^4}{45}A + O(\varepsilon t^4 + t^6)] - [-\frac{t^2}{6}R''(0,0) - \frac{t^3}{12}\dot{R}''(0,0) + \\ & \quad O(\varepsilon t^4 + t^6)][I_n - \frac{t^2}{3}R(0,0) - \frac{t^3}{12}\dot{R}(0,0) - \frac{t^4}{45}A + \\ & \quad O(\varepsilon t^4 + t^6)] - 2t[-\frac{t^2}{6}R'(0,0) - \frac{t^3}{12}\dot{R}'(0,0) + \\ & \quad O(\varepsilon t^4 + t^6)][-\frac{t}{2}R'(0,0) - \frac{t^2}{6}\dot{R}'(0,0) + \\ & \quad O(\varepsilon t^3 + t^5)] - \frac{t^2}{2}R''(0,0) - \frac{t^3}{6}\dot{R}''(0,0) + \\ & \quad O(\varepsilon t^4 + t^6) \\ &= \frac{t^2}{6}R''(0,0) + \frac{t^3}{12}\dot{R}''(0,0) - \frac{t^2}{2}R''(0,0) - \frac{t^3}{6}\dot{R}''(0,0) + \\ & \quad O(\varepsilon t^4 + t^6) \\ &= -\frac{t^2}{3}R''(0,0) - \frac{t^3}{12}\dot{R}''(0,0) + O(\varepsilon t^4 + t^6) \end{aligned}$$

.

This ends the proof of Lemma 3.3. □

From now on, we come back to the proof of Theorem 3.2. Under the above preparations, we can obtain the expansion of the *MTW tensor* near the origin. From (3.29)(3.36)(3.44), we have

$$
\begin{aligned}
\mathcal{C}(m,\nu)(\xi,\eta) &= [R(\xi,E_1,\xi,E_1)+\frac{3}{4}\tau(\nabla_1R_{1212}\xi_2^2+2\nabla_1R_{1213}\xi_2\xi_3+\\
&\quad \nabla_1R_{1313}\xi_3^2)+\frac{2}{5}\tau^2(\xi_2^2+\xi_3^2)+O(\varepsilon\tau^2+\tau^4)(\xi_2^2+\xi_3^2)]\eta_1^2+\\
&\quad \{2R(\xi,E_1,\xi,E_2)+\tau[\frac{1}{2}\nabla_2R_{1212}\xi_2^2+(\nabla_1R_{1223}+\\
&\quad \nabla_2R_{1213})\xi_2\xi_3+(\nabla_1R_{1323}+\frac{1}{2}\nabla_2R_{1313})\xi_3^2-\\
&\quad \nabla_1R_{1212}\xi_1\xi_2-\nabla_1R_{1213}\xi_1\xi_3]-\frac{2}{5}\tau^2\xi_1\xi_2+\\
&\quad O(\varepsilon\tau^2+\tau^4)(\xi_2^2+\xi_3^2+\xi_1\xi_2+\xi_1\xi_3)\}\eta_1\eta_2+\\
&\quad \{R(\xi,E_2,\xi,E_2)+\tau[\frac{1}{4}\nabla_1R_{1212}\xi_1^2-\frac{1}{2}\nabla_2R_{1212}\xi_1\xi_2-\\
&\quad \frac{1}{2}(\nabla_1R_{1223}+\nabla_2R_{1213})\xi_1\xi_3+\frac{1}{2}\nabla_2R_{1223}\xi_2\xi_3+\\
&\quad (\frac{1}{4}\nabla_1R_{2323}+\frac{1}{2}\nabla_2R_{1323})\xi_3^2]+\frac{\tau^2}{15}(\xi_1^2+\xi_2^2+2\xi_3^2)+\\
&\quad O(\varepsilon\tau^2+\tau^4)\}\eta_2^2\\
&= R(\xi,E_1,\xi,E_1)\eta_1^2+2R(\xi,E_1,\xi,E_2)\eta_1\eta_2+R(\xi,E_2,\xi,E_2)\eta_2^2+\\
&\quad \frac{3}{4}\tau(\nabla_1R_{1212}\xi_2^2+2\nabla_1R_{1213}\xi_2\xi_3+\nabla_1R_{1313}\xi_3^2)\eta_1^2+\\
&\quad \tau[\frac{1}{2}\nabla_2R_{1212}\xi_2^2+(\nabla_1R_{1223}+\nabla_2R_{1213})\xi_2\xi_3+\\
&\quad (\nabla_1R_{1323}+\frac{1}{2}\nabla_2R_{1313})\xi_3^2-\nabla_1R_{1212}\xi_1\xi_2-\nabla_1R_{1213}\xi_1\xi_3]\eta_1\eta_2+\\
&\quad \tau[\frac{1}{4}\nabla_1R_{1212}\xi_1^2-\frac{1}{2}\nabla_2R_{1212}\xi_1\xi_2-\frac{1}{2}(\nabla_1R_{1223}+\nabla_2R_{1213})\xi_1\xi_3+\\
&\quad \frac{1}{2}\nabla_2R_{1223}\xi_2\xi_3+(\frac{1}{4}\nabla_1R_{2323}+\frac{1}{2}\nabla_2R_{1323})\xi_3^2]\eta_2^2+\\
&\quad \tau^2(\frac{1}{15}\xi_1^2\eta_2^2-\frac{2}{5}\xi_1\xi_2\eta_1\eta_2+\frac{2}{5}\xi_2^2\eta_1^2+\frac{1}{15}\xi_2^2\eta_2^2+\frac{2}{5}\xi_3^2\eta_1^2+\frac{2}{15}\xi_3^2\eta_2^2)+\\
&\quad O(\varepsilon\tau^2+\tau^4)(\xi_1\xi_2\eta_1\eta_2+\xi_1\xi_3\eta_1\eta_2+\xi_2^2\eta_1^2+\\
&\quad \xi_2^2\eta_1\eta_2+\xi_3^2\eta_1^2+\xi_3^2\eta_1\eta_2+\eta_2^2)\\
&= I+II+III+IV.
\end{aligned}
$$

We will estimate each term from I to IV. The combination of the zero order term and the second order term will control all the negative parts.

The term I It is readily to see that the zero order term $I=R_m(\xi,\eta,\xi,\eta)$. Recalling the curvature assumption (1.3), by the definition of sectional curvature, the term I can be bounded from below:

$$
\begin{aligned}
I &\geq |\xi|^2|\eta|^2-\langle\xi,\eta\rangle^2 \\
&= (\xi_1\eta_2-\xi_2\eta_1)^2+\xi_3^2(\eta_1^2+\eta_2^2).
\end{aligned}
\tag{3.47}
$$

The term II The term II involves the first order derivatives of the curvature as coefficients for the terms $\tau\xi_1^2\eta_2^2,\tau\xi_2^2\eta_1^2,\tau\xi_1\xi_2\eta_2^2$ and $\tau\xi_2^2\eta_1\eta_2$. They can not be directly controlled by the related terms of the second order in ξ and η, but the combination of them will compose good terms, more precisely, the term II can be stated as:

$$\begin{aligned}
II &= \frac{3}{4}\tau\nabla_1 R_{1212}\xi_2^2\eta_1^2 - \tau\nabla_1 R_{1212}\xi_1\xi_2\eta_1\eta_2 + \frac{\tau}{4}\nabla_1 R_{1212}\xi_1^2\eta_2^2 + \\
&\quad \frac{\tau}{2}\nabla_2 R_{1212}(\xi_2^2\eta_1\eta_2 - \xi_1\xi_2\eta_2^2) + \frac{3}{2}\tau\nabla_1 R_{1213}\xi_2\xi_3\eta_1^2 + \frac{3}{4}\tau\nabla_1 R_{1313}\xi_3^2\eta_1^2 + \\
&\quad \tau[(\nabla_1 R_{1223} + \nabla_2 R_{1213})\xi_2\xi_3 + (\nabla_1 R_{1323} + \frac{1}{2}\nabla_2 R_{1313})\xi_3^2 - \\
&\quad \nabla_1 R_{1213}\xi_1\xi_3]\eta_1\eta_2 + \tau[-\frac{1}{2}(\nabla_1 R_{1223} + \nabla_2 R_{1213})\xi_1\xi_3 + \\
&\quad \frac{1}{2}\nabla_2 R_{1223}\xi_2\xi_3 + (\frac{1}{4}\nabla_1 R_{2323} + \frac{1}{2}\nabla_2 R_{1323})\xi_3^2]\eta_2^2 + \\
&= \frac{\tau}{4}\nabla_1 R_{1212}(\xi_2\eta_1 - \xi_1\eta_2)^2 + \frac{\tau}{2}\nabla_1 R_{1212}(\xi_2\eta_1 - \xi_1\eta_2)\xi_2\eta_1 + \\
&\quad \frac{\tau}{2}\nabla_2 R_{1212}(\xi_2\eta_1 - \xi_1\eta_2)\xi_2\eta_2 + \frac{3}{2}\tau\nabla_1 R_{1213}\xi_2\xi_3\eta_1^2 + \frac{3}{4}\tau\nabla_1 R_{1313}\xi_3^2\eta_1^2 + \\
&\quad \tau[(\nabla_1 R_{1223} + \nabla_2 R_{1213})\xi_2\xi_3 + (\nabla_1 R_{1323} + \frac{1}{2}\nabla_2 R_{1313})\xi_3^2 - \\
&\quad \nabla_1 R_{1213}\xi_1\xi_3]\eta_1\eta_2 + \tau[-\frac{1}{2}(\nabla_1 R_{1223} + \nabla_2 R_{1213})\xi_1\xi_3 + \\
&\quad \frac{1}{2}\nabla_2 R_{1223}\xi_2\xi_3 + (\frac{1}{4}\nabla_1 R_{2323} + \frac{1}{2}\nabla_2 R_{1323})\xi_3^2]\eta_2^2.
\end{aligned}$$

Using the curvature assumption (1.4) and the parallel property (1.1), the term II can be estimated as follows:

$$\begin{aligned}
II &\geq -\frac{\varepsilon}{4}\tau(\xi_2\eta_1 - \xi_1\eta_2)^2 - \frac{\varepsilon}{2}\tau|(\xi_2\eta_1 - \xi_1\eta_2)\xi_2\eta_1| - \\
&\quad \frac{\varepsilon}{2}\tau|(\xi_2\eta_1 - \xi_1\eta_2)\xi_2\eta_2| - \frac{3}{2}\varepsilon\tau|\xi_2\xi_3\eta_1^2| - \frac{3}{4}\varepsilon\tau\xi_3^2\eta_1^2 - \\
&\quad \varepsilon\tau(2|\xi_2\xi_3\eta_1\eta_2| + \frac{3}{2}|\xi_3^2\eta_1\eta_2| + |\xi_1\xi_3\eta_1\eta_2|) - \\
&\quad \varepsilon\tau(|\xi_1\xi_3\eta_2^2| + \frac{1}{2}|\xi_2\xi_3\eta_2^2| + \frac{3}{4}\xi_3^2\eta_2^2).
\end{aligned}$$

In view of the Cauchy Schwartz inequality, it follows that

$$\begin{aligned}
2\tau|(\xi_2\eta_1 - \xi_1\eta_2)\xi_2\eta_1| &\leq (\xi_2\eta_1 - \xi_1\eta_2)^2 + \tau^2\xi_2^2\eta_1^2, \\
2\tau|(\xi_2\eta_1 - \xi_1\eta_2)\xi_2\eta_2| &\leq (\xi_2\eta_1 - \xi_1\eta_2)^2 + \tau^2\xi_2^2\eta_2^2, \\
2\tau|\xi_2\xi_3\eta_1^2| &\leq \tau^2\xi_2^2\eta_1^2 + \xi_3^2\eta_1^2, \\
2\tau|\xi_2\xi_3\eta_1\eta_2| &\leq \tau^2\xi_2^2\eta_1^2 + \xi_3^2\eta_2^2, \\
2|\xi_3^2\eta_1\eta_2| &\leq \xi_3^2\eta_1^2 + \xi_3^2\eta_2^2, \\
2\tau|\xi_1\xi_3\eta_1\eta_2| &\leq \tau^2\xi_1^2\eta_2^2 + \xi_3^2\eta_1^2, \\
2\tau|\xi_1\xi_3\eta_2^2| &\leq \tau^2\xi_1^2\eta_2^2 + \xi_3^2\eta_2^2, \\
2\tau|\xi_2\xi_3\eta_2^2| &\leq \tau^2\xi_2^2\eta_2^2 + \xi_3^2\eta_2^2.
\end{aligned}$$

Finally,

$$\begin{aligned}
II &\geq -\varepsilon(\frac{\tau}{4} + \frac{1}{2})(\xi_1\eta_2 - \xi_2\eta_1)^2 - \varepsilon\tau^2\xi_1^2\eta_2^2 - 2\varepsilon\tau^2\xi_2^2\eta_1^2 - \\
&\quad \frac{1}{2}\varepsilon\tau^2\xi_2^2\eta_2^2 - \varepsilon(\frac{5}{4} + \frac{3}{2}\tau)\xi_3^2\eta_1^2 - \varepsilon(\frac{7}{4} + \frac{3}{2}\tau)\xi_3^2\eta_2^2.
\end{aligned}$$

Now assume $0 < \tau < 2$, the following inequality holds:

$$\begin{aligned}
II &\geq -\varepsilon(\xi_1\eta_2 - \xi_2\eta_1)^2 - \varepsilon\tau^2\xi_1^2\eta_2^2 - 2\varepsilon\tau^2\xi_2^2\eta_1^2 - \\
&\quad \frac{1}{2}\varepsilon\tau^2\xi_2^2\eta_2^2 - \varepsilon(\frac{5}{4} + \frac{3}{2}\tau)\xi_3^2\eta_1^2 - \varepsilon(\frac{7}{4} + \frac{3}{2}\tau)\xi_3^2\eta_2^2.
\end{aligned} \tag{3.48}$$

The term III The term III consists of all terms whose coefficients involving the second order derivatives of the curvature. The term $\xi_1\xi_2\eta_1\eta_2\tau^2$ is bad one. It will be handled in the following. The others are all good ones. They are used to control the negative terms in the following.
The term IV We now handle the remainder IV. By definition, there exists a small positive number $\bar{\delta}_3$ which is independent of ε and a positive constant C such that, for any $0<\tau<\bar{\delta}_3$,

$$IV \quad \geq \quad -C\tau^2(\varepsilon+\tau^2)(\xi_1^2\eta_2^2+\xi_2^2\eta_1^2+\xi_2^2\eta_2^2+\xi_3^2\eta_1^2+\xi_3^2\eta_2^2+\eta_2^2).$$

Given a small positive real number ε_1 which will be determined later, assume that $0<\varepsilon<\frac{\varepsilon_3}{2C}, 0<\tau<\min\{\sqrt{\frac{\varepsilon_3}{2C}},\bar{\delta}_3\}$. We have:

$$IV \quad \geq \quad -\varepsilon_3\tau^2(\xi_1^2\eta_2^2+\xi_2^2\eta_1^2+\xi_2^2\eta_2^2+\xi_3^2\eta_1^2+\xi_3^2\eta_2^2+\eta_2^2). \tag{3.49}$$

Under the above estimations, we can imply the lower bound of the $MTW tensor$. Assume that $0<\varepsilon<\frac{\varepsilon_3}{2C}, 0<\tau<\min\{\sqrt{\frac{\varepsilon_3}{2C}},\bar{\delta}_3\}$ with ε_3 small enough.

Substituting the lower bounds (3.47),(3.48) and (3.49) into (3.47), we can derive:

$$\begin{aligned}\mathcal{C}(m,\nu)(\xi,\eta) \quad \geq \quad & (\xi_1\eta_2-\xi_2\eta_1)^2+\xi_3^2(\eta_1^2+\eta_2^2)-\frac{1}{2}\varepsilon_3(\xi_1\eta_2-\xi_2\eta_1)^2- \\ & \frac{1}{2}\varepsilon_3\tau^2\xi_1^2\eta_2^2-\varepsilon_3\tau^2\xi_2^2\eta_1^2-\frac{1}{4}\varepsilon_3\tau^2\xi_2^2\eta_2^2-\varepsilon_3(\frac{5}{8}+\frac{3}{4}\tau)\xi_3^2\eta_1^2- \\ & \varepsilon_3(\frac{7}{8}+\frac{3}{4}\tau)\xi_3^2\eta_2^2+\tau^2(\frac{1}{15}\xi_1^2\eta_2^2-\frac{2}{5}\xi_1\xi_2\eta_1\eta_2+\frac{2}{5}\xi_2^2\eta_1^2+ \\ & \frac{1}{15}\xi_2^2\eta_2^2+\frac{2}{5}\xi_3^2\eta_1^2+\frac{2}{15}\xi_3^2\eta_2^2)-\varepsilon_3\tau^2(\xi_1^2\eta_2^2+ \\ & \xi_2^2\eta_1^2+\xi_2^2\eta_2^2+\xi_3^2\eta_1^2+\xi_3^2\eta_2^2+\eta_2^2).\end{aligned}$$

Gathering similar terms, we get

$$\begin{aligned}\mathcal{C}(m,\nu)(\xi,\eta) \quad \geq \quad & (1-\frac{\varepsilon_3}{2})(\xi_1\eta_2-\xi_2\eta_1)^2+(\frac{1}{15}-\frac{3}{2}\varepsilon_3)\tau^2\xi_1^2\eta_2^2- \\ & \frac{2}{5}\tau^2\xi_1\xi_2\eta_1\eta_2+(\frac{2}{5}-2\varepsilon_3)\tau^2\xi_2^2\eta_1^2+(\frac{1}{15}-\frac{5}{4}\varepsilon_3)\tau^2\xi_2^2\eta_2^2+ \\ & [\frac{2}{5}\tau^2-\varepsilon_3(\frac{13}{8}+\frac{3}{4}\tau)+1]\xi_3^2\eta_1^2+ \\ & [\frac{2}{15}\tau^2-\varepsilon_3(\frac{15}{8}+\frac{3}{4}\tau)+1]\xi_3^2\eta_2^2-\varepsilon_3\tau^2\eta_2^2 \\ = \quad & (1-\frac{3}{4}\varepsilon_3)(\xi_1\eta_2-\xi_2\eta_1)^2+(\frac{1}{15}-3\varepsilon_3)\tau^2\xi_1^2\eta_2^2- \\ & \frac{2}{5}\tau^2\xi_1\xi_2\eta_1\eta_2+(\frac{2}{5}-3\varepsilon_3)\tau^2\xi_2^2\eta_1^2+ \\ & \frac{\varepsilon_3}{4}(\xi_1\eta_2-\xi_2\eta_1)^2+\frac{3}{2}\varepsilon_3\tau^2\xi_1^2\eta_2^2+\varepsilon_3\tau^2\xi_2^2\eta_1^2+ \\ & (\frac{1}{15}-\frac{5}{4}\varepsilon_3)\tau^2\xi_2^2\eta_2^2+ \\ & [1-\varepsilon_3(\frac{13}{8}+\frac{3}{4}\tau)]\xi_3^2\eta_1^2+[1-\varepsilon_3(\frac{15}{8}+\frac{3}{4}\tau)]\xi_3^2\eta_2^2+ \\ & \frac{2}{5}\tau^2\xi_3^2\eta_1^2+\frac{2}{15}\tau^2\xi_3^2\eta_2^2-\varepsilon_3\tau^2\eta_2^2\end{aligned}$$

$$\begin{aligned}
&= [1-\frac{3}{4}\varepsilon_3+(\frac{1}{15}-3\varepsilon_3)\tau^2]\xi_1^2\eta_2^2-2(1-\frac{3}{4}\varepsilon_3+\frac{\tau^2}{5})\xi_1\xi_2\eta_1\eta_2+\\
&\quad [1-\frac{3}{4}\varepsilon_3+(\frac{2}{5}-3\varepsilon_3)\tau^2]\xi_2^2\eta_1^2+\frac{\varepsilon_3}{4}(\xi_1\eta_2-\xi_2\eta_1)^2+\\
&\quad \frac{3}{2}\varepsilon_3\tau^2\xi_1^2\eta_2^2+\varepsilon_3\tau^2\xi_2^2\eta_1^2+(\frac{1}{15}-\frac{5}{4}\varepsilon_3)\tau^2\xi_2^2\eta_2^2+\\
&\quad [1-\varepsilon_3(\frac{13}{8}+\frac{3}{4}\tau)]\xi_3^2\eta_1^2+[1-\varepsilon_3(\frac{15}{8}+\frac{3}{4}\tau)]\xi_3^2\eta_2^2+\\
&\quad \frac{2}{5}\tau^2\xi_3^2\eta_1^2+\frac{2}{15}\tau^2\xi_3^2\eta_2^2-\varepsilon_3\tau^2\eta_2^2.
\end{aligned}$$

Note that the discriminant of the quadratic polynomial $h_{\varepsilon,\tau}(t)=[1-\frac{3}{4}\varepsilon+(\frac{1}{15}-3\varepsilon)\tau^2]t^2-2(1-\frac{3}{4}\varepsilon+\frac{\tau^2}{5})t+1-\frac{3}{4}\varepsilon+(\frac{2}{5}-3\varepsilon)\tau^2$ is polynomial with arguments (ε,τ). It is given by

$$\Lambda=4\tau^2\left[-\frac{1}{15}+\frac{121}{20}\varepsilon-\frac{9}{2}\varepsilon^2+\left(\frac{1}{75}+\frac{7}{5}\varepsilon-9\varepsilon^2\right)\tau^2\right].$$

Assume $\varepsilon<\frac{7}{45}$, there exists a small positive constant $\tilde{\varepsilon}_3<\frac{1}{30}$, such that the discriminant is non-positive in $[0,\tilde{\varepsilon}_3]\times[0,1]$, i.e. for any $(\varepsilon,\tau)\in[0,\tilde{\varepsilon}_3]\times[0,1]$,

$$\Lambda\le 4\tau^2(-\frac{4}{75}+\frac{149}{20}\varepsilon-\frac{27}{2}\varepsilon^2)\le 0.$$

So if $\varepsilon_3<\tilde{\varepsilon}_3$,

$$\begin{aligned}
\mathcal{C}(m,\nu)(\xi,\eta) &\ge \frac{\varepsilon_3}{4}(\xi_1\eta_2-\xi_2\eta_1)^2+[1-\varepsilon_3(\frac{13}{8}+\frac{3}{4}\tau)]\xi_3^2\eta_1^2+\\
&\quad [1-\varepsilon_3(\frac{15}{8}+\frac{3}{4}\tau)]\xi_3^2\eta_2^2+\frac{3}{2}\varepsilon_3\tau^2\xi_1^2\eta_2^2+\varepsilon_3\tau^2\xi_2^2\eta_1^2+\\
&\quad (\frac{1}{15}-\frac{5}{4}\varepsilon_3)\tau^2\xi_2^2\eta_2^2+\frac{2}{5}\tau^2\xi_3^2\eta_1^2+\frac{2}{15}\tau^2\xi_3^2\eta_2^2-\varepsilon_3\tau^2\eta_2^2.
\end{aligned}$$

Assume that $\varepsilon_3<\frac{C}{2}$. For all $\tau\le\min\{\sqrt{\frac{\varepsilon_3}{2C}},\bar{\delta}_3\}$,

$$\begin{aligned}
\mathcal{C}(m,\nu)(\xi,\eta) &\ge \frac{\varepsilon_3}{4}(\xi_1\eta_2-\xi_2\eta_1)^2+(1-2\varepsilon_3)\xi_3^2\eta_1^2+\\
&\quad (1-\frac{9}{4}\varepsilon_3)\xi_3^2\eta_2^2+\frac{3}{2}\varepsilon_3\tau^2\xi_1^2\eta_2^2+\varepsilon_3\tau^2\xi_2^2\eta_1^2+\\
&\quad (\frac{1}{15}-\frac{5}{4}\varepsilon_3)\tau^2\xi_2^2\eta_2^2+\frac{2}{5}\tau^2\xi_3^2\eta_1^2+\frac{2}{15}\tau^2\xi_3^2\eta_2^2-\varepsilon_3\tau^2\eta_2^2.
\end{aligned}$$

Replacing η_2^2 by $\eta_2^2(\xi_1^2+\xi_2^2+\xi_3^2)$, we get

$$\begin{aligned}
\mathcal{C}(m,\nu)(\xi,\eta) &\ge \frac{\varepsilon_3}{4}(\xi_1\eta_2-\xi_2\eta_1)^2+(1-2\varepsilon_3)\xi_3^2\eta_1^2+\\
&\quad (1-\frac{9}{4}\varepsilon_3)\xi_3^2\eta_2^2+\frac{1}{2}\varepsilon_3\tau^2\xi_1^2\eta_2^2+\varepsilon_3\tau^2\xi_2^2\eta_1^2+\\
&\quad (\frac{1}{15}-\frac{9}{4}\varepsilon_3)\tau^2\xi_2^2\eta_2^2+\frac{2}{5}\tau^2\xi_3^2\eta_1^2+(\frac{2}{15}-\varepsilon_3)\tau^2\xi_3^2\eta_2^2.
\end{aligned}$$

Assume that $\varepsilon_3<\min\{\frac{2}{135},\frac{C}{2},\tilde{\varepsilon}_3\}$, then for any $\tau\le\min\{\sqrt{\frac{\varepsilon_3}{2C}},\bar{\delta}_3\}$, we have

$$\begin{aligned}
\mathcal{C}(m,\nu)(\xi,\eta) &\ge \frac{\varepsilon_3}{4}(\xi_1\eta_2-\xi_2\eta_1)^2+\frac{131}{135}\xi_3^2\eta_1^2+\\
&\quad \frac{29}{30}\xi_3^2\eta_2^2+\frac{1}{2}\varepsilon_3\tau^2\xi_1^2\eta_2^2+\varepsilon_3\tau^2\xi_2^2\eta_1^2+\\
&\quad \frac{1}{30}\tau^2\xi_2^2\eta_2^2+\frac{2}{5}\tau^2\xi_3^2\eta_1^2+\frac{1}{10}\tau^2\xi_3^2\eta_2^2.
\end{aligned}$$

We infer the existence of positive constants $\bar{\varepsilon}_3,\kappa_3$ and δ_3 such that for $\forall\nu\in T_mM$ with $|\nu|<\delta_3$,

$$\mathcal{C}(m,\nu)(\xi,\eta)\ \ge\ \kappa_3\left[(|\xi|^2|\eta|^2-\langle\xi,\eta\rangle^2)+|\nu|^2(\xi_2^2+\xi_3^2)|\eta|^2+|\nu|^2|\xi|^2\eta_2^2\right].$$

The proof of Theorem 3.2 is thus complete. □

Chapter 4

The inverse for the Hessian of the squared distance

This chapter is devoted to the asymptotic property of the inverse of the Hessian of the squared distance. Firstly, we deal with the derivatives of geodesic motion from order one to order three. Secondly, we give a basic formula to calculate the Hessian of the squared distance. Finally, we examine the approximation for the inverse of the Hessian of the squared distance. We begin by some notations.

Let (M, g) be a closed Riemaniann manifold of dimension $n \geq 2$. Assume that (M, g) satisfies the curvature assumptions (1.3) and (1.4). Fix $m_0 \in M, \nu_0 \in I(m_0)\backslash\{0\}$. Set $\gamma(t) = \exp_{m_0}(t\nu_0)$. Let $\{e_1(t), e_2(t), \cdots, e_n(t)\}$ be a parallel orthonormal moving frame of vector fields along the geodesic γ with $e_1(t) = \frac{\dot\gamma(t)}{|\dot\gamma(t)|}$. Taking the corresponding Fermi coordinate system x along the geodesic $\exp_{m_0}(t\frac{\nu_0}{|\nu_0|})$, the fiber coordinate of $TM \to M$ naturally related to x is denoted by $v = (v^1, v^2, \cdots, v^n)$.

For each $m \in M, \nu \in I(m)$ with m in the domain of the Fermi coordinate system x, we set:

$$X = X(x, v, t) = (X^1(x, v, t), X^2(x, v, t), \cdots, X^n(x, v, t)) = x(\exp_m(t\nu)),$$

where $x = x(m)$ and $\nu = v^i\partial_i$.

Thus, $X(x, v, t)$ is the coordinate of the geodesic $\exp_m t\nu$. Then the n-tuple $X = X(x, v, t)$ is the solution of the Cauchy problem (2.1).

Note that (m_0, ν_0) corresponds to $(0, v_0)$ where $v_0 = (|\nu_0|, 0)$ in the Fermi coordinate system x. On the axis, set for short $X_0(t) := X(0, v_0, t)$.

4.1 The derivatives of the geodesic motion

4.1.1 The first derivatives of the geodesic motion

In this subsection, we are concerned with the first derivatives of the geodesic motion. Recall that $X(x, v, t)$ is the coordinate of the geodesic $\exp_m t\nu$ and $X = X(x, v, t)$ is the solution of the Cauchy problem (2.1). It is clear that $\dot X(0, v_0, t) = (|\nu_0|t, 0, \cdots, 0)^T$. Let J_a be $\partial_x X$ or $D_v X$.

Differentiating (2.1) once with respect to the variable x(or v), on the axis, we obtain the following equation:

$$\ddot J^i_a + \partial_l\Gamma^i_{jk}(X)\dot X^j\dot X^k J^l_a + 2\Gamma^i_{jk}(X)\dot X^j\dot J^k_a = 0,$$

with the initial conditions, namely either

$$\partial_a X^i(0) = \delta^i_a, \partial_a \dot X^i(0) = 0,$$

or

$$D_a X^i(0) = 0, D_a \dot{X}^i(0) = \delta_a^i.$$

Evaluating on the axis, and using (1.15), the relations $\dot{X}(0, v_0, t) = (|\nu_0|t, 0, \cdots, 0)^T$ and (1.12), we get the following equation

$$\ddot{J}_a^i + |\nu_0|^2 R_{1\beta 1}^i(X_0) J_a^\beta = 0,$$

This equation is exactly (1.7). Thus from Lemma 1.1, we get

Lemma 4.1. *[29] On the axis, for $t \in [0,1]$, the terms:*

$$|\partial_x X(0, v_0, t)|, |\partial_x \dot{X}(0, v_0, t)|, |D_v X(0, v_0, t)|, |D_v \dot{X}(0, v_0, t)|,$$

are all bounded from above by a positive constant $C > 0$.

We require the notation $\overline{\partial_x X_0}(t)$ and $\overline{D_v X_0}(t)$ for the solution $\bar{J}_a$ of the unperturbed equation

$$\ddot{\bar{J}}_a^i + |\nu_0|^2 \bar{R}_{1\beta 1}^i \bar{J}_a^\beta = 0,$$

with the initial conditions, namely either

$$\overline{\partial_a X_0^i}(0) = \delta_a^i, \dot{\overline{\partial_a X_0^i}}(0) = 0,$$

or

$$\overline{D_a X_0^i}(0) = 0, \dot{\overline{D_a X_0^i}}(0) = \delta_a^i.$$

It is clear that $\overline{\partial_x X_0}(t)$ and $\overline{D_v X_0}(t)$ go back to $\bar{J}_1$ and $\bar{J}_0$ respectively on the axis. From Lemma 1.2, we obtain

Lemma 4.2. *[29] There exists a positive number $C > 0$ such that on the axis, for all $t \in [0,1]$,, we have*

$$\begin{aligned}
&|\partial_x X(0, v_0, t) - \overline{\partial_x X_0}(t)| \leq C\varepsilon, \\
&|\partial_x \dot{X}(0, v_0, t) - \dot{\overline{\partial_x X_0}}(t)| \leq C\varepsilon, \\
&|D_v X(0, v_0, t) - \overline{D_v X_0}(t)| \leq C\varepsilon, \\
&|D_v \dot{X}(0, v_0, t) - \dot{\overline{D_v X_0}}(t)| \leq C\varepsilon.
\end{aligned}$$

4.1.2 The second derivatives of the geodesic motion

In this subsection, we handle the second derivatives of geodesic motion. Let J_{ab} be $\partial_{ab}^2 X, \partial_a D_b X, D_a \partial_b X$ or $D_{ab}^2 X$.

Differentiating (2.1) twice with respect to the parameters x and v, we get

$$\begin{aligned}
&\ddot{J}_{ab}^i + \partial_l \Gamma_{jk}^i \dot{X}^j \dot{X}^k J_{ab}^l + 2\Gamma_{jk}^i \dot{X}^j \dot{J}_{ab}^k = \\
&-\partial_{lp}^2 \Gamma_{jk}^i \dot{X}^j \dot{X}^k J_a^l J_b^p - 2\partial_l \Gamma_{jk}^i \dot{X}^j (\dot{J}_b^k J_a^l + \dot{J}_a^k J_b^l) - 2\Gamma_{jk}^i \dot{J}_b^j \dot{J}_a^k.
\end{aligned}$$

with homogenuous initial conditions:

$$J_{ab}^i(0) = \dot{J}_{ab}^i(0) = 0.$$

Evaluating on the axis, and using (1.15), the relations $\dot{X}(0, v_0, t) = (|\nu_0|t, 0, \cdots, 0)^T$ and (1.12), we obtain

$$\ddot{J}_{ab}^i + |\nu_0|^2 R_{1\alpha 1}^i(X_0) J_{ab}^\alpha = -|\nu_0|^2 \partial_{lp}^2 \Gamma_{11}^i J_a^l J_b^p - 2|\nu_0| R_{k\beta 1}^i(X_0)(\dot{J}_b^k J_a^\beta + \dot{J}_a^k J_b^\beta). \quad (4.1)$$

Using (1.15),(1.16) and Lemma 4.1, we record a standard result of the second order differential equations, namely:

Lemma 4.3. *[29] On the axis, for $t \in [0,1]$, the terms:*

$$
\begin{array}{l}
|\partial^2_{xx}X(0,v_0,t)|, |\partial^2_{xx}\dot{X}(0,v_0,t)|,\\
|\partial_x D_v X(0,v_0,t)|, |\partial_x D_v \dot{X}(0,v_0,t)|,\\
|D^2_{vv}X(m,v_0,t)|, |D^2_{vv}\dot{X}(0,v_0,t)|
\end{array}
$$

are all bounded from above by a positive constant $C > 0$.

Let us introduce the solutions $\overline{\partial^2_{xx}X_0}, \overline{\partial_x D_v X_0}, \overline{D_v\partial_x X_0}$ and $\overline{D^2_{vv}X_0}$ along the axis of the unperturbed equation:

$$
\ddot{\bar{J}}^i_{ab} + |\nu_0|^2\delta^i_\alpha \bar{J}^\alpha_{ab} = -2|\nu_0|(\delta^i_\beta\delta^1_k - \delta^i_1\delta^\beta_k)(\bar{J}^\beta_a\dot{\bar{J}}^k_b + \bar{J}^\beta_b\dot{\bar{J}}^k_a), \tag{4.2}
$$

with homogenuous initial conditions

$$
\bar{J}^i_{ab}(0) = \dot{\bar{J}}^i_{ab}(0) = 0. \tag{4.3}
$$

Lemma 4.4. *[29] There exists a positive number $C > 0$ such that on the axis, for all $t \in [0,1]$, we have*

$$
\begin{array}{l}
|\partial^2_{xx}X(0,v_0,t) - \overline{\partial^2_{xx}X_0}(t)| \le C\varepsilon,\\
|\partial^2_{xx}\dot{X}(0,v_0,t) - \dot{\overline{\partial^2_{xx}X}}_0(t)| \le C\varepsilon,\\
|\partial_x D_v X(0,v_0,t) - \overline{\partial_x D_v X_0}(t)| \le C\varepsilon,\\
|\partial_x D_v \dot{X}(0,v_0,t) - \dot{\overline{\partial_x D_v X_0}}(t)| \le C\varepsilon,\\
|D^2_{vv}X(0,v_0,t) - \overline{D^2_{vv}X_0}(t)| \le C\varepsilon,\\
|D^2_{vv}\dot{X}(0,v_0,t) - \dot{\overline{D^2_{vv}X_0}}(t)| \le C\varepsilon.
\end{array}
$$

Furthermore,$|\partial^2_{1x}X(0,v_0,t)| \le C\varepsilon, |\partial_1 D_v X(0,v_0,t)| \le C\varepsilon.$

Proof. Let $\mathcal{E}_a(t)$ be the matrix valued function whose elements $\mathcal{E}^i_{ab}(t)$ are given by the difference $J^i_{ab} - \bar{J}^i_{ab}$. Combining the equations (4.1) and (4.2) with (1.15),(1.16), Lemma 4.1, Lemma 4.2 and Lemma 4.3, we find that there exists a positive constant $C > 0$ such that $|\ddot{\mathcal{E}}_a + \bar{R}\mathcal{E}_a| \le C\varepsilon$. Applying a representation formula [29] to $\mathcal{E}^i_{ab}(t)$, it yields $|\mathcal{E}_a| \le C\varepsilon$. The remaining approximations come from the representation:

$$
\dot{\mathcal{E}}^i_{ab}(t) = \int_0^t \ddot{\mathcal{E}}^i_{ab}(s)ds.
$$

The last two approximations are just the consequences of the facts $\overline{\partial^2_{1x}X_0}(t) = 0$ and $\overline{\partial_1 D_v X_0}(t) = 0$ which follow from the equation (4.2) and the initial condition (4.3).

4.1.3 The third derivatives of the geodesic motion

In this section, we address the third derivatives of the geodesic motion. Let $J^i_{abc}(t)$ be $\partial^3_{xxx}X(0,v_0,t), \partial^2_{xx}D_v X(0,v_0,t)$, $\partial_x D^2_{vv}X(0,v_0,t)$ or $D^3_{vvv}X(0,v_0,t)$.

Differentiating (2.1) three times with respect to the variable x and v :

$$
\begin{aligned}
\ddot{J}^i_{abc} + |\nu_0|^2 R^i_{1\alpha 1}(X_0)J^\alpha_{abc} \quad = \quad & -|\nu_0|^2\partial^3_{lpq}\Gamma^i_{11}J^l_aJ^p_bJ^q_c - |\nu_0|^2\partial^2_{lp}\Gamma^i_{11}\sum_{(a,b,c)} J^l_{ab}J^p_c\\
& -2|\nu_0|\partial^2_{lp}\Gamma^i_{1k}\sum_{(a,b,c)} \dot{J}^k_a J^l_b J^p_c\\
& -2|\nu_0|R^i_{k\beta 1}(X_0)\sum_{(a,b,c)} (\dot{J}^k_a J^\beta_{bc} + \dot{J}^k_{ab}J^\beta_c)\\
& -2\partial_\beta\Gamma^i_{jk}\sum_{(a,b,c)} \dot{J}^j_a\dot{J}^k_b J^\beta_c.
\end{aligned}
$$

with homogenuous initial conditions

$$J^i_{abc}(0) = J^i_{abc}(0) = 0.$$

Repeating the procedure in Lemma 4.3 we get:

Lemma 4.5. *[29] On the axis, for $t \in [0,1]$, the terms:*

$$\begin{aligned}
&|\partial^3_{xxx}X(0,v_0,t)|, |\partial^3_{xxx}\dot{X}(0,v_0,t)|,\\
&|\partial^2_{xx}D_vX(0,v_0,t)|, |\partial^2_{xx}D_v\dot{X}(0,v_0,t)|,\\
&|\partial_x D^2_{vv}X(0,v_0,t)|, |\partial_x D^2_{vv}\dot{X}(0,v_0,t)|,\\
&|D^3_{vvv}X(0,v_0,t)|, |D^3_{vvv}\dot{X}(0,v_0,t)|
\end{aligned}$$

are bounded from above by a positive constant $C > 0$ (independent of t, v_0).

Let us introduce the solutions $\overline{\partial^3_{xxx}X_0}$, $\overline{\partial^2_{xx}D_vX_0}$, $\overline{\partial_xD_v\partial_xX_0}$, $\overline{D_v\partial^2_{xx}X_0}$, $\overline{\partial_xD^2_{vv}X_0}$, $\overline{D_v\partial_xD_vX_0}$, $\overline{D^2_{vv}\partial_xX_0}$ and $\overline{D^3_{vvv}X_0}$ along the axis of the unperturbed equation:

$$\begin{aligned}
&\ddot{\bar{J}}^i_{abc} + |\nu|^2\delta^i_\alpha \bar{J}^\alpha_{abc} \qquad (4.4)\\
&= \frac{4}{3}(\delta^i_k - \delta^i_1\delta^1_k)\sum_{(a,b,c)}(|\nu|^2\bar{J}^k_a\bar{J}^\beta_b\bar{J}^\beta_c - 2\dot{\bar{J}}^\beta_a\dot{\bar{J}}^\beta_b\bar{J}^k_c) -\\
&\quad 2|\nu|(\delta^i_\beta\delta^1_k - \delta^i_1\delta^\beta_k)\sum_{(a,b,c)}(\dot{\bar{J}}^k_a\bar{J}^\beta_{bc} + \dot{\bar{J}}^k_{ab}\bar{J}^\beta_c) -\\
&\quad 2(\delta^i_k - \delta^i_1\delta^1_k)\sum_{(a,b,c)}\dot{\bar{J}}^1_a\dot{\bar{J}}^1_b\bar{J}^k_c +\\
&\quad [2\delta^i_1\delta^1_k + \frac{2}{3}(\delta^i_k - \delta^i_1\delta^1_k)]\sum_{(a,b,c)}(\dot{\bar{J}}^k_a\dot{\bar{J}}^\beta_b\bar{J}^\beta_c + \dot{\bar{J}}^\beta_a\dot{\bar{J}}^k_b\bar{J}^\beta_c).
\end{aligned}$$

with homogenuous initial conditions

$$\bar{J}^i_{abc}(0) = \dot{\bar{J}}^i_{abc}(0) = 0.$$

Repeating the procedure in Lemma 4.4, we get

Lemma 4.6. *[29] There exists a positive number $C > 0$ such that on the axis, for all $t \in [0,1]$, we have*

$$\begin{aligned}
&|\partial^3_{xxx}X(0,v_0,t) - \overline{\partial^3_{xxx}X_0}(t)| \le C\varepsilon,\\
&|\partial^2_{xx}D_vX(0,v_0,t) - \overline{\partial^2_{xx}D_vX_0}(t)| \le C\varepsilon,\\
&|\partial_xD^2_{vv}X(0,v_0,t) - \overline{\partial_xD^2_{vv}X_0}(t)| \le C\varepsilon,\\
&|D^3_{vvv}X(0,v_0,t) - \overline{D^3_{vvv}X_0}(t)| \le C\varepsilon.
\end{aligned}$$

4.2 The Hessian of the squared distance

In this section, we compute the local expression of the Hessian of the squared distance. We start from the well-known identity(p.156 [59]):

$$p_2 = \exp_{p_1}[-\mathrm{grad}_{p_1}c(\cdot,p_2)], \qquad (4.5)$$

where the identity makes sense whenever $(p_1,p_2) \in M \times M$ are no cut points of each other.

Suppose that the points p_1 and p_2 lie in the domain of the Fermi coordinate system x. The coordinates of p_1 and p_2 are given by $x_1 = x(p_1)$ and $x_2 = x(p_2)$ respectively. Let m be in the domain of the Fermi coordinate system x. Set $\nu \in I(m)$. The coordinate of ν is denoted by $v = v^i\partial_i$ i.e.$v^i = dx^i(\nu)$.

Differentiating (4.5) with respect to the coordinates x_1 at $x_1 = x(m)$, we get for $X(x_1, v, t)$ at $x_1 = x(m), t = 1$ and at $v = v^i\partial_i$ given by $\exp_m \nu = p_2$, the following identity:

$$D_k X^i(x, v, 1)\nabla^k_j c(m, \exp_m \nu) = \delta^\nabla_j X^i(x, v, 1), \tag{4.6}$$

where $\delta^\nabla_j X^i = \partial_j X^i - \Gamma^p_{jl}(x)v^l D_p X^i$.

This is the fundamental formula to compute the approximation of the inverse for the Hessian of the squared distance.

4.3 The inverse for the Hessian of the squared distance

In the last section, we consider the approximation of the inverse of Hessian of the squared distance. The result is presented as follows.

Theorem 4.1. *Let (M, g) be a closed n-dimensional Riemannian manifold satisfying (1.3) and (1.4). Then there exists positive numbers $C > 0$ and $\varepsilon_0 > 0$ depending only n such that, for any $\varepsilon < \varepsilon_0$ and for any $m_0 \in M, \nu_0 \in I(m_0), |\nu_0|_{m_0} \geq \frac{3\pi}{4}$, the following inequalities hold*

1) $|\mathcal{S}^{-1}(m_0, \nu_0, 1) - \bar{\mathcal{S}}^{-1}(m_0, \nu_0, 1)| \leq C\varepsilon$;

2) $|\partial_x \mathcal{S}^{-1}(m_0, \nu_0, 1) - \partial_x \bar{\mathcal{S}}^{-1}(m_0, \nu_0, 1)| \leq C\varepsilon$,
$|D_v \mathcal{S}^{-1}(m_0, \nu_0, 1) - D_v \bar{\mathcal{S}}^{-1}(m_0, \nu_0, 1)| \leq C\varepsilon$;

3) $|\partial^2_{xx} \mathcal{S}^{-1}(m_0, \nu_0, 1) - \partial^2_{xx} \bar{\mathcal{S}}^{-1}(m_0, \nu_0, 1)| \leq C\varepsilon$,
$|\partial_x D_v \mathcal{S}^{-1}(m_0, \nu_0, 1) - \partial_x D_v \bar{\mathcal{S}}^{-1}(m_0, \nu_0, 1)| \leq C\varepsilon$,
$|D^2_{vv} \mathcal{S}^{-1}(m_0, \nu_0, 1) - D^2_{vv} \bar{\mathcal{S}}^{-1}(m_0, \nu_0, 1)| \leq C\varepsilon$.

Proof. Set $m_0 \in M, \nu_0 \in I(m_0), |\nu_0|_{m_0} \geq \frac{3\pi}{4}$. Let x be the Fermi coordinate system along the geodesic $\exp_{m_0}(t\frac{\nu_0}{|\nu_0|})$ and v be the fiber coordinate of $TM \to M$ naturally associated to x. Set $r_0 = |\nu_0|$.
1) Making use of (b) in Proposition 1.1, we know that the matrix of the linear operator $\mathcal{S}(m_0, \nu_0, 1)$ from $T_m M$ to $T_m M$ in the orthonormal basis $\{e_1(0), e_2(0), \cdots, e_n(0)\}$ is given by $J_0^{-1} J_1$. As a consequence, the matrix to the inverse of $\mathcal{S}$ is given by $J_1^{-1} J_0$. It is clear that 1) is just a result of Lemma 4.1 and Lemma 4.2.
2) The inverse of $\mathcal{S}$ is denoted by A. The components of A in the Fermi coordinate system x is denoted by A^i_j. From the formula (4.6) and (b) in Proposition 1.1, we obtain the formula

$$A^i_j(x, v) = Z^i_k D_j X^k, \tag{4.7}$$

where $Z^i_k \delta^\nabla_j X^k = \delta^i_j$.
Differentiating (4.7) once with respect to x :

$$\begin{aligned} \partial_a A^i_j &= -Z^i_p \partial_a \delta^\nabla_q X^p Z^q_k D_j X^k + Z^i_k \partial_a D_j X^k \\ &= -Z^i_p [\partial^2_{aq} X^p - \partial_a \Gamma^d_{ql}(x) v^l D_d X^p - \\ &\quad \Gamma^d_{ql}(x) v^l \partial_a D_d X^p] Z^q_k D_j X^k + Z^i_k \partial_a D_j X^k. \end{aligned}$$

Evaluating on the axis, we have

$$\partial_a A^i_j = -Z^i_p(\partial^2_{aq} X^p - r_0 R^d_{qa1} D_d X^p) Z^q_k D_j X^k + Z^i_k \partial_a D_j X^k.$$

Applying Lemma 4.1, Lemma 4.2, Lemma 4.3 and Lemma 4.4, and together with the facts

$$\overline{\partial^2_{aj} X^i_0}(t) = (-r_0 t + \sin r_0 t \cos r_0 t)\delta^i_1(\delta_{aj} - \delta_{1a}\delta_{1j}).$$

$$\overline{\partial_a D_j X_0^i}(t) = -\delta_{1j}(\delta_a^i - \delta_1^i\delta_{1a})t\sin r_0 t + \frac{\sin^2 r_0 t}{r_0}\delta_1^i(\delta_{aj} - \delta_{1a}\delta_{1j}),$$

we get

$$\partial_a A_j^i = \mathcal{B}(\varepsilon).$$

To calculate the corresponding derivatives on the sphere, we need to give the formula of $\bar{\mathcal{S}}^{-1}$. From the Remark 1.1(2), we see that

$$\bar{S}(m,\nu,1)(\xi) = \xi - (1 - |\nu|\cot(|\nu|))(\xi - \langle \xi, \frac{\nu}{|\nu|}\rangle \frac{\nu}{|\nu|}).$$

It can be checked that the inverse of $\bar{\mathcal{S}}$ which is denoted by $\bar{A}$ takes the form

$$\bar{A}(\xi) = \xi - (1 - \frac{|\nu|}{\tan|\nu|})(\xi - \langle \xi, \frac{\nu}{|\nu|}\rangle \frac{\nu}{|\nu|}).$$

Differentiating $\bar{A}_j^i = \delta_j^i - (1 - \frac{|\nu|}{\tan|\nu|})(\delta_j^i - \frac{g_{jk}(x)v^iv^k}{|\nu|^2})$ with respect to x and evaluating at (m,ν), we get $\partial_a \bar{A}_j^i = \mathcal{B}(\varepsilon)$.

Differentiating (4.7) once with respect to v, we infer

$$\begin{aligned} D_a A_j^i &= -Z_p^i D_a \delta_q^\nabla X^p Z_k^q D_j X^k + Z_k^i D_{aj}^2 X^k \\ &= -Z_p^i[D_a\partial_q X^p - \Gamma_{qa}^d(x) D_d X^p - \\ &\quad \Gamma_{ql}^d(x) v^l D_{ad}^2 X^p] Z_k^q D_j X^k + Z_k^i D_{aj}^2 X^k \end{aligned}$$

Evaluating on the axis, this gives

$$D_a A_j^i = -Z_p^i D_a \partial_q X^p Z_k^q D_j X^k + Z_k^i D_{aj}^2 X^k$$

Applying Lemma 4.1, Lemma 4.2, Lemma 4.3 and Lemma 4.4, and combining with the facts

$$\overline{D_a\partial_j X_0^i}(t) = \overline{\partial_j D_a X_0^i}(t).$$

$$\begin{aligned} \overline{D_{aj}^2 X_0^i}(t) &= (\frac{1}{r_0}t\cos r_0 t - \frac{1}{r_0^2}\sin r_0 t)(\delta_a^i\delta_{1j} + \delta_1^i\delta_{aj} - 2\delta_1^i\delta_{1a}\delta_{1j}) + \\ &\quad (\frac{t}{r_0} - \frac{1}{r_0^2}\sin r_0 t\cos r_0 t)\delta_{1a}(\delta_j^i - \delta_1^i\delta_{1j}), \end{aligned}$$

we obtain

$$\begin{aligned} D_a A_j^i &= (\frac{1}{r_0} - \frac{\tan r_0}{r_0^2})(\delta_a^i\delta_{1j} + \delta_1^i\delta_{aj} - 2\delta_1^i\delta_{1a}\delta_{1j}) + \\ &\quad (\frac{\sec^2 r_0}{r_0} - \frac{\tan r_0}{r_0^2})\delta_{1a}(\delta_j^i - \delta_1^i\delta_{1j}) + \mathcal{B}(\varepsilon) \\ &= D_a\bar{A}_j^i + \mathcal{B}(\varepsilon). \end{aligned}$$

3)Differentiating (4.7) twice with respect to x, we have

$$\begin{aligned} \partial_{ab}^2 A_j^i &= Z_c^i\partial_a\delta_d^\nabla X^c Z_p^d\partial_b\delta_q^\nabla X^p Z_k^q D_j X^k + \\ &\quad Z_c^i\partial_b\delta_d^\nabla X^c Z_p^d\partial_a\delta_q^\nabla X^p Z_k^q D_j X^k - \\ &\quad Z_p^i\partial_{ab}^2\delta_q^\nabla X^p Z_k^q D_j X^k - Z_p^i\partial_a\delta_q^\nabla X^p Z_k^q\partial_b D_j X^k - \\ &\quad Z_p^i\partial_b\delta_q^\nabla X^p Z_k^q\partial_a D_j X^k + Z_k^i\partial_{ab}^2 D_j X^k \end{aligned} \tag{4.8}$$

Using Lemma 4.1, Lemma 4.2, Lemma 4.3 and Lemma 4.4,Lemma 4.5, Lemma 4.6, together with the facts

$$\begin{aligned} \overline{\partial_{abj}^3 X_0^i}(t) &= (\frac{r_0}{2}t\sin r_0 t + \frac{1}{6}\sin^2 r_0 t\cos r_0 t)(\delta_k^i - \delta_1^i\delta_k^1)\sum_{(a,b,j)}\delta_a^k\delta_b^\beta\delta_j^\beta + \\ &\quad \frac{1}{2}(r_0 t\sin r_0 t - \sin^2 r_0 t\cos r_0 t)(\delta_k^i - \delta_1^i\delta_k^1)\sum_{(a,b,j)}\delta_a^k(\delta_{bj} - \delta_b^1\delta_j^1) \end{aligned}$$

$$\overline{\partial^2_{ab} D_j X^i_0}(t) =$$
$$(\frac{\sin^2 r_0 t}{6r_0} - \frac{t\cos r_0 t}{2} + \frac{\sin r_0 t}{2r_0})(\delta^i_k - \delta^i_1\delta^1_k)(\delta^k_a\delta^\beta_b\delta^\beta_j + \delta^\beta_a\delta^k_b\delta^\beta_c) +$$
$$[(\frac{\sin^2 r_0 t}{6r_0} + \frac{t\cos r_0 t}{2} - \frac{\sin r_0 t}{2r_0})(\delta^i_k - \delta^i_1\delta^1_k) - 2\delta^i_1\delta^1_k t\sin r_0 t]\delta^\beta_a\delta^\beta_b\delta^k_j +$$
$$\frac{1}{2}(-\frac{\sin^2 r_0 t}{r_0} + t\cos r_0 t - \frac{\sin r_0 t}{r_0})(\delta^i_k - \delta^i_1\delta^1_k)[\delta^k_a(\delta_{bj} - \delta^1_b\delta^1_j) +$$
$$\delta^k_b(\delta_{aj} - \delta^1_a\delta^1_j)] +$$
$$(-\frac{\sin^2 r_0 t}{2r_0} - \frac{3t\cos r_0 t}{2} + \frac{3\sin r_0 t}{2r_0})(\delta^i_k - \delta^i_1\delta^1_k)\delta^k_j(\delta_{ab} - \delta^1_a\delta^1_b),$$

we obtain

$$\begin{aligned}\partial^2_{ab} A^i_j &= (\frac{\tan r_0}{r_0} - \sec^2 r_0)(\delta_{ab} - \delta_{1a}\delta_{1b})(\delta^i_j - \delta^i_1\delta_{1j}) + \mathcal{B}(\varepsilon)\\ &= \partial^2_{ab}\bar{A}^i_j + \mathcal{B}(\varepsilon).\end{aligned}$$

Differentiating (4.7) with respect to x and v respectively, we infer

$$\begin{aligned}\partial_a D_b A^i_j &= Z^i_c \partial_a \delta^\nabla_d X^c Z^d_p D_b \delta^\nabla_q X^p Z^q_k D_j X^k +\\ &\quad Z^i_c D_b \delta^\nabla_d X^c Z^d_p \partial_a \delta^\nabla_q X^p Z^q_k D_j X^k -\\ &\quad Z^i_p \partial_a D_b \delta^\nabla_q X^p Z^q_k D_j X^k - Z^i_p \partial_a \delta^\nabla_q X^p Z^q_k D^2_{bj} X^k -\\ &\quad Z^i_p D_b \delta^\nabla_q X^p Z^q_k \partial_a D_j X^k + Z^i_k \partial_a D^2_{bj} X^k\end{aligned} \tag{4.9}$$

In view of Lemma 4.1, Lemma 4.2, Lemma 4.3 and Lemma 4.4,Lemma 4.5, Lemma 4.6, and using the facts

$$\overline{\partial_a D_b \partial_j X^i_0}(t) = \overline{\partial^2_{aj} D_b X^i_0}(t).$$

$$\overline{\partial_a D^2_{bj} X^i_0}(t) =$$
$$(-\frac{t\sin r_0 t}{2r_0} - \frac{\sin^2 r_0 t\cos r_0 t}{6r_0^2})(\delta^i_k - \delta^i_1\delta^1_k)\delta^k_a\delta^\beta_b\delta^\beta_j +$$
$$[(\frac{t\sin r_0 t}{2r_0} - \frac{\sin^2 r_0 t\cos r_0 t}{6r_0^2})(\delta^i_k - \delta^i_1\delta^1_k) +$$
$$(\frac{2t}{r_0}\sin r_0 t\cos r_0 t - \frac{\sin^2 r_0 t}{r_0^2})\delta^i_1\delta^1_k](\delta^\beta_a\delta^k_b\delta^\beta_j + \delta^\beta_a\delta^\beta_b\delta^k_j) +$$
$$(-\frac{t\sin r_0 t}{2r_0} + \frac{1}{2r_0^2}\sin^2 r_0 t\cos r_0 t)(\delta^i_k - \delta^i_1\delta^1_k)\sum_{(a,b,j)}\delta^k_a(\delta_{bj} - \delta^1_b\delta^1_j) +$$
$$-t^2\cos r_0 t(\delta^i_k - \delta^i_1\delta^1_k)\delta^k_a\delta^1_b\delta^1_j,$$

we have

$$\partial_a D_b A^i_j = \mathcal{B}(\varepsilon)$$

Similarly, differentiating (4.7) twice with respect to v, we obtain

$$\begin{aligned}D^2_{ab} A^i_j &= Z^i_c D_a \delta^\nabla_d X^c Z^d_p D_b \delta^\nabla_q X^p Z^q_k D_j X^k +\\ &\quad Z^i_c D_b \delta^\nabla_d X^c Z^d_p D_a \delta^\nabla_q X^p Z^q_k D_j X^k -\\ &\quad Z^i_p D^2_{ab} \delta^\nabla_q X^p Z^q_k D_j X^k - Z^i_p D_a \delta^\nabla_q X^p Z^q_k D^2_{bj} X^k -\\ &\quad Z^i_p D_b \delta^\nabla_q X^p Z^q_k D^2_{aj} X^k + Z^i_k D^3_{abj} X^k\end{aligned} \tag{4.10}$$

Thanks of Lemma 4.1, Lemma 4.2, Lemma 4.3 and Lemma 4.4,Lemma 4.5, Lemma 4.6, together with the facts

$$\overline{D^2_{ab}\partial_j X^i_0}(t) = \overline{\partial_j D^2_{ab} X^i_0}(t).$$

$$\begin{aligned}
&\overline{\partial_a D^2_{bj} X^i_0}(t) = \\
&[(-\frac{\sin^3 r_0 t}{6r_0^3} - \frac{t\cos r_0 t}{2r_0^2} + \frac{\sin r_0 t}{2r_0^3})(\delta^i_k - \delta^i_1\delta^1_k) + \\
&(-\frac{2}{r_0^2} t\cos^2 r_0 t + \frac{2}{r_0^3}\sin r_0 t\cos r_0 t)\delta^i_1\delta^1_k] \sum_{(a,b,j)} \delta^k_a \delta^\beta_b \delta^\beta_j + \\
&(\frac{\sin^3 r_0 t}{2r_0^3} + \frac{3t\cos r_0 t}{2r_0^2} - \frac{3\sin r_0 t}{2r_0^3})(\delta^i_k - \delta^i_1\delta^1_k) \sum_{(a,b,j)} \delta^k_a(\delta_{bj} - \delta^1_b\delta^1_j) + \\
&(-\frac{t^2}{r_0}\sin r_0 t + \frac{2}{r_0^3}\sin r_0 t - \frac{2t\cos r_0 t}{r_0^2})(\delta^i_k - \delta^i_1\delta^1_k) \sum_{(a,b,j)} \delta^k_a\delta^1_b\delta^1_j,
\end{aligned}$$

we get

$$D^2_{ab}A^i_j \quad = \quad D^2_{ab}\bar{A}^i_j + \mathcal{B}(\varepsilon).$$

This finishes the proof of Theorem 4.1. □

Chapter 5

The smoothness of the optimal transport map

This chapter is concerned with the smoothness of the optimal transport map on two classes of compact Riemannian manifolds which are nearly spherical manifolds and Riemannian products of nearly spherical manifolds. The optimal transport map is given by exp(grad u), where the potential function u satisfies a Monge-Ampère type equation. By the method of maximum principle, we prove that the Jacobian of the exponential map at grad u has an uniform positive lower bound. Then the Ma-Trudinger-Wang's device, together with the method of continuity implies the smoothness of the optimal transport map.

5.1 Introduction

Let (M, g) be a closed Riemannian manifold of dimension $n \geq 2$. Assume that (M, g) is smooth.

Let $\mathrm{Hess}^{(c)} u$ be c-Hessian of u, namely,

$$\mathrm{Hess}^{(c)}_m u = \nabla^2_m u + \nabla^2_m c(\cdot, \exp_m \nabla_m u).$$

As in subsection 0.3.1, the C^2 potential function u of the optimal transport map $G(m) = \exp_m \nabla_m u$, pushing forward $\rho_0 dvol$ to $\rho_1 dvol$, satisfies the following Monge-Ampère type equation:

$$\det(d_{\nabla_m u} \exp_m) \det \mathrm{Hess}^{(c)} u = \frac{\rho_0(m)}{\rho_1(G(m))}. \tag{5.1}$$

Conversely, a classical C^2 solution of the above equation is the potential function of the optimal transport map G pushing forward $\rho_0 dvol$ to $\rho_1 dvol$.

We say that a C^2 function $u : M \to \mathbb{R}$ is admissible if for every point $m \in M$, $\nabla_m u \in I(m)$ and $\mathrm{Hess}^{(c)}_m u > 0$.

It is known that the C^2 solutions of (5.1) are unique up to a constant(see [27]).

We consider the regularity of the potential function u, that is, given $(k, \alpha) \in \mathbb{N} \times (0, 1)$, with $k \geq 2$, if both ρ_0 and ρ_1 are $C^{k,\alpha}$, we want to know whether the solution u is $C^{k+2,\alpha}$.

We will address the above problem by the continuity method.

Let $\mathcal{I}$ be the set of the parameter $t \in [0, 1]$ for which there exists a $C^{k+2,\alpha}$ admissible solution u_t of the equation (5.1) by replacing ρ_1 by $\rho_t(\cdot) = (1-t)\rho_0(\cdot) + t\rho_1 \circ G(\cdot)$. To ensure the uniqueness, the solutions u_t are normalized by $\int_M u_t dvol = 0$.

It is clear that $0 \in \mathcal{I}$, so the set $\mathcal{I}$ is nonempty. The openness is derived by an implicit function theorem [53]. The connectedness of the interval $[0, 1]$ will imply that the equation (5.1) admits a $C^{k,\alpha}$ solution if $\mathcal{I}$ is closed.

Delanoë(see Proposition 4.1 in [27]) reduced the closedness to an uniform upper bound on the Hessian of the classical solutions u_t for all $t \in \mathcal{I}$. Moreover, Delanoë(see [27]p.50) also showed that the existence of an uniform upper bound on the Hessian of the classical solutions u_t is equivalent to the following two estimates:

(1) There exists a positive constant δ_0, such that

$$\forall (t,m) \in \mathcal{I} \times M, \det(d_{\nabla_m u_t} \exp_m) \geq \delta_0, \tag{5.2}$$

(2) There exists a positive constant C, such that

$$\mathrm{Hess}_m\, u_t + \mathcal{S}(m, \nabla_m u_t, 1) \leq CId_m, \tag{5.3}$$

for any $(t,m) \in \mathcal{I} \times M$.

In conclusion, Delanoë(see [27]) derived the following fact.

Lemma 5.1. *Given a closed Riemannian manifold and given $(k,\alpha) \in \mathbb{N} \times (0,1)$, with $k \geq 2$, the potential function u is $C^{k+2,\alpha}$ for every couple of $C^{k,\alpha}$ positive probability measures $(\rho_0 dvol, \rho_1 dvol)$, if, for each such couple, the requirements (5.2) and (5.3) are fulfilled. Moreover, if either (1) or (2) fails, there exists a number $t_0 \in [0,1]$ such that, the potential function u_{t_0} is not C^2.*

The first result of this chapter shows the regularity of the optimal transport map on nearly spherical manifold.

Theorem 5.1. *Let (M,g) be a closed n-dimensional Riemannian manifold satisfying the curvature assumption (1.3). Then there exists a positive constant ε_0 depending only n such that if*

$$\|Riem - \frac{1}{2} g \owedge g\|_{C^2(M,g)} < \varepsilon_0,$$

then for any couple $(k,\alpha) \in \mathbb{N} \times (0,1)$, with $k \geq 2$, the potential function of the optimal transport map is $C^{k+2,\alpha}$ for every couple $(\rho_0 dvol, \rho_1 dvol)$ of $C^{k,\alpha}$ positive Borel probability measures on M.

A direct result of Theorem 5.1 is the smoothness of the optimal transport maps on nearly spherical manifold.

Corallary 5.1. *Under the same assumptions as in Theorem 5.1, let $\rho_0 dvol$ and $\rho_1 dvol$ be two smooth positive Borel probability measures on M. Then the optimal transport map is smooth.*

The second result of this chapter concentrates on the regularity of the optimal transport map on the product of nearly spherical manifold.

Theorem 5.2. *Let M_1 and M_2 be two closed Riemannian manifolds of dimension $n_1 \geq 2$ and $n_2 \geq 2$ respectively. Suppose that both M_1 and M_2 satisfy the assumptions (1.3). There exists some positive constant $\varepsilon_0 > 0$ such that if (1.4) holds on M_1 and M_2 with $\varepsilon < \varepsilon_0$, then for any couple $(k,\alpha) \in \mathbb{N} \times (0,1)$, with $k \geq 2$, the potential function of the optimal transport map is $C^{k+2,\alpha}$ for every couple $(\rho_0 dvol, \rho_1 dvol)$ of $C^{k,\alpha}$ positive Borel probability measures on the Riemannian product $M_1 \times M_2$.*

As a direct consequence of Theorem 5.2, we get the smoothness of the optimal transport maps on the product of nearly spherical manifolds.

Corallary 5.2. *Under the same hypothesis as in Theorem 5.2, let $\rho_0 dvol$ and $\rho_1 dvol$ be two smooth positive Borel probability measures on $M_1 \times M_2$. Then the corresponding optimal transport map is smooth.*

At the end of this chapter, we derive that the optimal transport map may not be smooth on some manifolds sufficiently close to the product of the standard spheres in C^4 norm. More generally, we prove the following result.

Theorem 5.3. *Let (M_1, g_1) and (M_2, g_2) be two closed Riemannian manifolds of dimension $n_1 \geq 2$ and $n_2 \geq 2$ respectively. The product metric produced by g_1 and g_2 is denoted by $g^\times$. Then, for any given $\varepsilon > 0$, there exists a metric g on $M_1 \times M_2$ that is conformal to $g^\times$ and satisfies*

$$\|g - g^\times\|_{C^4} < \varepsilon,$$

such that there exist $\rho_0 dvol$ and $\rho_1 dvol$ two smooth positive Borel probability measures on $M_1 \times M_2$, the corresponding optimal transport map on $(M_1 \times M_2, g)$ is not smooth.

If (M_1, g_1) and (M_2, g_2) satisfy the conditions of Theorem 5.1, we know that the optimal transport map is smooth on $(M_1 \times M_2, g^\times)$. By Theorem 5.3, we also know that the smoothness of the optimal transport map is not stable on the perturbed metric of $(M_1 \times M_2, g^\times)$.

5.2 The smoothness of the optimal transport map on nearly spherical manifold

5.2.1 Preliminary

In this subsection, we will establish some a priori estimates of the Monge-Ampère type equation (5.1) and a key proposition.

Lemma 5.2. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 2$. The diameter of (M, g) is denoted by D. Then for all admissible function u with $\int_M u dvol = 0$, the following estimates hold*

$$\max_M |u| \leq D^2,$$
$$\max_M |\nabla u| \leq D.$$

Proof. Let the function u be admissible with $\int_M u dvol = 0$. Fix $m \in M$. By definition of the admissible function, the curve $\exp_m(t\nabla_m u)$ is a minimizing geodesic from m to $\exp_m \nabla_m u$. Thus

$$|\nabla_m u| = d(m, \exp_m \nabla_m u) \leq \mathrm{D}.$$

As a consequence, the oscillation of u is bounded above by D^2. The vanishing average gives the results. □

As mentioned in section 5.1, the existence of a uniform upper bound on the Hessian of the classical solutions u_t is equivalent to estimates (5.2) and (5.3). The Ma-Trudinger-Wang's estimate reduces the existence of a uniform upper bound on the Hessian of the classical solutions u_t to the estimate(5.2) under the assumption that the *MTW tensor* satisfies the *A3S* condition.

Lemma 5.3. *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 2$. Assume that the MTW tensor of M satisfies the A3S condition. Let $\rho_0 dvol$ and $\rho_1 dvol$ be C^2 positive Borel probability measures on M. If the requirement (5.2) is fulfilled, then there exists a positive constant C depending on $n, \delta, \max_{\mathcal{I}\times M} |d\log\rho_t|_{C^2}$, such that the norm of $Hess_m^{(c)} u_t$ is bounded from above C, for every couple $(t, m) \in \mathcal{I} \times M$.*

Proof. The proof is given in Delanoë [27].

To conclude this subsection, we give a key proposition which is used in next subsection.

Let $\mathcal{J}(m,\nu) = -|\nu|^2\mathcal{S}^{-1}(m,\nu,1)$. We are interested in the behaviour of $\mathcal{J}$ when $|\nu| \geq \frac{3\pi}{4}$. So we assume that $|\nu| \geq \frac{3\pi}{4}$ in the following. It is clear that $-|\nu|^2$ is the first eigenvalue of $\mathcal{J}$ with the associated eigenvector $\frac{\nu}{|\nu|}$. By virtue of 1) in Theorem 4.1, there exists a positive number $C > 0$ such that

$$|\mathcal{J}(m,\nu) - \bar{\mathcal{J}}(m,\nu)| \leq C\varepsilon. \tag{5.4}$$

Let us now describe the behavior for the derivatives of $\mathcal{J}$.

Proposition 5.1. *Let (M,g) be a closed n-dimensional Riemannian manifold satisfying the curvature assumptions (1.3) and (1.4). Then there exist positive constants ε_0 and $C > 0$ such that, for any $\varepsilon < \varepsilon_0$, and for any $m_0 \in M, \nu_0 \in I(m_0), |\nu_0| \geq \pi - \delta, 0 < \delta < \frac{\pi}{4}$, the absolute value of the first and second partial derivatives of the components $\mathcal{J}^i_j$ with respect to (x,v) at the point $(0,|\nu_0|,0)$, are all bounded from above by $C\varepsilon$, except the following partial derivatives:*

$$\begin{aligned}
&D_1\mathcal{J}^i_i, D_\beta\mathcal{J}^1_\beta = D_\beta\mathcal{J}^\beta_1, \partial^2_{\beta\beta}\mathcal{J}^i_i, D^2_{11}\mathcal{J}^i_i,\\
&D^2_{1\beta}\mathcal{J}^\beta_1 = D^2_{\beta 1}\mathcal{J}^\beta_1 = D^2_{1\beta}\mathcal{J}^1_\beta = D^2_{\beta 1}\mathcal{J}^1_\beta,\\
&D^2_{\beta\beta}\mathcal{J}^i_i, D^2_{\mu\kappa}\mathcal{J}^\mu_\kappa = D^2_{\kappa\mu}\mathcal{J}^\mu_\kappa = D^2_{\mu\kappa}\mathcal{J}^\kappa_\mu = D^2_{\kappa\mu}\mathcal{J}^\kappa_\mu, \mu \neq \kappa.
\end{aligned}$$

and the following estimates hold:

$$|D_1\mathcal{J}^1_1 + 2\pi| \leq C(\varepsilon+\delta), |D_1\mathcal{J}^\alpha_\alpha + \pi| \leq C(\varepsilon+\delta), \tag{5.5}$$
$$|D_\beta\mathcal{J}^\beta_1 + \pi| \leq C(\varepsilon+\delta), \tag{5.6}$$
$$|\partial^2_{\beta\beta}\mathcal{J}^1_1 - 2\pi^2| \leq C(\varepsilon+\delta), |\partial^2_{\beta\beta}\mathcal{J}^\alpha_\alpha - \pi^2| \leq C(\varepsilon+\delta), \tag{5.7}$$
$$|D^2_{11}\mathcal{J}^i_i + 2| \leq C(\varepsilon+\delta), |D^2_{1\beta}\mathcal{J}^\beta_1| \leq C(\varepsilon+\delta), \tag{5.8}$$
$$|D^2_{\beta\beta}\mathcal{J}^1_1| \leq C(\varepsilon+\delta), |D^2_{\beta\beta}\mathcal{J}^\alpha_\alpha + 1 + 2\delta^\alpha_\beta| \leq C(\varepsilon+\delta), \tag{5.9}$$
$$|D^2_{\mu\kappa}\mathcal{J}^\mu_\kappa + 1| \leq C(\varepsilon+\delta), \mu \neq \kappa. \tag{5.10}$$

Proof. Let x be the Fermi coordinate system associated to the geodesic $\exp_{m_0} s\frac{\nu_0}{|\nu_0|}$ and v be the fiber coordinates of $TM \to M$ naturally associated to x. Set $g = g_{ij}(x)dx^i dx^j, \nu = v^i\partial x_i, \bar{\mathcal{J}} = -|\nu|^2\bar{\mathcal{S}}^{-1}(m,\nu,1)$. The components of $\bar{\mathcal{J}}$ in the Fermi coordinate system are denoted by $\bar{\mathcal{J}}^i_j$, i.e.$\bar{\mathcal{J}} = \bar{\mathcal{J}}^i_j dx^j \otimes \frac{\partial}{\partial x^i}$.

In view of Theorem 4.1, there exists a positive constant $C > 0$ such that

$$|\partial_x\mathcal{J}(m_0,\nu_0) - \partial_x\bar{\mathcal{J}}(m_0,\nu_0)| \leq C\varepsilon, \tag{5.11}$$
$$|D_v\mathcal{J}(m_0,\nu_0) - D_v\bar{\mathcal{J}}(m_0,\nu_0)| \leq C\varepsilon, \tag{5.12}$$
$$|\partial^2_{xx}\mathcal{J}(m_0,\nu_0) - \partial^2_{xx}\bar{\mathcal{J}}(m_0,\nu_0)| \leq C\varepsilon, \tag{5.13}$$
$$|\partial_x D_v\mathcal{J}(m_0,\nu_0) - \partial_x D_v\bar{\mathcal{J}}(m_0,\nu_0)| \leq C\varepsilon, \tag{5.14}$$
$$|D^2_{vv}\mathcal{J}(m_0,\nu_0) - D^2_{vv}\bar{\mathcal{J}}(m_0,\nu_0)| \leq C\varepsilon. \tag{5.15}$$

Thus we only need to calculate the following derivatives:

$$\partial_a\bar{\mathcal{J}}^i_j, D_a\bar{\mathcal{J}}^i_j, \partial^2_{ab}\bar{\mathcal{J}}^i_j, \partial_a D_b\bar{\mathcal{J}}^i_j, D^2_{ab}\bar{\mathcal{J}}^i_j.$$

Let us compute the above derivatives of $\bar{\mathcal{J}}$. To differentiate the components of $\bar{\mathcal{J}}^i_j$, we need an explicit formula for $\bar{\mathcal{J}}^i_j$. By virtue of (7), the map $\bar{\mathcal{J}}$ has the expression

$$\bar{\mathcal{J}}(\xi) = -|\nu|^2\bar{\mathcal{S}}^{-1}(\xi) = -|\nu|^2\xi + (|\nu|^2 - |\nu|\tan|\nu|)(\xi - \langle\xi, \frac{\nu}{|\nu|}\rangle\frac{\nu}{|\nu|}).$$

As a consequence, the components of $\bar{\mathcal{J}}$ in the Fermi coordinate system x are given by

$$\begin{aligned}\bar{\mathcal{J}}^i_j &= -|\nu|^2\delta^i_j + (|\nu|^2 - |\nu|\tan|\nu|)(\delta^i_j - g_{jk}(x)\frac{v^i}{|\nu|}\frac{v^k}{|\nu|}) \\ &= -|\nu|^2\delta^i_j + \varphi(|\nu|)(|\nu|^2\delta^i_j - g_{jk}v^iv^k). \qquad (5.16)\end{aligned}$$

where $\varphi(|\nu|) = 1 - \frac{\tan|\nu|}{|\nu|}$.

Let us compute the first order derivatives of $\bar{\mathcal{J}}^i_j$. Differentiating (5.16) with respect to x and v respectively, one has

$$\begin{aligned}\partial_a\bar{\mathcal{J}}^i_j &= -\partial_a g_{kl}v^kv^l\delta^i_j + \frac{\dot{\varphi}}{2|\nu|}\partial_a g_{pq}v^pv^q(|\nu|^2\delta^i_j - g_{jk}v^iv^k) + \\ &\quad \varphi(\partial_a g_{kl}v^kv^l\delta^i_j - \partial_a g_{jk}v^iv^k), \\ D_a\bar{\mathcal{J}}^i_j &= -2g_{ak}v^k\delta^i_j + \frac{\dot{\varphi}}{|\nu|}g_{ap}v^p(|\nu|^2\delta^i_j - g_{jk}v^iv^k) + \\ &\quad \varphi(2g_{ak}v^k\delta^i_j - g_{jk}\delta^i_av^k - g_{ja}v^i).\end{aligned}$$

Set $r_0 = |\nu_0|$. As $\partial_i g_{kl} = 0$ at the point m_0, we have at the point $(m_0, \nu_0) = (0, r_0, 0)$,

$$\partial_a\bar{\mathcal{J}}^i_j = 0.$$

Combining with (5.11), there holds that

$$|\partial_a\mathcal{J}| \le C\varepsilon.$$

Recalling $g_{ij} = \delta_{ij}$ at the point m_0, we obtain

$$\begin{aligned}D_a\bar{\mathcal{J}}^i_j &= -2r_0\delta^1_a\delta^i_j + r_0^2\dot{\varphi}\delta^1_a(\delta^i_j - \delta^i_1\delta^1_j) + \\ &\quad r_0\varphi(2\delta^1_a\delta^i_j - \delta^i_a\delta^1_j - \delta^i_1\delta^a_j) \\ &= -2r_0\delta^1_a\delta^i_1\delta^1_j + (r_0^2\dot{\varphi} - 2r_0)\delta^1_a(\delta^i_j - \delta^i_1\delta^1_j) + \\ &\quad r_0\varphi(2\delta^1_a\delta^i_j - \delta^i_a\delta^1_j - \delta^i_1\delta^a_j) \\ &= -2r_0\delta^1_a\delta^i_1\delta^1_j + (\tan r_0 - r_0\sec^2 r_0 - 2r_0)\delta^1_a(\delta^i_j - \delta^i_1\delta^1_j) + \\ &\quad (r_0 - \tan r_0)(2\delta^1_a\delta^i_j - \delta^i_a\delta^1_j - \delta^i_1\delta^a_j).\end{aligned}$$

If $a = 1$,

$$\begin{aligned}D_1\bar{\mathcal{J}}^i_j &= -2r_0\delta^i_1\delta^1_j + (\tan r_0 - r_0\sec^2 r_0 - 2r_0)(\delta^i_j - \delta^i_1\delta^1_j) + \\ &\quad 2(r_0 - \tan r_0)(\delta^i_j - \delta^i_1\delta^1_j) \\ &= -2r_0\delta^i_1\delta^1_j + (-\tan r_0 - r_0\sec^2 r_0)(\delta^i_j - \delta^i_1\delta^1_j).\end{aligned}$$

Observe that $D_1\bar{\mathcal{J}}^i_j = 0$ if $i \ne j$. Thus we only consider the case $i = j$.

$$\begin{aligned}D_1\bar{\mathcal{J}}^i_i &= -2r_0\delta^i_1 + (-\tan r_0 - r_0\sec^2 r_0)(1 - \delta^i_1) \\ &= -2\pi\delta^i_1 - \pi(1 - \delta^i_1) + \mathcal{B}(\delta).\end{aligned}$$

Together with (5.12), this gives (5.5).
If $a > 1$ (we note by β)

$$D_\beta\bar{\mathcal{J}}^i_j = (-r_0 + \tan r_0)(\delta^i_\beta\delta^1_j + \delta^i_1\delta^\beta_j).$$

In particular, $D_\beta\bar{\mathcal{J}}^i_j \ne 0$ except when $i = 1, j = \beta$ or $i = \beta, j = 1$. By symmetry of $\mathcal{J}$, it remains to prove the case $i = \beta, j = 1$.

$$\begin{aligned}D_\beta\bar{\mathcal{J}}^\beta_1 &= -r_0 + \tan r_0 \\ &= -\pi + \mathcal{B}(\delta).\end{aligned}$$

Now by (5.12), we get (5.6).

Let us compute the second order derivatives of $\bar{\mathcal{J}}^i_j$. Differentiating (5.16) twice with respect to x, one has

$$
\begin{aligned}
\partial^2_{ab}\bar{\mathcal{J}}^i_j &= -\partial^2_{ab}g_{kl}v^kv^l\delta^i_j + [(\frac{\ddot{\varphi}}{2|\nu|^2} - \frac{\dot{\varphi}}{2|\nu|^3})\partial_a g_{lh}v^lv^h\partial_b g_{pq}v^pv^q + \\
&\quad \frac{\dot{\varphi}}{2|\nu|}\partial^2_{ab}g_{pq}v^pv^q](|\nu|^2\delta^i_j - g_{jk}v^iv^k) + \\
&\quad \frac{\dot{\varphi}}{2|\nu|}\partial_b g_{pq}v^pv^q(\partial_a g_{kl}v^kv^l\delta^i_j - \partial_a g_{jk}v^iv^k) + \\
&\quad \frac{\dot{\varphi}}{2|\nu|}\partial_a g_{pq}v^pv^q(\partial_b g_{kl}v^kv^l\delta^i_j - \partial_b g_{jk}v^iv^k) + \\
&\quad \varphi(\partial^2_{ab}g_{kl}v^kv^l\delta^i_j - \partial^2_{ab}g_{jk}v^iv^k).
\end{aligned}
$$

Evaluating at the point $(m_0, \nu_0) = (0, r_0, 0)$, and by virtue of (1.12), we have

$$
\begin{aligned}
\partial^2_{ab}\bar{\mathcal{J}}^i_j &= -r_0^2\partial^2_{ab}g_{11}\delta^i_j + \frac{r_0^3\dot{\varphi}}{2}\partial^2_{ab}g_{11}(\delta^i_j - \delta^i_1\delta^1_j) + \\
&\quad r_0^2\varphi(\partial^2_{ab}g_{11}\delta^i_j - \partial^2_{ab}g_{j1}\delta^i_1).
\end{aligned}
$$

If $a = 1, 1 \leq b \leq n$ or $1 \leq a \leq n, b = 1$, since $\partial^2_{1i}g_{kl} = 0$ at the point (m_0, ν_0), we thus obtain $\partial^2_{1b}\bar{\mathcal{J}}^i_j = 0$ or $\partial^2_{a1}\bar{\mathcal{J}}^i_j = 0$. From (5.13), we derive $|\partial^2_{1b}\mathcal{J}| \leq C\varepsilon$ or $|\partial^2_{a1}\mathcal{J}| \leq C\varepsilon$. If $a, b > 1$, for $j = 1$,

$$
\partial^2_{ab}\bar{\mathcal{J}}^i_1 = -r_0^2\partial^2_{ab}g_{11}\delta^i_1.
$$

Together with (1.4) and the first expression in (1.13), it follows that

$$
\partial^2_{ab}\bar{\mathcal{J}}^i_1 = 2r_0^2\delta_{ab}\delta^i_1 + \mathcal{B}(\varepsilon).
$$

As a consequence, $\partial^2_{ab}\bar{\mathcal{J}}^i_1 = \mathcal{B}(\varepsilon)$ if $i > 1$ or $a \neq b$, thus $|\partial^2_{ab}\mathcal{J}^i_1| \leq C\varepsilon$. It suffices to consider the case $i = 1$ and $a = b$.

$$
\begin{aligned}
\partial^2_{\beta\beta}\bar{\mathcal{J}}^1_1 &= 2r_0^2 + \mathcal{B}(\varepsilon) \\
&= 2\pi^2 + \mathcal{B}(\varepsilon + \delta).
\end{aligned}
$$

Making use of (5.13), we infer the first inequality in (5.7).
For $j > 1$, by the symmetry of $\mathcal{J}$, we assume that $i > 1$. Recalling (1.13),(1.4), we derive

$$
\begin{aligned}
\partial^2_{ab}\bar{\mathcal{J}}^i_j &= 2r_0^2\delta_{ab}\delta^i_j - r_0^3\dot{\varphi}\delta_{ab}\delta^i_j - 2r_0^2\varphi\delta_{ab}\delta^i_j + \mathcal{B}(\varepsilon) \\
&= (2r_0^2 - 2r_0^2\varphi - r_0^3\dot{\varphi})\delta_{ab}\delta^i_j + \mathcal{B}(\varepsilon) \\
&= (r_0\tan r_0 + r_0^2\sec^2 r_0)\delta_{ab}\delta^i_j + \mathcal{B}(\varepsilon).
\end{aligned}
$$

Note that $\partial^2_{ab}\bar{\mathcal{J}}^i_j = \mathcal{B}(\varepsilon)$ if $a \neq b$ or $i \neq j$, thus it suffices to consider the case $a = b$ and $i = j$.

$$
\begin{aligned}
\partial^2_{\beta\beta}\bar{\mathcal{J}}^\alpha_\alpha &= r_0\tan r_0 + r_0^2\sec^2 r_0 + \mathcal{B}(\varepsilon) \\
&= \pi^2 + \mathcal{B}(\varepsilon + \delta).
\end{aligned}
$$

Exploiting (5.13), we infer the second expression in (5.7).

Differentiating (5.16) twice with respect to x and v respectively, one has

$$\begin{aligned}\partial_a D_b \bar{\mathcal{J}}^i_j &= -2\partial_a g_{bk} v^k \delta^i_j + [(\frac{\ddot{\varphi}}{|\nu|^2} - \frac{\dot{\varphi}}{|\nu|^3})\partial_a g_{lh} v^l v^h g_{bp} v^p + \\ &\quad \frac{\dot{\varphi}}{|\nu|}\partial_a g_{bp} v^p](|\nu|^2 \delta^i_j - g_{jk} v^i v^k) + \\ &\quad \frac{\dot{\varphi}}{|\nu|} g_{bp} v^p (2\partial_a g_{kl} v^k v^l \delta^i_j - \partial_a g_{jk} v^i v^k) + \\ &\quad \frac{\dot{\varphi}}{2|\nu|}\partial_a g_{pq} v^p v^q (2 g_{bk} v^k \delta^i_j - g_{jk}\delta^i_a v^k - g_{jb} v^i) + \\ &\quad \varphi(2\partial_a g_{bk} v^k \delta^i_j - \partial_a g_{jk}\delta^i_b v^k - \partial_a g_{jb} v^i).\end{aligned}$$

Evaluating at the point $(m_0, \nu_0) = (0, r_0, 0)$, and together with the fact $\partial_i g_{kl} = 0$ at the point m_0, this yields

$$\partial_a D_b \bar{\mathcal{J}}^i_j = 0.$$

Combining with (5.11), we get

$$|\partial_a D_b \mathcal{J}| \leq C\varepsilon.$$

Differentiating (5.16) twice with respect to v, one has

$$\begin{aligned}D^2_{ab}\bar{\mathcal{J}}^i_j &= -2 g_{ab}\delta^i_j + [(\frac{\ddot{\varphi}}{|\nu|^2} - \frac{\dot{\varphi}}{|\nu|^3}) g_{ap} v^p g_{bq} v^q + \\ &\quad \frac{\dot{\varphi}}{|\nu|} g_{ab}](|\nu|^2 \delta^i_j - g_{jk} v^i v^k) + \\ &\quad \frac{\dot{\varphi}}{|\nu|} g_{bp} v^p (2 g_{ak} v^k \delta^i_j - g_{jk}\delta^i_a v^k - g_{ja} v^i) + \\ &\quad \frac{\dot{\varphi}}{|\nu|} g_{ap} v^p (2 g_{bk} v^k \delta^i_j - g_{jk}\delta^i_b v^k - g_{jb} v^i) + \\ &\quad \varphi(2 g_{ab}\delta^i_j - g_{ja}\delta^i_b - g_{jb}\delta^i_a).\end{aligned}$$

Evaluating at the point (m_0, ν_0), and using the fact $g_{ij} = \delta_{ij}$ at the point m_0, there holds

$$\begin{aligned}D^2_{ab}\bar{\mathcal{J}}^i_j &= -2\delta_{ab}\delta^i_j + [(r_0^2\ddot{\varphi} - r_0\dot{\varphi})\delta^1_a\delta^1_b + r_0\dot{\varphi}\delta_{ab}](\delta^i_j - \delta^i_1\delta^1_j) + \\ &\quad r_0\dot{\varphi}\delta^1_b(2\delta^1_a\delta^i_j - \delta^i_a\delta^1_j - \delta^i_1\delta^a_j) + \\ &\quad r_0\dot{\varphi}\delta^1_a(2\delta^1_b\delta^i_j - \delta^i_b\delta^1_j - \delta^i_1\delta^b_j) + \\ &\quad \varphi(2\delta_{ab}\delta^i_j - \delta^i_b\delta^a_j - \delta^i_a\delta^b_j).\end{aligned}$$

If $a = b = 1$,

$$\begin{aligned}D^2_{11}\bar{\mathcal{J}}^i_j &= -2\delta^i_j + r_0^2\ddot{\varphi}(\delta^i_j - \delta^i_1\delta^1_j) + 4r_0\dot{\varphi}(\delta^i_j - \delta^i_1\delta^1_j) + \\ &\quad 2\varphi(\delta^i_j - \delta^i_1\delta^1_j) \\ &= -2\delta^i_1\delta^1_j + (r_0^2\ddot{\varphi} + 4r_0\dot{\varphi} + 2\varphi - 2)(\delta^i_j - \delta^i_1\delta^1_j) \\ &= -2\delta^i_1\delta^1_j + (-2r_0\tan r_0\sec^2 r_0 - 2\sec^2 r_0)(\delta^i_j - \delta^i_1\delta^1_j).\end{aligned}$$

Notice that $D^2_{11}\bar{\mathcal{J}}^i_j = 0$ if $i \neq j$. Therefore, it suffices to consider the case $i = j$.

$$\begin{aligned}D^2_{11}\bar{\mathcal{J}}^i_i &= -2\delta^i_1 - 2(1 - \delta^i_1) + \mathcal{B}(\delta) \\ &= -2 + \mathcal{B}(\delta).\end{aligned}$$

Together with (5.15), we deduce the first inequality in (5.8).
If $a = 1, b > 1$, or $a > 1, b = 1$. In view of $D^2_{ab}\bar{\mathcal{J}}^i_j = D^2_{ba}\bar{\mathcal{J}}^i_j$, we only need to consider the case $a = 1, b > 1$.

$$D^2_{1b}\bar{\mathcal{J}}^i_j = -(r_0\dot{\varphi} + \varphi)(\delta^i_b\delta^1_j + \delta^i_1\delta^b_j).$$

As a consequence, $D^2_{1b}\bar{\mathcal{J}}^i_j \neq 0$ except when $i=1, j=b$ or $i=b, j=1$. By the symmetry of $\mathcal{J}$, it suffices to consider the case $i=b, j=1$.

$$\begin{aligned} D^2_{1b}\bar{\mathcal{J}}^b_1 &= -(r_0\dot{\varphi}+\varphi) \\ &= \sec^2 r_0 - 1 \\ &= \mathcal{B}(\delta). \end{aligned}$$

Combining with (5.15), we get the second inequality in (5.8).
If $a, b > 1$,

$$\begin{aligned} D^2_{ab}\bar{\mathcal{J}}^i_j &= 2(\varphi-1)\delta_{ab}\delta^i_j + r_0\dot{\varphi}\delta_{ab}(\delta^i_j - \delta^i_1\delta^1_j) - \\ & \quad \varphi(\delta^i_b\delta^a_j + \delta^i_a\delta^b_j). \end{aligned}$$

Observe that $D^2_{ab}\bar{\mathcal{J}}^i_j \neq 0$ except when $a=b, i=j$ or $a\neq b, i=a, j=b$ or $a\neq b, i=b, j=a$.
The case $a=b, i=j$:

$$\begin{aligned} D^2_{aa}\bar{\mathcal{J}}^i_i &= 2(\varphi-1) + r_0\dot{\varphi}(1-\delta^i_1) - 2\varphi\delta^i_a \\ &= -2\frac{\tan r_0}{r_0} + (\frac{\tan r_0}{r_0} - \sec^2 r_0)(1-\delta^i_1) + \\ & \quad 2(\frac{\tan r_0}{r_0} - 1)\delta^i_a \\ &= -(1-\delta^i_1) - \delta^i_a + \mathcal{B}(\delta). \end{aligned}$$

Combining with (5.15), we get (5.9).
The case $a\neq b, i\neq j, i=a, j=b$ or $a\neq b, i\neq j, i=b, j=a$:

$$\begin{aligned} D^2_{ab}\bar{\mathcal{J}}^a_b = D^2_{ab}\bar{\mathcal{J}}^b_a &= -\varphi \\ &= \frac{\tan r_0}{r_0} - 1 \\ &= -1 + \mathcal{B}(\delta). \end{aligned}$$

Together with (5.15), we derive (5.10). This ends the proof of Proposition 5.1. □

5.2.2 Proof of Theorem 5.1

In this subsection, we are going to prove Theorem 5.1. Assume that the curvatures of M satisfy (1.3) and (1.4). Fix any couple $(k,\alpha) \in \mathbb{N}\times(0,1)$, with $k \geq 2$. Let $(\rho_0 dvol, \rho_1 dvol)$ be a couple of $C^{k,\alpha}$ positive Borel probability measures on M.

As mentioned in Chapter 3, the *MTW tensor* on nearly spherical manifold satisfies the *A3S* condition. Thus the condition of Lemma 5.3 is satisfied. Granted Lemma 5.1, to complete Theorem 5.1, it is sufficient to prove (5.2).

Fix $(t,m) \in \mathcal{I}\times M$. Note that the left side in (5.2) is related to the initial Jacobi matrix J_0, i.e.

$$\det d_{\nabla_m u_t}\exp_m = \det J_0(m, \nabla_m u_t, 1).$$

It is also useful to recall that the gradient of u_t at m locates in the injectivity domain at m. Then by the Bishop's theorem, $\det d_{\nabla_m u_t}\exp_m$ is uniformly bounded from above by 1 if M has non-negative Ricci curvature. As mentioned in Section 1.2.1, we know that $\det d_{\nabla_m u_t}\exp_m$ is positive. But $\det d_{\nabla_m u_t}\exp_m$ may not has a positive lower bound. Recall $\det J_0(m,\nu,1)$ vanishes if (and only if) $\exp_m \nu$ is conjugate to m. Hence, the estimate (5.2) is not obvious. For instance, on the round sphere $\mathbb{S}^n, \det d_{\nabla_m u_t}\exp_m = (\frac{\sin|\nabla_m u_t|}{|\nabla_m u_t|})^{n-1}$ is close to zero as $|\nabla_m u_t|$ approaches π.

Observe that the assumption (1.3) infers that the length of gradient $\nabla_m u_t$ is strictly less than π. Making use of Lemma 1.2, the estimate (5.2) is obvious if $\max\{|\nabla_m u_t| : m\in M\} \leq \frac{3\pi}{4}$. Thus, without loss of generality, we assume that there exists at least a point such that the length of gradient ∇u_t at that point is not less than $\frac{3\pi}{4}$.

Proof of Theorem 5.1. We assume that there exists at least a point such that the length of gradient ∇u_t at that point is not less than $\frac{3\pi}{4}$. We will use the method of maximum principle to prove (5.2). We need to constructed an appropriate test function.

Let $\mathcal{J}(m, \nabla_m u_t) = -|\nabla_m u_t|^2 \mathcal{S}^{-1}(m, \nabla_m u_t, 1)$. Consider the minimization problem:

$$\min\{\langle \mathcal{J}\xi, \xi\rangle : (m, \xi) \in TM, \frac{3\pi}{4} \le |\nabla u_t|_m, |\xi|_m = 1, \xi \perp \nabla_m u_t\}.$$

Suppose that the minimum is attained at the point (m_0, ξ_0). We consider the test function:

$$h\,(m, \xi) = \frac{\langle \mathcal{J}\xi, \xi\rangle + \langle \xi, \nabla u_t\rangle^2}{|\xi|^2 - \frac{\langle \xi, \nabla u_t\rangle^2}{|\nabla u_t|^2}}.$$

Then h attains the minimum at the point (m_0, ξ_0) in a neighborhood of the point (m_0, ξ_0) in TM. To see this, let $\xi^\perp$ be the orthonormal part of ξ. Then

$$\frac{\langle \mathcal{J}\xi^\perp, \xi^\perp\rangle}{|\xi^\perp|^2} = h\,(m, \xi)\,.$$

By continuity, the test function h attains the local minimum at the point (m_0, ξ_0) in a neighborhood of the point (m_0, ξ_0) in TM.

The minimum $h(m_0, \xi_0)$ has a nice explanation, that is the second eigenvalue of the self-adjoint operator $\mathcal{J}$. Specifically, as h is bilinear on the orthogonal complement subspaces $(\nabla_m u_t)^\perp$ with respect to ξ, thus the minimum $h(m_0, \xi_0)$ is the second eigenvalue of the self-adjoint operator $\mathcal{J}(m_0, \nabla_{m_0} u_t)$ with the associated eigenvector ξ_0.

As a consequence of the above explanation, a necessary condition for (5.2) is that the minimum $h(m_0, \xi_0)$ has a positive lower bound. Thus (5.2) is transformed into the positive lower bound of $h(m_0, \xi_0)$. Notice that the minimum $h(m_0, \xi_0)$ has to be positive. To see this, from the Hessian Comparison Theorem, we know that $-\mathcal{S}^\perp$ is not less than $-\frac{r_0 \cos r_0}{\sin r_0} I_{n-1}$ which is positive definite when $r_0 = |\nabla_{m_0} u_t| \in (\frac{\pi}{2}, \pi)$. Thus the minimum $h(m_0, \xi_0)$ is positive.

In view of (5.4), we deduce

$$-\frac{r_0 \sin r_0}{\cos r_0} - C\varepsilon \le h(m_0, \xi_0) \le -\frac{r_0 \sin r_0}{\cos r_0} + C\varepsilon, \tag{5.17}$$

where $r_0 \ge \frac{3\pi}{4}$.

Since the real value function $-\frac{r \sin r}{\cos r}$ is decreasing in $(\frac{\pi}{2}, \pi)$, thus the right inequality infers that the minimum $h(m_0, \xi_0)$ has a positive upper bound. If $h(m_0, \xi_0)$ has a positive lower bound which is independent of the densities, by choosing ε sufficiently small, the right inequality also infers that $r_0 \le \pi - \hat{\delta}$ for some $\hat{\delta} > 0$. This is the uniform gradient estimate.

In order to differentiate the test function h, it needs to rule out the boundary case. Since the function $-\frac{r \sin r}{\cos r}$ is decreasing in $(\frac{\pi}{2}, \pi)$, the left inequality in (5.17) ensures that we can assume that $r_0 > \pi - \delta, 0 < \delta < \frac{\pi}{4}$.

Henceforth, we will drop freely the subscript t.

Take the Fermi coordinate system x along the geodesic $\exp_{m_0}(s\frac{\nabla_{m_0} u}{|\nabla_{m_0} u|})$.

Some local notations

Components of some tensors in x will be denoted by:

$$\text{grad } u = \nabla^i u(m)\frac{\partial}{\partial x^i}, \nabla^2_m u = \nabla^i_j u(m) dx^j \otimes \frac{\partial}{\partial x^i}, S = S^i_j(m, \nu, 1) dx^j \otimes \frac{\partial}{\partial x^i},$$

$$\mathcal{J} = \mathcal{J}^i_j(m, \nu) dx^j \otimes \frac{\partial}{\partial x^i}, \mathcal{H} = \mathcal{H}^i_j(m) dx^j \otimes \frac{\partial}{\partial x^i}, \mathcal{F} = \mathcal{F}^i_j(m) dx^j \otimes \frac{\partial}{\partial x^i},$$

where, $\mathcal{H}^i_j = \nabla^i_j u + S^i_j, \mathcal{H}^i_k \mathcal{F}^k_j = \delta^i_j$.

Fix $\xi \in T_m M$. The coordinate of ξ in the Fermi coordinate system x is denoted by $\xi = \xi^i \partial_i$. Then

$$\langle \mathcal{J}\xi, \xi\rangle = \mathcal{J}^a_b g_{ap}\xi^b\xi^p, |\xi|^2 = g_{ab}\xi^a\xi^b, \langle \xi, \nabla u\rangle = \xi^a \nabla_a u. \tag{5.18}$$

In the following all terms are evaluated at the point $(x, v) = (0, r_0, 0)$. It will be implicitly understood throughout the calculations. The components of ξ_0 are denoted by ξ_0^i, i.e. $\xi_0 = \xi_0^i \partial_i, \xi_0^1 = 0$.

The first derivative condition

Differentiating the test function h with respect to x^i, the first derivative condition for the critical point could be read as:

$$(\partial_i \mathcal{J}^\alpha_\beta + D_k \mathcal{J}^\alpha_\beta \nabla^k_i u)\xi_0^\alpha \xi_0^\beta = 0. \tag{5.19}$$

The second derivative condition

Differentiating twice on the test function h with respect to x^i and x^j respectively, the second derivative condition read as follows:

$$0 \le I_1 + II_1 + III_1 + IV_1 + V_1, \tag{5.20}$$

where

$$\begin{aligned}
I_1 &= -\langle \mathcal{J}\xi_0, \xi_0\rangle \mathcal{F}^i_j \partial^2_{ij} g_{\alpha\beta}\xi_0^\alpha\xi_0^\beta + \mathcal{F}^i_j \partial^2_{ij} g_{\alpha k}\mathcal{J}^k_\beta \xi_0^\alpha \xi_0^\beta,\\
II_1 &= -\mathcal{F}^i_j \partial_j \Gamma^k_{il}\nabla^l u D_k \mathcal{J}^\alpha_\beta \xi_0^\alpha\xi_0^\beta + \mathcal{F}^i_j \partial^2_{ij}\mathcal{J}^\alpha_\beta \xi_0^\alpha\xi_0^\beta,\\
III_1 &= 2\mathcal{F}^i_j \nabla^j_k u \partial_i D_k \mathcal{J}^\alpha_\beta \xi_0^\alpha \xi_0^\beta,\\
IV_1 &= 2(1 + \frac{1}{r_0^2}\langle \mathcal{J}\xi_0, \xi_0\rangle)\mathcal{F}^i_j \nabla^\alpha_i u \nabla^j_\beta u \xi_0^\alpha\xi_0^\beta +\\
&\quad \mathcal{F}^i_j \nabla^k_i u \nabla^j_l u D^2_{kl}\mathcal{J}^\alpha_\beta \xi_0^\alpha\xi_0^\beta,\\
V_1 &= \mathcal{F}^i_j \partial_j \nabla^k_i u D_k \mathcal{J}^\alpha_\beta \xi_0^\alpha \xi_0^\beta.
\end{aligned}$$

At the point m_0, the potential function u satisfies the equation:

$$\det J_0 \det(H^i_j) = \frac{\rho_0}{\rho_t}. \tag{5.21}$$

It is clear that the positive definiteness of the the matrix (H^i_j) implies that $\nabla^1_1 u$ is strictly greater than -1. We will also require the following expression:

$$\mathcal{S}(0, r_0, 0, 1) = \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{S}^\alpha_\beta \end{bmatrix}. \tag{5.22}$$

Note that $-\mathcal{S}^\perp = -(\mathcal{S}^\alpha_\beta)$ has a uniform lower bound under the curvature assumption (1.3). Indeed, from the Hessian Comparison Theorem, we know that $-\mathcal{S}^\perp$ is not less than $-\frac{r_0 \cos r_0}{\sin r_0} I_{n-1}$. Making use of the fact that the real function $-\frac{t\cos t}{\sin t}$ is increasing in $(\frac{\pi}{2}, \pi)$, we have for $r_0 \ge \frac{3\pi}{4}$,

$$-\mathcal{S}^\perp \ge \frac{3\pi}{4} I_{n-1}. \tag{5.23}$$

We will calculate each term from I_1 to V_1.

The term I_1 Since $g_{ij} = \delta_{ij}$ on the axis and $\mathcal{J}^1_\alpha = 0$ at the point $(0, r_0, 0)$, the term I_1 can be written as:

$$I_1 = -\langle \mathcal{J}\xi_0, \xi_0\rangle \mathcal{F}^\varphi_\psi \partial^2_{\varphi\psi} g_{\alpha\beta}\xi_0^\alpha\xi_0^\beta + \mathcal{F}^\varphi_\psi \partial^2_{\varphi\psi} g_{\alpha\iota}\mathcal{J}^\iota_\beta \xi_0^\alpha\xi_0^\beta.$$

Making use of (1.14), it follows that

$$I_1 = \frac{2}{3}\langle \mathcal{J}\xi_0, \xi_0\rangle \mathcal{F}^\varphi_\psi R_{\varphi\alpha\psi\beta}\xi_0^\alpha\xi_0^\beta - \frac{2}{3}\mathcal{F}^\varphi_\psi R_{\varphi\alpha\psi\iota}\mathcal{J}^\iota_\beta \xi_0^\alpha\xi_0^\beta.$$

Together with the curvature assumption (1.4), the positive definiteness of $(\mathcal{F}^i_j)$ and the uniform bound for the norm of $\mathcal{J}$, we have

$$\begin{aligned} I_1 &\leq \frac{2}{3}\langle \mathcal{J}\xi_0,\xi_0\rangle \mathcal{F}^\alpha_\alpha - \frac{2}{3}\langle \mathcal{J}\xi_0,\xi_0\rangle \mathcal{F}^\alpha_\beta \xi_0^\alpha \xi_0^\beta - \\ &\quad \frac{2}{3}\langle \mathcal{J}\xi_0,\xi_0\rangle \mathcal{F}^\alpha_\alpha + \frac{2}{3}\mathcal{F}^\alpha_\varphi \mathcal{J}^\varphi_\beta \xi_0^\alpha \xi_0^\beta + C\varepsilon \mathcal{F}^\alpha_\alpha \\ &= -\frac{2}{3}\langle \mathcal{J}\xi_0,\xi_0\rangle \mathcal{F}^\alpha_\beta \xi_0^\alpha \xi_0^\beta + \frac{2}{3}\mathcal{F}^\alpha_\varphi \mathcal{J}^\varphi_\beta \xi_0^\alpha \xi_0^\beta + C\varepsilon \mathcal{F}^\alpha_\alpha \\ &\leq -\frac{2}{3}\langle (\mathcal{J}-\bar{\mathcal{J}})\xi_0,\xi_0\rangle \mathcal{F}^\alpha_\beta \xi_0^\alpha \xi_0^\beta + C\varepsilon \mathcal{F}^\alpha_\alpha, \end{aligned}$$

where the last inequality follows from (5.4).

By virtue of (5.4) and the positive definiteness of $(\mathcal{F}^i_j)$ again, we derive that there exists a universal constant such that the following upper bound holds:

$$I_1 \leq C\varepsilon \mathcal{F}^\alpha_\alpha. \tag{5.24}$$

The term II_1 There are two terms in II_1. Proposition 5.1 derives that the second term is bounded by $\pi^2\mathcal{F}^\alpha_\alpha + C(\varepsilon+\delta)\mathcal{F}^i_i$. We mainly deal with the first term. By the first expression in (1.15) and $\nabla u = (r_0, 0)$ at the point m_0, we get

$$\begin{aligned} -\mathcal{F}^i_j \partial_j \Gamma^k_{il} \nabla^l u D_k \mathcal{J}^\alpha_\beta \xi_0^\alpha \xi_0^\beta &= -r_0 \mathcal{F}^i_j R^k_{ij1} D_k \mathcal{J}^\alpha_\beta \xi_0^\alpha \xi_0^\beta \\ &= -r_0 \mathcal{F}^\varphi_\psi R^1_{\varphi\psi 1} D_1 \mathcal{J}^\alpha_\beta \xi_0^\alpha \xi_0^\beta - r_0 \mathcal{F}^i_j R^\varphi_{ij1} D_\varphi \mathcal{J}^\alpha_\beta \xi_0^\alpha \xi_0^\beta \\ &\leq -\pi r_0 \mathcal{F}^\alpha_\alpha + C(\varepsilon+\delta)\mathcal{F}^i_i, \end{aligned}$$

where the last inequality holds because of (1.4) and Proposition 5.1.

Thus we derive that there exists a positive constant $C > 0$ such that the following estimate holds:

$$\begin{aligned} II_1 &\leq \pi(\pi - r_0)\mathcal{F}^\alpha_\alpha + C(\varepsilon+\delta)\mathcal{F}^i_i \\ &\leq C(\varepsilon+\delta)\mathcal{F}^i_i. \end{aligned} \tag{5.25}$$

The term III_1 Using that $\mathcal{F}^i_j \nabla^j_k u = \delta^i_k - \mathcal{F}^i_j \mathcal{S}^j_k$, the term III_1 becomes:

$$III_1 = 2\partial_k D_k \mathcal{J}^\alpha_\beta \xi_0^\alpha \xi_0^\beta - 2\mathcal{F}^i_j \mathcal{S}^j_k \partial_i D_k \mathcal{J}^\alpha_\beta \xi_0^\alpha \xi_0^\beta.$$

It is easy to see that the first term is bounded by $C\varepsilon$ by Proposition 5.1. For the second term, $\mathcal{S}^j_k$ is unbounded as the gradient goes to the conjugate locus. From $\mathcal{F}\mathcal{S} = \mathcal{S}^{-1}\mathcal{S}\mathcal{F}\mathcal{S}$ and the boundedness of $\mathcal{S}^{-1}$, we infer the existence of a positive constant C such that:

$$III_1 \leq C\varepsilon(1 + \mathcal{F}^i_j \mathcal{S}^k_i \mathcal{S}^j_k).$$

Let us observe that the following identity holds:

$$\mathcal{F}^i_j \nabla^k_i u \nabla^j_l u = \nabla^k_l u - \mathcal{S}^k_l + \mathcal{F}^i_j \mathcal{S}^k_i \mathcal{S}^j_l. \tag{5.26}$$

In particular,

$$\mathcal{F}^i_j \nabla^\alpha_i u \nabla^j_\beta u = \mathcal{H}^\alpha_\beta - 2\mathcal{S}^\alpha_\beta + \mathcal{F}^i_j S^\alpha_i S^j_\beta.$$

By (5.23),(5.22) and the positive definiteness of $(\mathcal{H}^i_j), (-\mathcal{S}^\alpha_\beta)$, we get

$$III_1 \leq C\varepsilon(\mathcal{F}^1_1 + \mathcal{F}^i_j \nabla^\alpha_i u \nabla^j_\alpha u). \tag{5.27}$$

The term IV_1 Splitting the negative term $\mathcal{F}^i_j \nabla^k_i u \nabla^j_l u D^2_{kl} \mathcal{J}^\alpha_\beta \xi_0^\alpha \xi_0^\beta$ into four parts, we get

$$\begin{aligned} IV_1 &= \mathcal{F}^i_j \nabla^1_i u \nabla^j_1 u D^2_{11} \mathcal{J}^\alpha_\beta \xi_0^\alpha \xi_0^\beta + 2\mathcal{F}^i_j \nabla^1_i u \nabla^j_\iota u D^2_{1\iota} \mathcal{J}^\alpha_\beta \xi_0^\alpha \xi_0^\beta + \\ &\quad \mathcal{F}^i_j \nabla^\varphi_i u \nabla^j_\varphi u D^2_{\varphi\varphi} \mathcal{J}^\alpha_\beta \xi_0^\alpha \xi_0^\beta + \sum_{\varphi\neq\psi} \mathcal{F}^i_j \nabla^\varphi_i u \nabla^j_\psi u D^2_{\varphi\psi} \mathcal{J}^\alpha_\beta \xi_0^\alpha \xi_0^\beta + \\ &\quad 2\left(1 + \frac{1}{r_0^2}\langle \mathcal{J}\xi_0,\xi_0\rangle\right)\mathcal{F}^i_j \nabla^\alpha_i u \nabla^j_\beta u \xi_0^\alpha \xi_0^\beta. \end{aligned}$$

Using Proposition 5.1, we infer that there exists a positive constant C such that

$$\begin{aligned} IV_1 &\leq [C(\varepsilon+\delta)-2]\mathcal{F}^i_j\nabla^1_i u\nabla^j_1 u + \\ &\quad [C(\varepsilon+\delta)+\frac{2}{r_0^2}\langle\mathcal{J}\xi_0,\xi_0\rangle-1]\mathcal{F}^i_j\nabla^\alpha_i u\nabla^j_\alpha u. \end{aligned}$$

In view of (5.26), it follows that

$$\mathcal{F}^i_j\nabla^1_i u\nabla^j_1 u = \mathcal{H}^1_1 - 2 + \mathcal{F}^1_1.$$

For ε and δ small enough($\varepsilon,\delta<\frac{1}{C}$), by the positive definiteness of $(\mathcal{H}^i_j)$, we get

$$\begin{aligned} IV_1 &\leq [C(\varepsilon+\delta)-2](\mathcal{F}^1_1-2) + \\ &\quad [C(\varepsilon+\delta)+\frac{2}{r_0^2}\langle\mathcal{J}\xi_0,\xi_0\rangle-1]\mathcal{F}^i_j\nabla^\alpha_i u\nabla^j_\alpha u. \end{aligned} \tag{5.28}$$

The term V_1 The term V_1 involves the third derivatives of u. After commuting the third derivatives of u, the term V_1 can be written:

$$V_1 = \mathcal{F}^i_j(\partial_k\nabla^j_i u + r_0 R^k_{i1j})D_k\mathcal{J}^\alpha_\beta\xi^\alpha_0\xi^\beta_0.$$

We first compute the third derivative. In the Fermi coordinate system x, the determinant of the positive matrix $(g_{ij})_{1\leq i,j\leq m}$ is denoted by $|g|$. Recall the equation (3), by definition of $\det d_{\nabla_m u}\exp_m$, the potential function u satisfies the Monge-Ampère type equation

$$\det \mathrm{Hess}^{(c)} u = \frac{\sqrt{|g|(x)}\rho_0(x)}{\sqrt{|g|(X)}\det(D_v X)\rho_t(x)}. \tag{5.29}$$

By taking the logarithm and differentiating the associated equation with respect to the variable x^k at the point $(0,r_0,0)$, we obtain

$$\begin{aligned} \mathcal{F}^i_j\partial_k\nabla^j_i u &= \frac{\partial_k\rho_0}{\rho_0} - \frac{(1-t)\partial_k\rho_0 + t\partial_p\rho_1(\partial_k X^p + D_l X^p\nabla^l_k u)}{\rho_t} - \\ &\quad B^p_q(\partial_k D_p X^q + D^2_{pl}X^q\nabla^l_k u) + \mathcal{F}^i_j\mathcal{S}^q_i\mathcal{S}^j_p\partial_k A^p_q + \\ &\quad \mathcal{F}^i_j\mathcal{S}^q_i\mathcal{S}^j_p D_l A^p_q\nabla^l_k u \\ &= \frac{\partial_k\rho_0}{\rho_0} - \frac{(1-t)\partial_k\rho_0 + t\partial_p\rho_1\partial_k X^p}{\rho_t} - B^p_q\partial_k D_p X^q + \\ &\quad \mathcal{F}^i_j\mathcal{S}^q_i\mathcal{S}^j_p\partial_k A^p_q + (-\frac{t\partial_p\rho_1 D_l X^p}{\rho_t} - B^p_q D^2_{pl}X^q + \\ &\quad \mathcal{F}^i_j\mathcal{S}^q_i\mathcal{S}^j_p D_l A^p_q)\nabla^l_k u, \end{aligned}$$

where the matrix (B^p_q) is the inverse of the matrix $(D_j X^i)$ and (A^p_q) is the inverse of the matrix $(\mathcal{S}^i_j)$.

Thus we get the following simplified expressions:

$$\begin{aligned} V_1 &= [\frac{\partial_k\rho_0}{\rho_0} - \frac{(1-t)\partial_k\rho_0 + t\partial_p\rho_1\partial_k X^p}{\rho_t} - B^p_q\partial_k D_p X^q + \\ &\quad \mathcal{F}^i_j\mathcal{S}^q_i\mathcal{S}^j_p\partial_k A^p_q]D_k\mathcal{J}^\alpha_\beta\xi^\alpha_0\xi^\beta_0 + (-\frac{t\partial_p\rho_1 D_l X^p}{\rho_t} - B^p_q D^2_{pl}X^q + \\ &\quad \mathcal{F}^i_j\mathcal{S}^q_i\mathcal{S}^j_p D_l A^p_q)\nabla^l_k u D_k\mathcal{J}^\alpha_\beta\xi^\alpha_0\xi^\beta_0 + r_0\mathcal{F}^i_j R^k_{i1j}D_k\mathcal{J}^\alpha_\beta\xi^\alpha_0\xi^\beta_0. \end{aligned}$$

By the critical condition (5.19), Lemma 4.3, Lemma 4.4, Proposition 5.1, Cauchy-Schwarz inequality, we deduce that there exists a positive constant C such that

$$\begin{aligned} V_1 &\leq C\mathfrak{M} + C(\varepsilon+\delta)\mathcal{F}^1_1 + [C(\varepsilon+\delta)-\pi^2]\mathcal{F}^\alpha_\alpha + \\ &\quad C(\varepsilon+\delta)\mathcal{F}^i_j\nabla^\alpha_i u\nabla^j_\alpha u, \end{aligned} \tag{5.30}$$

where $\mathfrak{M} = \max\{\max_M |d\log\rho_0|, \max_M |d\log\rho_1|\}$.
Plugging the upper bounds (5.24),(5.25),(5.27),(5.28),(5.30) into the inequality (5.20), we obtain the following inequality:

$$\begin{aligned} 0 \;\le\;& 4 + C\mathfrak{M} + [4C(\varepsilon+\delta) - 2]\mathcal{F}_1^1 + [3C(\varepsilon+\delta) - \pi^2]\mathcal{F}_\alpha^\alpha + \\ & [3C(\varepsilon+\delta) + \frac{2}{r_0^2}\langle \mathcal{J}\xi_0, \xi_0\rangle - 1]\mathcal{F}_j^i \nabla_i^\alpha u \nabla_\alpha^j u \\ =\;& 4 + C\mathfrak{M} + [4C(\varepsilon+\delta) - 2]\mathcal{F}_1^1 + [3C(\varepsilon+\delta) - \pi^2]\mathcal{F}_\alpha^\alpha + \\ & [3C(\varepsilon+\delta) - \frac{1}{2}]\mathcal{F}_j^i \nabla_i^\alpha u \nabla_\alpha^j u + (\frac{2}{r_0^2}\langle \mathcal{J}\xi_0, \xi_0\rangle - \frac{1}{2})\mathcal{F}_j^i \nabla_i^\alpha u \nabla_\alpha^j u. \end{aligned}$$

Fix $\varepsilon < \frac{1}{24C}, \delta < \frac{1}{24C}$. Therefore, we get the inequality:

$$\begin{aligned} 0 \;\le\;& 4 + C\mathfrak{M} - \frac{5}{3}\mathcal{F}_1^1 + (\frac{1}{4} - \pi^2)\mathcal{F}_\alpha^\alpha - \frac{1}{4}\mathcal{F}_j^i \mathcal{S}_i^\alpha \mathcal{S}_\alpha^j + \\ & (\frac{2}{r_0^2}\langle \mathcal{J}\xi_0, \xi_0\rangle - \frac{1}{2})\mathcal{F}_j^i \nabla_i^\alpha u \nabla_\alpha^j u \\ \le\;& 4 + C\mathfrak{M} - \frac{1}{4}\mathcal{F}_j^i \mathcal{S}_i^k \mathcal{S}_k^j + (\frac{2}{r_0^2}\langle \mathcal{J}\xi_0, \xi_0\rangle - \frac{1}{2})\mathcal{F}_j^i \nabla_i^\alpha u \nabla_\alpha^j u. \end{aligned}$$

If $\langle \mathcal{J}\xi_0, \xi_0\rangle \ge \frac{(\pi-\delta)^2}{4}$, there is nothing to prove. We only have to consider the case $\langle \mathcal{J}\xi_0, \xi_0\rangle \le \frac{(\pi-\delta)^2}{4}$. Observe that the last term is non-positive in this case, thus

$$0 \;\le\; 4 + C\mathfrak{M} - \frac{1}{4}\mathcal{F}_j^i \mathcal{S}_i^k \mathcal{S}_k^j.$$

By the inequality of arithmetic and geometric means and the equation (5.21), we get that there exists a positive number $\tilde{C}$ such that

$$\mathcal{F}_j^i \mathcal{S}_i^k \mathcal{S}_k^j \ge \tilde{C}(\min\{1, \frac{\rho_1}{\rho_0}\})^{\frac{1}{n}} \frac{1}{\langle \mathcal{J}\xi_0, \xi_0\rangle^{\frac{1}{n}}}.$$

Thus $\langle \mathcal{J}\xi_0, \xi_0\rangle$ is bounded below by $\delta_0 = \frac{\tilde{C}^n \min\{1, \frac{\rho_1}{\rho_0}\}}{(16+4C\mathfrak{M})^n}$. In conclusion, we just need to choose ε, δ such that $0 < \varepsilon < \frac{1}{24C}, 0 < \delta < \min\{\frac{1}{24C}, \pi - 2\sqrt{\delta_0}\}$, then the estimate (5.2) is proved. By the continuity method, we prove Theorem 5.1.

5.3 The smoothness of the optimal transport map on product manifolds

In this section, we investigate the smoothness of the optimal transport map on Riemannian product manifold of nearly spherical manifolds. By Lemma 5.1, the regularity of the optimal transport map reduces to (5.2) and (5.3).

Let $(M_1, \tilde{g})$ and $(M_2, \hat{g})$ be two closed Riemannian manifolds of dimension $n_1 \ge 2$ and $n_2 \ge 2$ respectively. Suppose that both M_1 and M_2 satisfy (1.3) and (1.4).

It is known that the sectional curvatures of $M_1 \times M_2$ are non-negative and may vanish. Using Corollary 3.1, the corresponding *MTW tensor* satisfies *A3W* condition and may vanish on some directions. Moreover, the *MTW tensor* is non-negative.

The non-trivial cut locus and the vanishing of *MTW tensor* are the main obstacles of the smoothness. The strategy is to establish that the optimal transport map uniformly stay away from the cut locus by the maximum principle. This result allows to derive the uniformly second order derivative estimate. Then the method of continuity implies the smoothness of the optimal transport map.

5.3.1 Preliminary

In this subsection, we recall some facts about product Riemannian manifolds.

Let $(M_1, \tilde{g})$ and $(M_2, \hat{g})$ be two complete smooth Riemannian manifolds of dimension $n_1 \geq 2$ and $n_2 \geq 2$ respectively. It is well known that the tangent vector space has the decomposition

$$\forall (\tilde{m}, \hat{m}) \in M_1 \times M_2,\ T_{(\tilde{m},\hat{m})}(M_1 \times M_2) = T_{\tilde{m}} M_1 \oplus T_{\hat{m}} M_2.$$

Denote by $(M_1 \times M_2, g)$ the Riemannian product of $(M_1, \tilde{g})$ and $(M_2, \hat{g})$. We denote by $\sim$ and $\wedge$ the projection mappings of $T(M_1 \times M_2)$ to TM_1 and TM_2 respectively. For vector fields $X = \widetilde{X} + \widehat{X}, Y = \widetilde{Y} + \widehat{Y}$ on $M_1 \times M_2$. And the Levi-Civita connection ∇ of $(M_1 \times M_2, g)$ is given by

$$\nabla_X Y = \widetilde{\nabla}_{\widetilde{X}} \widetilde{Y} + \widehat{\nabla}_{\widehat{X}} \widehat{Y},$$

where $\widetilde{\nabla}, \widehat{\nabla}$ denote the Levi-Civita connections of M_1, M_2 respectively.

In addition, the Riemannian curvature tensor Riem of $M_1 \times M_2$ takes the form

$$\mathrm{Riem}(X, Y, Z, W) = \widetilde{\mathrm{Riem}}(\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{W}) + \widehat{\mathrm{Riem}}(\widehat{X}, \widehat{Y}, \widehat{Z}, \widehat{W}), \tag{5.31}$$

where $\widetilde{\mathrm{Riem}}, \widehat{\mathrm{Riem}}$ denote the Riemannian curvature tensors of M_1, M_2 respectively.

Let $\gamma(t) = (\widetilde{\gamma}(t), \widehat{\gamma}(t))$ be a curve on $M_1 \times M_2$. By definition, γ is geodesic on $M_1 \times M_2$ if and only if $\widetilde{\gamma}, \widehat{\gamma}$ are geodesics on M_1, M_2 respectively. Moreover, the tangent vector field $J = (\widetilde{J}, \widehat{J}) \in T_\gamma(M_1 \times M_2)$ is Jacobi field along the geodesic γ if and only if $\widetilde{J} \in T_{\widetilde{\gamma}} M_1, \widehat{J} \in T_{\widehat{\gamma}} M_2$ are Jacobi fields along the geodesic $\widetilde{\gamma}, \widehat{\gamma}$ respectively.

The Jacobi matrices with the initial conditions on Riemannian product $M_1 \times M_2$ can be described by ones on M_1 and M_2. More precisely, we have the following result.

Lemma 5.4. *Let $\gamma(t) = (\tilde{\gamma}(t), \hat{\gamma}(t))$ be a geodesic on $M_1 \times M_2$. Let the orthonormal vector frame field $\{\tilde{e}_1, \tilde{e}_2, \cdots, \tilde{e}_{n_1}\}$ in $T_{\tilde{\gamma}} M_1$ be the parallel transport along $\tilde{\gamma}$ with $\tilde{e}_1 = \frac{\dot{\tilde{\gamma}}}{|\dot{\tilde{\gamma}}|}$ and the orthonormal vector frame field $\{\hat{e}_{n_1+1}, \hat{e}_{n_1+2}, \cdots, \hat{e}_{n_1+n_2}\}$ in $T_{\hat{\gamma}} M_2$ be the parallel transport along $\hat{\gamma}$ with $\hat{e}_{n_1+1} = \frac{\dot{\hat{\gamma}}}{|\dot{\hat{\gamma}}|}$. Then the vector frame field $\{e_1 = \frac{\dot{\gamma}}{|\dot{\gamma}|}, e_2 = \tilde{e}_2, \cdots, e_{n_1} = \tilde{e}_{n_1}, e_{n_1+1} = \frac{1}{|\dot{\gamma}|}(|\dot{\hat{\gamma}}|\tilde{e}_1 - |\dot{\tilde{\gamma}}|\hat{e}_{n_1+1}), e_{n_1+2} = \hat{e}_{n_1+2}, \cdots, e_{n_1+n_2} = \hat{e}_{n_1+n_2}\}$ in $T_\gamma(M_1 \times M_2)$ is an orthonormal parallel transport moving frame along γ. Moreover, the Jacobi matrices with the initial conditions J_0 and J_1 takes the form:*

$$J_a(t) = \begin{pmatrix} \tilde{J}_a(t) & 0 \\ 0 & \hat{J}_a(t) \end{pmatrix}, a = 0, 1,$$

where $\tilde{J}_a$ and $\hat{J}_a$ are the Jacobi matrices with the initial conditions on M_1 and M_2 respectively.

Proof. The Lemma follows from (1.7), (1.8) and the decomposition (5.31). □

As a direct consequence, the Hessian of squared distance on $M_1 \times M_2$ can be decomposed into the Hessian of squared distance on M_1 and M_2.

Corallary 5.3. *Suppose the same assumptions as in Lemma 5.4 are satisfied. Given $m = (\tilde{m}, \hat{m}) \in M_1 \times M_2, \nu = (\tilde{\nu}, \hat{\nu}) \in I(m)$, let $\widetilde{\mathcal{S}}(\tilde{m}, \tilde{\nu}, t)$ be the linear operator from $T_{\tilde{m}} M_1$ to $T_{\tilde{m}} M_1$ whose matrix in the orthonormal basis $\{\tilde{e}_1(0), \tilde{e}_2(0), \cdots, \tilde{e}_{n_1}(0)\}$ is given by $t\tilde{J}_0(t)^{-1}\tilde{J}_1(t)$ and $\widehat{\mathcal{S}}(\hat{m}, \hat{\nu}, t)$ be the linear operator from $T_{\hat{m}} M_2$ to $T_{\hat{m}} M_2$ whose matrix in the orthonormal basis $\{\hat{e}_1(0), \hat{e}_2(0), \cdots, \hat{e}_{n_2}(0)\}$ is given by $t\hat{J}_0(t)^{-1}\hat{J}_1(t)$. Then*

$$\mathcal{S}(m, \nu, 1) = \widetilde{\mathcal{S}}(\tilde{m}, \tilde{\nu}, 1) + \widehat{\mathcal{S}}(\hat{m}, \hat{\nu}, 1),$$

where $\mathcal{S}(m, \nu, t)$ is the linear operator from $T_m(M_1 \times M_2)$ to $T_m(M_1 \times M_2)$ whose matrix in the orthonormal basis $\{e_1(0), e_2(0), \cdots, e_{n_1+n_2}(0)\}$ is given by $tJ_0(t)^{-1}J_1(t)$.

5.3.2 Uniform stay away estimate

In this subsection, we will settle the estimate (5.2). Even through the cut locus of $M_1 \times M_2$ is non-trivial, we will prove that the optimal map uniformly stay away from the cut locus by the method of maximum principle.

Fix any couple $(k, \alpha) \in \mathbb{N} \times (0, 1)$, with $k \geq 2$. Let $(\rho_0 dvol, \rho_1 dvol)$ be $C^{k,\alpha}$ positive Borel probability measures on $M_1 \times M_2$. Fix $t \in \mathcal{I}, m = (\tilde{m}, \hat{m}) \in M_1 \times M_2$.

Note that the left side in (5.2) is related to the initial matrix J_0. Actually,

$$\begin{aligned}\det d_{\nabla_m u_t} \exp_m &= \det J_0(m, \nabla_m u_t, 1) \\ &= \det \tilde{J}_0(\tilde{m}, \widetilde{\nabla}_{\tilde{m}} u_t, 1) \det \hat{J}_0(\hat{m}, \widehat{\nabla}_{\hat{m}} u_t, 1),\end{aligned}$$

where the second equality follows from Lemma 5.4.

It is also useful to mention that the gradient of u_t at m locates in the injectivity domain at m. Then by the Bishop's theorem, $\det \tilde{J}_0(\tilde{m}, \widetilde{\nabla}_{\tilde{m}} u_t, 1)$ and $\det \hat{J}_0(\hat{m}, \widehat{\nabla}_{\hat{m}} u_t, 1)$ are all uniformly bounded from above by 1 if both M_1 and M_2 have non-negative Ricci curvatures.

By the description of Jacobi fields in Section 1.2.1, we know that the discriminants $\det \tilde{J}_0(\tilde{m}, \widetilde{\nabla}_{\tilde{m}} u_t, 1)$ and $\det \hat{J}_0(\hat{m}, \widehat{\nabla}_{\hat{m}} u_t, 1)$ are all positive. Thus the positive lower bound of $\det d_{\nabla_m u_t} \exp_m$ is equivalent to the positive lower bounds of the discriminants $\det \tilde{J}_0(\tilde{m}, \widetilde{\nabla}_{\tilde{m}} u_t, 1)$ and $\det \hat{J}_0(\hat{m}, \widehat{\nabla}_{\hat{m}} u_t, 1)$. But $\det d_{\nabla_m u_t} \exp_m$ may not has a positive lower bound in general. Since $\det J_0(m, \nu, 1)$ vanishes if (and only if) $\exp_m \nu$ is conjugate to m, so the estimate (5.2) is not obvious, for instance, on the Riemannian products of the round sphere $\mathbb{S}^{n_1} \times \mathbb{S}^{n_2}, \det d_{\nabla_m u_t} \exp_m = (\frac{\sin |\widetilde{\nabla}_{\tilde{m}} u_t|}{|\widetilde{\nabla}_{\tilde{m}} u_t|})^{n_1 - 1} (\frac{\sin |\widehat{\nabla}_{\hat{m}} u_t|}{|\widehat{\nabla}_{\hat{m}} u_t|})^{n_2 - 1}$ is close to zero as $|\widetilde{\nabla}_{\tilde{m}} u_t|$ approaches π or $|\widehat{\nabla}_{\hat{m}} u_t|$ approaches π.

We will use the method of maximum principle to prove (5.2). It needs to construct an appropriate test function.

Henceforth, we will drop freely the subscript t.

Before showing the estimate (5.2), we give a Claim first.

Claim 5.1. *Set $\mathcal{J}(\tilde{m}, \hat{m}, \tilde{\nu}, \hat{\nu}) = \widetilde{\mathcal{J}}(\tilde{m}, \tilde{\nu}) + \widehat{\mathcal{J}}(\hat{m}, \hat{\nu})$ where $\widetilde{\mathcal{J}}(\tilde{m}, \tilde{\nu}) = -|\tilde{\nu}|^2 \widetilde{\mathcal{S}}^{-1}(\tilde{m}, \tilde{\nu}, 1)$ and $\widehat{\mathcal{J}}(\hat{m}, \hat{\nu}) = -|\hat{\nu}|^2 \widehat{\mathcal{S}}^{-1}(\hat{m}, \hat{\nu}, 1)$. Then the minimum*

$$\begin{aligned}&\min\{(\langle \mathcal{J}\tilde{\xi}, \tilde{\xi}\rangle + \pi^2 - |\widehat{\nabla}_{\hat{m}} u|^2)(\langle \mathcal{J}\hat{\xi}, \hat{\xi}\rangle + \pi^2 - |\widetilde{\nabla}_{\tilde{m}} u|^2) : \\ &(\tilde{\xi}, \hat{\xi}) \in T_{\tilde{m}} M_1 \times T_{\hat{m}} M_2, |\tilde{\xi}|_{\tilde{m}} = 1 = |\hat{\xi}|_{\hat{m}}, \tilde{g}(\tilde{\xi}, \widetilde{\nabla}_{\tilde{m}} u) = 0 = \hat{g}(\hat{\xi}, \widehat{\nabla}_{\hat{m}} u), \\ &|\widetilde{\nabla}_{\tilde{m}} u| \geq \frac{3\pi}{4}, |\widehat{\nabla}_{\hat{m}} u| \geq \frac{3\pi}{4}\}.\end{aligned}$$

has a positive lower bound δ_1 which depends on the densities and n_1, n_2.

Notice that both $\langle \mathcal{J}\tilde{\xi}, \tilde{\xi}\rangle$ and $\langle \mathcal{J}\hat{\xi}, \hat{\xi}\rangle$ have to be positive. To see this, from the Hessian Comparison Theorem, it follows that $-\widetilde{\mathcal{S}}^{\perp}$(the restriction of $-\widetilde{\mathcal{S}}$ to $(\mathbb{R}\widetilde{\nabla}_{\tilde{m}} u)^{\perp}$ in $T_{\tilde{m}} M_1$) is not less than $-\frac{r_1 \cos r_1}{\sin r_1} I_{n_1 - 1}$ which is positive definite when $r_1 = |\widetilde{\nabla}_{\tilde{m}} u| \in (\frac{\pi}{2}, \pi)$. By definition of $\mathcal{J}$, we know that $\langle \mathcal{J}\tilde{\xi}, \tilde{\xi}\rangle$ is positive. Similarly, $\langle \mathcal{J}\hat{\xi}, \hat{\xi}\rangle$ is also positive.

By virtue of 1) in Theorem 4.1, there exists a positive number C such that, for any $(\tilde{\xi}, \hat{\xi}) \in T_{\tilde{m}} M_1 \times T_{\hat{m}} M_2, |\tilde{\xi}|_{\tilde{m}} = 1 = |\hat{\xi}|_{\hat{m}}$,

$$-\frac{r_1 \sin r_1}{\cos r_1} - C\varepsilon \leq \langle \mathcal{J}\tilde{\xi}, \tilde{\xi}\rangle \leq -\frac{r_1 \sin r_1}{\cos r_1} + C\varepsilon, \tag{5.32}$$

$$-\frac{s_1 \sin s_1}{\cos s_1} - C\varepsilon \leq \langle \mathcal{J}\hat{\xi}, \hat{\xi}\rangle \leq -\frac{s_1 \sin s_1}{\cos s_1} + C\varepsilon. \tag{5.33}$$

where $r_1 = |\tilde{\nabla}_{\tilde{m}} u_t| \geq \frac{3\pi}{4}$ and $s_1 = |\hat{\nabla}_{\hat{m}} u_t| \geq \frac{3\pi}{4}$.

Since the function $-\frac{r \sin r}{\cos r}$ is decreasing in $(\frac{\pi}{2}, \pi)$, thus the right inequalities in (5.32) and (5.33) infer that both $\langle \mathcal{J}\tilde{\xi}, \tilde{\xi}\rangle$ and $\langle \mathcal{J}\hat{\xi}, \hat{\xi}\rangle$ are bounded from above by some positive constant. In such case, we infer that $\max\{r_1, s_1\} \le \pi - \hat{\delta}$ for some $\hat{\delta} > 0$. These are the uniform gradient estimates.

Proof of Claim 5.1. It is clear that the minimum of $(\langle \mathcal{J}\tilde{\xi}, \tilde{\xi}\rangle + \pi^2 - |\widehat{\nabla}_{\hat{m}} u|^2)(\langle \mathcal{J}\hat{\xi}, \hat{\xi}\rangle + \pi^2 - |\widetilde{\nabla}_{\tilde{m}} u|^2)$ is attained and finite. Let $(\tilde{m}_0, \hat{m}_0, \tilde{\xi}_0, \hat{\xi}_0)$ be the minimum point. We consider the test function:

$$\begin{aligned} h\,(m,\xi) &= \log(\frac{\langle \mathcal{J}\tilde{\xi}, \tilde{\xi}\rangle + \langle \tilde{\xi}, \tilde{\nabla} u\rangle^2}{|\tilde{\xi}|^2 - \dfrac{\langle \tilde{\xi}, \tilde{\nabla} u\rangle^2}{|\tilde{\nabla} u|^2}} + \pi^2 - |\hat{\nabla} u|^2) + \\ & \quad \log(\frac{\langle \mathcal{J}\hat{\xi}, \hat{\xi}\rangle + \langle \hat{\xi}, \hat{\nabla} u\rangle^2}{|\hat{\xi}|^2 - \dfrac{\langle \hat{\xi}, \hat{\nabla} u\rangle^2}{|\hat{\nabla} u|^2}} + \pi^2 - |\tilde{\nabla} u|^2). \end{aligned}$$

Then h attains the minimum at the point $(\tilde{m}_0, \hat{m}_0, \tilde{\xi}_0, \hat{\xi}_0)$ in a neighborhood of the point $(\tilde{m}_0, \hat{m}_0, \tilde{\xi}_0, \hat{\xi}_0)$ in $T(M_1 \times M_2)$. To see this, let $\tilde{\xi}^{\perp}$ be the orthonormal part of $\tilde{\xi}$ in $T_{\tilde{m}_0} M_1$ and $\hat{\xi}^{\perp}$ be the orthonormal part of $\hat{\xi}$ in $T_{\hat{m}_0} M_2$. Then

$$\log(\frac{\langle \mathcal{J}\tilde{\xi}^{\perp}, \tilde{\xi}^{\perp}\rangle}{|\tilde{\xi}^{\perp}|^2} + \pi^2 - |\hat{\nabla} u|^2) + \log(\frac{\langle \mathcal{J}\hat{\xi}^{\perp}, \hat{\xi}^{\perp}\rangle}{|\hat{\xi}^{\perp}|^2} + \pi^2 - |\tilde{\nabla} u|^2) = h\,(m,\xi)\,.$$

By continuity, we obtain that the test function h attains the local minimum at the point $(\tilde{m}_0, \hat{m}_0, \tilde{\xi}_0, \hat{\xi}_0)$ in a neighborhood of the point $(\tilde{m}_0, \hat{m}_0, \tilde{\xi}_0, \hat{\xi}_0)$ in $T(M_1 \times M_2)$.

Note that the terms $\langle \mathcal{J}\tilde{\xi}_0, \tilde{\xi}_0\rangle$ and $\langle \mathcal{J}\hat{\xi}_0, \hat{\xi}_0\rangle$ are all eigenvalues of the self adjoint operator $\mathcal{J}$.

Set for short:$r_1 = |\widetilde{\nabla}_{\tilde{m}_0} u|, s_1 = |\widehat{\nabla}_{\hat{m}_0} u|, r_2 = \langle \mathcal{J}\tilde{\xi}_0, \tilde{\xi}_0\rangle > 0, s_2 = \langle \mathcal{J}\hat{\xi}_0, \hat{\xi}_0\rangle > 0, r_3 = \pi^2 - r_1^2, s_3 = \pi^2 - s_1^2$.

Let x be the Fermi coordinate system in M_1 along the geodesic $\exp_{\tilde{m}_0}(s\frac{\widetilde{\nabla}_{\tilde{m}_0} u}{r_1})$ and y be the Fermi coordinate system in M_2 along the geodesic $\exp_{\hat{m}_0}(s\frac{\widehat{\nabla}_{\hat{m}_0} u}{s_1})$. Then $z = (x, y)$ is the coordinate system in $M_1 \times M_2$. The associated coordinate system in tangent bundle $T(M_1 \times M_2)$ is denoted by (z, v).

In the following all terms are evaluated at the point $(z, v) = (0, r_1, 0, s_1, 0)$. It will be implicitly understood throughout the calculations. The components of $\xi_0 = (\tilde{\xi}_0, \hat{\xi}_0)$ are denoted by $(\tilde{\xi}_0^{i_1}, \hat{\xi}_0^{i_2})$ i.e.

$$\tilde{\xi}_0 = \sum_{i_1=1}^{n_1} \tilde{\xi}_0^{i_1} \frac{\partial}{\partial x^{i_1}}, \hat{\xi}_0 = \sum_{i_2=n_1+1}^{n_1+n_2} \hat{\xi}_0^{i_2} \frac{\partial}{\partial y^{i_2}}. \tag{5.34}$$

It is clear that $\tilde{\xi}^1 = \hat{\xi}^{(n_1+1)} = 0$.

We are in position to calculate the derivatives of the test function h. It is clear that the Claim 5.1 is proved if $\max\{r_1, s_1\} \le \eta_0$ for $\eta_0 \in (0, \pi)$. Without loss of generality, we shall assume that $\min\{r_1, s_1\} > \pi - \delta$ where $\delta \in (0, \frac{\pi}{4})$ is determined later.

We give the following notations:the Latin indices run over $\{1, \cdots, n_1 + n_2\}$, the indices $i_1, j_1, \cdots$ run over $\{1, \cdots, n_1\}$, the indices $i_2, j_2, \cdots$ run over $\{n_1 + 1, \cdots, n_1 + n_2\}$, the indices $\alpha_1, \beta_1, \cdots$ run over $\{2, \cdots, n_1\}$, and the indices $\alpha_2, \beta_2, \cdots$ run over $\{n_1 + 2, \cdots, n_1 + n_2\}$.

The first derivative condition

By differentiating the test function h with respect to z^i, the first derivative condition could be read as:

$$\begin{aligned} &\frac{1}{r_2 + r_3}\left[(\partial_i \mathcal{J}^{\alpha_1}_{\beta_1} + D_k \mathcal{J}^{\alpha_1}_{\beta_1} \nabla^k_i u)\tilde{\xi}_0^{\alpha_1}\tilde{\xi}_0^{\beta_1} - 2s_1 \nabla^{n_1+1}_i u\right] + \\ &\frac{1}{s_2 + s_3}\left[(\partial_i \mathcal{J}^{\alpha_2}_{\beta_2} + D_k \mathcal{J}^{\alpha_2}_{\beta_2} \nabla^k_i u)\hat{\xi}_0^{\alpha_2}\hat{\xi}_0^{\beta_2} - 2r_1 \nabla^1_i u\right] = 0. \end{aligned} \tag{5.35}$$

The second derivative condition
Differentiating twice the test function h with respect to z^i and z^j respectively, the second derivative condition read as follows:

$$0 \leq I_2 + II_2 + III_2 + IV_2 + V_2 + VI_2, \tag{5.36}$$

where

$$\begin{aligned}
I_2 &= -\frac{1}{(r_2+r_3)^2}\mathcal{F}^i_j[(\partial_i \mathcal{J}^{\alpha_1}_{\beta_1} + D_k\mathcal{J}^{\alpha_1}_{\beta_1}\nabla^k_i u)\tilde{\xi}^{\alpha_1}_0\tilde{\xi}^{\beta_1}_0 - \\
&\quad 2s_1\nabla^{n_1+1}_i u][(\partial_j \mathcal{J}^{\phi_1}_{\psi_1} + D_l\mathcal{J}^{\phi_1}_{\psi_1}\nabla^l_j u)\tilde{\xi}^{\phi_1}_0\tilde{\xi}^{\psi_1}_0 - 2s_1\nabla^{n_1+1}_j u] - \\
&\quad \frac{1}{(s_2+s_3)^2}\mathcal{F}^i_j[(\partial_i \mathcal{J}^{\alpha_2}_{\beta_2} + D_k\mathcal{J}^{\alpha_2}_{\beta_2}\nabla^k_i u)\hat{\xi}^{\alpha_2}_0\hat{\xi}^{\beta_2}_0 - \\
&\quad 2r_1\nabla^1_i u][(\partial_j \mathcal{J}^{\phi_2}_{\psi_2} + D_l\mathcal{J}^{\phi_2}_{\psi_2}\nabla^l_j u)\hat{\xi}^{\phi_2}_0\hat{\xi}^{\psi_2}_0 - 2r_1\nabla^1_j u], \\
II_2 &= \frac{1}{r_2+r_3}\mathcal{F}^i_j[\partial^2_{ij}g_{\alpha_1 k}\mathcal{J}^k_{\beta_1}\tilde{\xi}^{\alpha_1}_0\tilde{\xi}^{\beta_1}_0 - r_2\partial^2_{ij}g_{\alpha_1\beta_1}\tilde{\xi}^{\alpha_1}_0\tilde{\xi}^{\beta_1}_0 + \\
&\quad (2\partial_j\Gamma^{p_2+\eta}_{iq_2} - \partial^2_{ij}g_{p_2q_2})\nabla^{p_2}u\nabla^{q_2}u] + \\
&\quad \frac{1}{s_2+s_3}\mathcal{F}^i_j[\partial^2_{ij}g_{\alpha_2 k}\mathcal{J}^k_{\beta_2}\hat{\xi}^{\alpha_2}_0\hat{\xi}^{\beta_2}_0 - s_2\partial^2_{ij}g_{\alpha_2\beta_2}\hat{\xi}^{\alpha_2}_0\hat{\xi}^{\beta_2}_0 + \\
&\quad (2\partial_j\Gamma^{p_1}_{iq_1} - \partial^2_{ij}g_{p_1q_1})\nabla^{p_1}u\nabla^{q_1}u], \\
III_2 &= \frac{1}{r_2+r_3}\mathcal{F}^i_j(\partial^2_{ij}\mathcal{J}^{\alpha_1}_{\beta_1}\tilde{\xi}^{\alpha_1}_0\tilde{\xi}^{\beta_1}_0 - \partial_j\Gamma^k_{il}\nabla^l u D_k\mathcal{J}^{\alpha_1}_{\beta_1}\tilde{\xi}^{\alpha_1}_0\tilde{\xi}^{\beta_1}_0) + \\
&\quad \frac{1}{s_2+s_3}\mathcal{F}^i_j(\partial^2_{ij}\mathcal{J}^{\alpha_2}_{\beta_2}\hat{\xi}^{\alpha_2}_0\hat{\xi}^{\beta_2}_0 - \partial_j\Gamma^k_{li}\nabla^l u D_k\mathcal{J}^{\alpha_2}_{\beta_2}\hat{\xi}^{\alpha_2}_0\hat{\xi}^{\beta_2}_0), \\
IV_2 &= \frac{2}{r_2+r_3}\mathcal{F}^i_j\nabla^j_k u\partial_i D_k\mathcal{J}^{\alpha_1}_{\beta_1}\tilde{\xi}^{\alpha_1}_0\tilde{\xi}^{\beta_1}_0 + \\
&\quad \frac{2}{s_2+s_3}\mathcal{F}^i_j\nabla^j_k u\partial_i D_k\mathcal{J}^{\alpha_2}_{\beta_2}\hat{\xi}^{\alpha_2}_0\hat{\xi}^{\beta_2}_0,
\end{aligned}$$

$$\begin{aligned}
V_2 &= \frac{1}{r_2+r_3}\mathcal{F}^i_j[\nabla^k_i u\nabla^j_l u D^2_{kl}\mathcal{J}^{\alpha_1}_{\beta_1}\tilde{\xi}^{\alpha_1}_0\tilde{\xi}^{\beta_1}_0 + \\
&\quad 2(1+\frac{r_2}{r_1^2})\nabla^{\alpha_1}_i u\nabla^j_{\beta_1}u\tilde{\xi}^{\alpha_1}_0\tilde{\xi}^{\beta_1}_0 - 2\nabla^{k_2}_i u\nabla^j_{k_2}u] + \\
&\quad \frac{1}{s_2+s_3}\mathcal{F}^i_j[\nabla^k_i u\nabla^j_l u D^2_{kl}\mathcal{J}^{\alpha_2}_{\beta_2}\hat{\xi}^{\alpha_2}_0\hat{\xi}^{\beta_2}_0 + \\
&\quad 2(1+\frac{s_2}{s_1^2})\nabla^{\alpha_2}_i u\nabla^j_{\beta_2}u\hat{\xi}^{\alpha_2}_0\hat{\xi}^{\beta_2}_0 - 2\nabla^{k_1}_i u\nabla^j_{k_1}u], \\
VI_2 &= \frac{1}{r_2+r_3}\mathcal{F}^i_j(\partial_j\nabla^k_i u D_k\mathcal{J}^{\alpha_1}_{\beta_1}\tilde{\xi}^{\alpha_1}_0\tilde{\xi}^{\beta_1}_0 - 2\partial_j\nabla^{k_2}_i u\nabla_{k_2}u) + \\
&\quad \frac{1}{s_2+s_3}\mathcal{F}^i_j(\partial_j\nabla^k_i u D_k\mathcal{J}^{\alpha_2}_{\beta_2}\hat{\xi}^{\alpha_2}_0\hat{\xi}^{\beta_2}_0 - 2\partial_j\nabla^{k_1}_i u\nabla_{k_1}u).
\end{aligned}$$

The potential function u at the point m_0 satisfies the equation:

$$\det J_0 \det(H^i_j) = \frac{\rho_0}{\rho_t}. \tag{5.37}$$

Notice that the positive definiteness of the matrix (H^i_j) implies that both $\nabla^1_1 u$ and $\nabla^{(n_1+1)}_{(n_1+1)}u$ are all greater than -1. We will also require the expression:

$$\mathcal{S}(0, r_1, 0, s_1, 0, 1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \widetilde{\mathcal{S}}^{\alpha_1}_{\beta_1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \widehat{\mathcal{S}}^{\alpha_2}_{\beta_2} \end{bmatrix}. \tag{5.38}$$

Note that $-(\widetilde{\mathcal{S}}^{\alpha_1}_{\beta_1})$ and $-(\widehat{\mathcal{S}}^{\alpha_2}_{\beta_2})$ are both uniformly bounded from below under the curvature assumption (1.3). In fact, from the Hessian Comparison Theorem, we know that $-(\widetilde{\mathcal{S}}^{\alpha_1}_{\beta_1}) \geq -\frac{r_1\cos r_1}{\sin r_1}I_{n-1}$ and $-(\widehat{\mathcal{S}}^{\alpha_2}_{\beta_2}) \geq -\frac{s_1\cos s_1}{\sin s_1}I_{n-1}$. Making use of the fact that the real function $-\frac{t\cos t}{\sin t}$ is increasing in $(\frac{\pi}{2},\pi)$, for $\min\{r_1,s_1\} \geq \frac{3\pi}{4}$, we have

$$-(\widetilde{\mathcal{S}}^{\alpha_1}_{\beta_1}) \geq \frac{3\pi}{4}I_{n-1}, -(\widehat{\mathcal{S}}^{\alpha_2}_{\beta_2}) \geq \frac{3\pi}{4}I_{n-1}. \tag{5.39}$$

We will cope with I_2 to VI_2 term by term.
The term I_2 It is obvious that I_2 is non-positive. That is

$$I_2 \leq 0. \tag{5.40}$$

The term II_2 Since $g_{ij}=\delta_{ij}$ on the axis and $\mathcal{J}^1_{\alpha_1}=\mathcal{J}^{(n_1+1)}_{\alpha_2}=\mathcal{J}^{j_2}_{i_1}=\mathcal{J}^{i_1}_{j_2}=0$ at the point $(0,r_1,0,s_1,0)$, the term II_1 can be written as:

$$\begin{aligned} II_2 \quad = \quad & \frac{1}{r_2+r_3}[\sum_{i,j,k=2}^{n_1}\mathcal{F}^i_j\partial^2_{ij}g_{\alpha_1 k}\mathcal{J}^k_{\beta_1}\tilde{\xi}^{\alpha_1}_0\tilde{\xi}^{\beta_1}_0 - r_2\sum_{i,j=2}^{n_1}\mathcal{F}^i_j\partial^2_{ij}g_{\alpha_1\beta_1}\tilde{\xi}^{\alpha_1}_0\tilde{\xi}^{\beta_1}_0 + \\ & s_1^2\sum_{i,j=n_1+2}^{n_1+n_2+\eta}\mathcal{F}^i_j(2\partial_j\Gamma^{(n_1+1)}_{i(n_1+1)} - \partial^2_{ij}g_{(n_1+1)(n_1+1)})] + \\ & \frac{1}{s_2+s_3}[\sum_{i,j,k=n_1+2}^{n_1+n_2}\mathcal{F}^i_j\partial^2_{ij}g_{\alpha_2 k}\mathcal{J}^k_{\beta_2}\hat{\xi}^{\alpha_2}_0\hat{\xi}^{\beta_2}_0 - s_2\sum_{i,j=n_1+2}^{n_1+n_2}\mathcal{F}^i_j\partial^2_{ij}g_{\alpha_2\beta_2}\hat{\xi}^{\alpha_2}_0\hat{\xi}^{\beta_2}_0 + \\ & r_1^2\sum_{i,j=2}^{n_1}\mathcal{F}^i_j(2\partial_j\Gamma^1_{i1} - \partial^2_{ij}g_{11})]. \end{aligned}$$

By (1.14), it follows that

$$\begin{aligned} II_2 \quad = \quad & \frac{1}{r_2+r_3}[-\frac{2}{3}\sum_{i,j,k=2}^{n_1}\mathcal{F}^i_jR_{i\alpha_1 jk}\mathcal{J}^k_{\beta_1}\tilde{\xi}^{\alpha_1}_0\tilde{\xi}^{\beta_1}_0 + \frac{2}{3}r_2\sum_{i,j=2}^{n_1}\mathcal{F}^i_jR_{i\alpha_1 j\beta_1}\tilde{\xi}^{\alpha_1}_0\tilde{\xi}^{\beta_1}_0 + \\ & s_1^2\sum_{i,j=n_1+2}^{n_1+n_2}\mathcal{F}^i_j(2R^{(n_1+1)}_{ij(n_1+1)} + 2R_{(n_1+1)i(n_1+1)j})] + \\ & \frac{1}{s_2+s_3}[-\frac{2}{3}\sum_{i,j,k=n_1+2}^{n_1+n_2}\mathcal{F}^i_jR_{i\alpha_2 jk}\mathcal{J}^k_{\beta_2}\hat{\xi}^{\alpha_2}_0\hat{\xi}^{\beta_2}_0 + \\ & \frac{2}{3}s_2\sum_{i,j=n_1+2}^{n_1+n_2}\mathcal{F}^i_jR_{i\alpha_2 j\beta_2}\hat{\xi}^{\alpha_2}_0\hat{\xi}^{\beta_2}_0 + r_1^2\sum_{i,j=2}^{n_1}\mathcal{F}^i_j(2R^1_{ij1} + 2R_{1i1j})] \\ = \quad & \frac{1}{r_2+r_3}(-\frac{2}{3}\sum_{i,j,k=2}^{n_1}\mathcal{F}^i_jR_{i\alpha_1 jk}\mathcal{J}^k_{\beta_1}\tilde{\xi}^{\alpha_1}_0\tilde{\xi}^{\beta_1}_0 + \frac{2}{3}r_2\sum_{i,j=2}^{n_1}\mathcal{F}^i_jR_{i\alpha_1 j\beta_1}\tilde{\xi}^{\alpha_1}_0\tilde{\xi}^{\beta_1}_0) + \\ & \frac{1}{s_2+s_3}(-\frac{2}{3}\sum_{i,j,k=n_1+2}^{n_1+n_2}\mathcal{F}^i_jR_{i\alpha_2 jk}\mathcal{J}^k_{\beta_2}\hat{\xi}^{\alpha_2}_0\hat{\xi}^{\beta_2}_0 + \\ & \frac{2}{3}s_2\sum_{i,j=n_1+2}^{n_1+n_2}\mathcal{F}^i_jR_{i\alpha_2 j\beta_2}\hat{\xi}^{\alpha_2}_0\hat{\xi}^{\beta_2}_0). \end{aligned}$$

Together with the curvature assumption (1.4), the positive definiteness of $(\mathcal{F}^i_j)$ and the uniform bound for the norm of $\mathcal{J}$, we have

$$\begin{aligned}
II_2 \le\ & \frac{1}{r_2+r_3}(-\frac{2}{3}r_2\mathcal{F}^{\alpha_1}_{\alpha_1}+\frac{2}{3}\mathcal{F}^{\alpha_1}_{\varphi_1}\mathcal{J}^{\varphi_1}_{\beta_1}\tilde{\xi}^{\alpha_1}_0\tilde{\xi}^{\beta_1}_0+\\
& \frac{2}{3}r_2\mathcal{F}^{\alpha_1}_{\alpha_1}-\frac{2}{3}r_2\mathcal{F}^{\alpha_1}_{\beta_1}\tilde{\xi}^{\alpha_1}_0\tilde{\xi}^{\beta_1}_0+C\varepsilon\mathcal{F}^{\alpha_1}_{\alpha_1})+\\
& \frac{1}{s_2+s_3}(-\frac{2}{3}s_2\mathcal{F}^{\alpha_2}_{\alpha_2}+\frac{2}{3}\mathcal{F}^{\alpha_2}_{\varphi_2}\mathcal{J}^{\varphi_2}_{\beta_2}\hat{\xi}^{\alpha_2}_0\hat{\xi}^{\beta_2}_0+\\
& \frac{2}{3}s_2\mathcal{F}^{\alpha_2}_{\alpha_2}-\frac{2}{3}s_2\mathcal{F}^{\alpha_2}_{\beta_2}\hat{\xi}^{\alpha_2}_0\hat{\xi}^{\beta_2}_0+C\varepsilon\mathcal{F}^{\alpha_2}_{\alpha_2})\\
=\ & \frac{1}{r_2+r_3}(\frac{2}{3}\mathcal{F}^{\alpha_1}_{\varphi_1}\mathcal{J}^{\varphi_1}_{\beta_1}\tilde{\xi}^{\alpha_1}_0\tilde{\xi}^{\beta_1}_0-\frac{2}{3}r_2\mathcal{F}^{\alpha_1}_{\beta_1}\tilde{\xi}^{\alpha_1}_0\tilde{\xi}^{\beta_1}_0+C\varepsilon\mathcal{F}^{\alpha_1}_{\alpha_1})+\\
& \frac{1}{s_2+s_3}(\frac{2}{3}\mathcal{F}^{\alpha_2}_{\varphi_2}\mathcal{J}^{\varphi_2}_{\beta_2}\hat{\xi}^{\alpha_2}_0\hat{\xi}^{\beta_2}_0-\frac{2}{3}s_2\mathcal{F}^{\alpha_2}_{\beta_2}\hat{\xi}^{\alpha_2}_0\hat{\xi}^{\beta_2}_0+C\varepsilon\mathcal{F}^{\alpha_2}_{\alpha_2})\\
\le\ & \frac{1}{r_2+r_3}(\frac{2}{3}\langle(\bar{\mathcal{J}}-\mathcal{J})\tilde{\xi}_0,\tilde{\xi}_0\rangle\mathcal{F}^{\alpha_1}_{\beta_1}\tilde{\xi}^{\alpha_1}_0\tilde{\xi}^{\beta_1}_0+C\varepsilon\mathcal{F}^{\alpha_1}_{\alpha_1})+\\
& \frac{1}{s_2+s_3}(\frac{2}{3}\langle(\mathcal{J}-\bar{\mathcal{J}})\hat{\xi}_0,\hat{\xi}_0\rangle\mathcal{F}^{\alpha_2}_{\beta_2}\hat{\xi}^{\alpha_2}_0\hat{\xi}^{\beta_2}_0+C\varepsilon\mathcal{F}^{\alpha_2}_{\alpha_2}).
\end{aligned}$$

where the last inequality follows from (5.4).

By virtue of (5.4) and the positive definiteness of $(\mathcal{F}^i_j)$ again, we derive that there exists a universal constant $C>0$ such that there holds:

$$II_2 \le \frac{C\varepsilon}{r_2+r_3}\mathcal{F}^{\alpha_1}_{\alpha_1}+\frac{C\varepsilon}{s_2+s_3}\mathcal{F}^{\alpha_2}_{\alpha_2}. \tag{5.41}$$

The term III_2 Noting $\nabla u=(r_1,0,s_1,0)$ at the point m_0, and using the first expression in (1.15) and the definition of $\mathcal{J}$, it follows that

$$\begin{aligned}
III_2 =\ & \frac{1}{r_2+r_3}(\sum_{i,j=2}^{n_1}\mathcal{F}^i_j\partial^2_{ij}\mathcal{J}^{\alpha_1}_{\beta_1}\tilde{\xi}^{\alpha_1}_0\tilde{\xi}^{\beta_1}_0-\\
& r_1\sum_{i,j,k=2}^{n_1}\mathcal{F}^i_jR^k_{ij1}D_k\mathcal{J}^{\alpha_1}_{\beta_1}\tilde{\xi}^{\alpha_1}_0\tilde{\xi}^{\beta_1}_0)+\\
& \frac{1}{s_2+s_3}(\sum_{i,j=n_1+2}^{n_1+n_2}\mathcal{F}^i_j\partial^2_{ij}\mathcal{J}^{\alpha_2}_{\beta_2}\hat{\xi}^{\alpha_2}_0\hat{\xi}^{\beta_2}_0-\\
& s_1\sum_{i,j,k=n_1+2}^{n_1+n_2}\mathcal{F}^i_jR^k_{ij(n_1+1)}D_k\mathcal{J}^{\alpha_2}_{\beta_2}\hat{\xi}^{\alpha_2}_0\hat{\xi}^{\beta_2}_0).
\end{aligned}$$

In light of Proposition 5.1 and the curvature assumption (1.4), we obtain

$$\begin{aligned}
III_2 \le\ & \frac{1}{r_2+r_3}[\pi(\pi-r_1)\mathcal{F}^{\alpha_1}_{\alpha_1}+C(\varepsilon+\delta)\mathcal{F}^{i_1}_{i_1}]+\\
& \frac{1}{s_2+s_3}[\pi(\pi-s_1)\mathcal{F}^{\alpha_1}_{\alpha_1}+C(\varepsilon+\delta)\mathcal{F}^{i_2}_{i_2}].
\end{aligned}$$

Thus we derive that there exists a positive constant C such that the following estimate holds:

$$III_2 \le \frac{C(\varepsilon+\delta)}{r_2+r_3}\mathcal{F}^{i_1}_{i_1}+\frac{C(\varepsilon+\delta)}{s_2+s_3}\mathcal{F}^{i_2}_{i_2}. \tag{5.42}$$

The term IV_2 Making use of $\mathcal{F}^i_j \nabla^j_k u = \delta^i_k - \mathcal{F}^i_j S^j_k$, the term IV_2 becomes:

$$\begin{aligned} IV_2 &= \frac{2}{r_2+r_3}(\partial_k D_k \mathcal{J}^{\alpha_1}_{\beta_1} \tilde{\xi}^{\alpha_1}_0 \tilde{\xi}^{\beta_1}_0 - \mathcal{F}^i_j S^j_k \partial_i D_k \mathcal{J}^{\alpha_1}_{\beta_1} \tilde{\xi}^{\alpha_1}_0 \tilde{\xi}^{\beta_1}_0) + \\ & \frac{2}{s_2+s_3}(\partial_k D_k \mathcal{J}^{\alpha_2}_{\beta_2} \hat{\xi}^{\alpha_2}_0 \hat{\xi}^{\beta_2}_0 - \mathcal{F}^i_j S^j_k \partial_i D_k \mathcal{J}^{\alpha_2}_{\beta_2} \hat{\xi}^{\alpha_2}_0 \hat{\xi}^{\beta_2}_0) \\ &= \frac{2}{r_2+r_3}(\sum_{k=1}^{n_1} \partial_k D_k \mathcal{J}^{\alpha_1}_{\beta_1} \tilde{\xi}^{\alpha_1}_0 \tilde{\xi}^{\beta_1}_0 - \sum_{i,j,k=1}^{n_1} \mathcal{F}^i_j S^j_k \partial_i D_k \mathcal{J}^{\alpha_1}_{\beta_1} \tilde{\xi}^{\alpha_1}_0 \tilde{\xi}^{\beta_1}_0) + \\ & \frac{2}{s_2+s_3}(\sum_{k=n_1+1}^{n_1+n_2} \partial_k D_k \mathcal{J}^{\alpha_2}_{\beta_2} \hat{\xi}^{\alpha_2}_0 \hat{\xi}^{\beta_2}_0 - \sum_{i,j,k=n_1+1}^{n_1+n_2} \mathcal{F}^i_j S^j_k \partial_i D_k \mathcal{J}^{\alpha_2}_{\beta_2} \hat{\xi}^{\alpha_2}_0 \hat{\xi}^{\beta_2}_0). \end{aligned}$$

According to Proposition 5.1, the identity $\mathcal{F}\mathcal{S} = \mathcal{S}^{-1}\mathcal{S}\mathcal{F}\mathcal{S}$ and the boundedness of $\mathcal{S}^{-1}$, it follows that

$$IV_2 \leq \frac{C\varepsilon}{r_2+r_3}(1 + \mathcal{F}^i_j S^{k_1}_i S^j_{k_1}) + \frac{C\varepsilon}{s_2+s_3}(1 + \mathcal{F}^i_j S^{k_2}_i S^j_{k_2}).$$

Let us observe that the following identities holds:

$$\begin{aligned} \mathcal{F}^i_j \nabla^k_i u \nabla^j_l u &= \nabla^k_l u - \mathcal{S}^k_l + \mathcal{F}^i_j \mathcal{S}^k_i \mathcal{S}^j_l \\ &= \mathcal{H}^k_l - 2\mathcal{S}^k_l + \mathcal{F}^i_j \mathcal{S}^k_i \mathcal{S}^j_l. \end{aligned} \tag{5.43}$$

By (5.38),(5.39) and the positive definiteness of $(\mathcal{H}^i_j), (-\widetilde{\mathcal{S}}^{\alpha_1}_{\beta_1}), (-\widehat{\mathcal{S}}^{\alpha_2}_{\beta_2})$, we derive the existence of a positive constant C such that:

$$\begin{aligned} IV_2 \leq & \frac{C\varepsilon}{r_2+r_3}(\mathcal{F}^1_1 + \mathcal{F}^i_j \nabla^{\alpha_1}_i u \nabla^j_{\alpha_1} u) + \\ & \frac{C\varepsilon}{s_2+s_3}(\mathcal{F}^{(n_1+1)}_{(n_1+1)} + \mathcal{F}^i_j \nabla^{\alpha_2}_i u \nabla^j_{\alpha_2} u). \end{aligned} \tag{5.44}$$

The term V_2 By definition of $\mathcal{J}$, we get

$$\begin{aligned} V_2 &= \frac{1}{r_2+r_3}[\sum_{k,l=1}^{n_1} \mathcal{F}^i_j \nabla^k_i u \nabla^j_l u D^2_{kl} \mathcal{J}^{\alpha_1}_{\beta_1} \tilde{\xi}^{\alpha_1}_0 \tilde{\xi}^{\beta_1}_0 + \\ & 2(1+\frac{r_2}{r_1^2})\mathcal{F}^i_j \nabla^{\alpha_1}_i u \nabla^j_{\beta_1} u \tilde{\xi}^{\alpha_1}_0 \tilde{\xi}^{\beta_1}_0 - 2\mathcal{F}^i_j \nabla^{k_2}_i u \nabla^j_{k_2} u] + \\ & \frac{1}{s_2+s_3}[\sum_{k,l=n_1+1}^{n_1+n_2} \mathcal{F}^i_j \nabla^k_i u \nabla^j_l u D^2_{kl} \mathcal{J}^{\alpha_2}_{\beta_2} \hat{\xi}^{\alpha_2}_0 \hat{\xi}^{\beta_2}_0 + \\ & 2(1+\frac{s_2}{s_1^2})\mathcal{F}^i_j \nabla^{\alpha_2}_i u \nabla^j_{\beta_2} u \hat{\xi}^{\alpha_2}_0 \hat{\xi}^{\beta_2}_0 - 2\nabla^{k_1}_i u \nabla^j_{k_1} u] \\ &= \frac{1}{r_2+r_3}[(\mathcal{F}^i_j \nabla^1_i u \nabla^j_1 u D^2_{11} \mathcal{J}^{\alpha_1}_{\beta_1} \tilde{\xi}^{\alpha_1}_0 \tilde{\xi}^{\beta_1}_0 + \\ & 2\mathcal{F}^i_j \nabla^1_i u \nabla^j_{\iota_1} u D^2_{1\iota_1} \mathcal{J}^{\alpha_1}_{\beta_1} \tilde{\xi}^{\alpha_1}_0 \tilde{\xi}^{\beta_1}_0 + \\ & \mathcal{F}^i_j \nabla^{\varphi_1}_i u \nabla^j_{\varphi_1} u D^2_{\varphi_1\varphi_1} \mathcal{J}^{\alpha_1}_{\beta_1} \tilde{\xi}^{\alpha_1}_0 \tilde{\xi}^{\beta_1}_0 + \\ & \sum_{\varphi_1 \neq \psi_1} \mathcal{F}^i_j \nabla^{\varphi_1}_i u \nabla^j_{\psi_1} u D^2_{\varphi_1\psi_1} \mathcal{J}^{\alpha_1}_{\beta_1} \tilde{\xi}^{\alpha_1}_0 \tilde{\xi}^{\beta_1}_0) + \end{aligned}$$

$$
\begin{aligned}
&2(1+\frac{r_2}{r_1^2})\mathcal{F}_j^i\nabla_i^{\alpha_1}u\nabla_{\beta_1}^j u\tilde{\xi}_0^{\alpha_1}\tilde{\xi}_0^{\beta_1}-2\mathcal{F}_j^i\nabla_i^{k_2}u\nabla_{k_2}^j u]+\\
&\frac{1}{s_2+s_3}[(\mathcal{F}_j^i\nabla_i^{n_1+1}u\nabla_{n_1+1}^j uD^2_{(n_1+1)(n_1+1)}\mathcal{J}_{\beta_2}^{\alpha_2}\hat{\xi}_0^{\alpha_2}\hat{\xi}_0^{\beta_2}+\\
&2\mathcal{F}_j^i\nabla_i^{n_1+1}u\nabla_{\iota_2}^j uD^2_{(n_1+1)\iota_2}\mathcal{J}_{\beta_2}^{\alpha_2}\hat{\xi}_0^{\alpha_2}\hat{\xi}_0^{\beta_2}+\\
&\mathcal{F}_j^i\nabla_i^{\varphi_2}u\nabla_{\varphi_2}^j uD^2_{\varphi_2\varphi_2}\mathcal{J}_{\beta_2}^{\alpha_2}\hat{\xi}_0^{\alpha_2}\tilde{\xi}_0^{\beta_2}+\\
&\sum_{\varphi_2\neq\psi_2}\mathcal{F}_j^i\nabla_i^{\varphi_2}u\nabla_{\psi_2}^j uD^2_{\varphi_2\psi_2}\mathcal{J}_{\beta_2}^{\alpha_2}\hat{\xi}_0^{\alpha_2}\hat{\xi}_0^{\beta_2}+\\
&2(1+\frac{s_2}{s_1^2})\mathcal{F}_j^i\nabla_i^{\alpha_2}u\nabla_{\beta_2}^j u\hat{\xi}_0^{\alpha_2}\hat{\xi}_0^{\beta_2}-2\mathcal{F}_j^i\nabla_i^{k_1}u\nabla_{k_1}^j u].
\end{aligned}
$$

Making use of Proposition 5.1, we infer that there exists a positive constant C such that the following esimate holds

$$
\begin{aligned}
V_2 \;\leq\;& \frac{1}{r_2+r_3}\{[C(\varepsilon+\delta)-2]\mathcal{F}_j^i\nabla_i^1u\nabla_1^j u+\\
&[C(\varepsilon+\delta)+\frac{2r_2}{r_1^2}-1]\mathcal{F}_j^i\nabla_i^{\alpha_1}u\nabla_{\alpha_1}^j u-2\mathcal{F}_j^i\nabla_i^{k_2}u\nabla_{k_2}^j u\}+\\
&\frac{1}{s_2+s_3}\{[C(\varepsilon+\delta)-2]\mathcal{F}_j^i\nabla_i^{n_1+1}u\nabla_{n_1+1}^j u+\\
&[C(\varepsilon+\delta)+\frac{2s_2}{s_1^2}-1]\mathcal{F}_j^i\nabla_i^{\alpha_2}u\nabla_{\alpha_2}^j u-2\mathcal{F}_j^i\nabla_i^{k_1}u\nabla_{k_1}^j u\}.
\end{aligned}
$$

In view of (5.26), it follows that

$$
\begin{aligned}
\mathcal{F}_j^i\nabla_i^1u\nabla_1^j u &= \mathcal{H}_1^1-2+\mathcal{F}_1^1,\\
\mathcal{F}_j^i\nabla_i^{n_1+1}u\nabla_{n_1+1}^j u &= \mathcal{H}_{n_1+1}^{n_1+1}-2+\mathcal{F}_{n_1+1}^{n_1+1}.
\end{aligned}
$$

Choose ε and δ small enough($\varepsilon,\delta<\frac{1}{C}$). By the positive definiteness of $(\mathcal{H}_j^i),(-\widetilde{\mathcal{S}}_{\beta_1}^{\alpha_1})$ and $(-\widehat{\mathcal{S}}_{\beta_2}^{\alpha_2})$, we have

$$
\begin{aligned}
V_2 \;\leq\;& \frac{1}{r_2+r_3}\{[C(\varepsilon+\delta)-2](\mathcal{F}_1^1-2)+\\
&[C(\varepsilon+\delta)+\frac{2r_2}{r_1^2}-1]\mathcal{F}_j^i\nabla_i^{\alpha_1}u\nabla_{\alpha_1}^j u-2(\mathcal{F}_{n_1+1}^{n_1+1}-2)-2\mathcal{F}_j^i\nabla_i^{\alpha_2}u\nabla_{\alpha_2}^j u\}+\\
&\frac{1}{s_2+s_3}\{[C(\varepsilon+\delta)-2](\mathcal{F}_{n_1+1}^{n_1+1}-2)+\\
&[C(\varepsilon+\delta)+\frac{2s_2}{s_1^2}-1]\mathcal{F}_j^i\nabla_i^{\alpha_2}u\nabla_{\alpha_2}^j u-2(\mathcal{F}_1^1-2)-2\mathcal{F}_j^i\nabla_i^{\alpha_1}u\nabla_{\alpha_1}^j u\}\\
\leq\;& \frac{1}{r_2+r_3}\{8+[C(\varepsilon+\delta)-2]\mathcal{F}_1^1+\\
&[C(\varepsilon+\delta)+\frac{2r_2}{r_1^2}-1]\mathcal{F}_j^i\nabla_i^{\alpha_1}u\nabla_{\alpha_1}^j u-2\mathcal{F}_{n_1+1}^{n_1+1}-2\mathcal{F}_j^i\nabla_i^{\alpha_2}u\nabla_{\alpha_2}^j u\}+\\
&\frac{1}{s_2+s_3}\{8+[C(\varepsilon+\delta)-2]\mathcal{F}_{n_1+1}^{n_1+1}+\\
&[C(\varepsilon+\delta)+\frac{2s_2}{s_1^2}-1]\mathcal{F}_j^i\nabla_i^{\alpha_2}u\nabla_{\alpha_2}^j u-2\mathcal{F}_1^1-2\mathcal{F}_j^i\nabla_i^{\alpha_1}u\nabla_{\alpha_1}^j u\}. \qquad (5.45)
\end{aligned}
$$

The term VI_2 The term VI_2 involves the third derivatives of u. After commuting the third derivatives of u, the term VI_2 can be read as:

$$\begin{aligned} VI_2 &= \frac{1}{r_2+r_3}[\mathcal{F}^i_j(\partial_k\nabla^j_i u + r_1 R^k_{i1j})D_k\mathcal{J}^{\alpha_1}_{\beta_1}\tilde{\xi}^{\alpha_1}_0\tilde{\xi}^{\beta_1}_0 - \\ &\quad 2s_1\mathcal{F}^i_j(\partial_{n_1+1}\nabla^j_i u + s_1 R^{n_1+1}_{i(n_1+1)j})] + \\ &\quad \frac{1}{s_2+s_3}[\mathcal{F}^i_j(\partial_k\nabla^j_i u + s_1 R^k_{i(n_1+1)j})D_k\mathcal{J}^{\alpha_2}_{\beta_2}\hat{\xi}^{\alpha_2}_0\hat{\xi}^{\beta_2}_0 - \\ &\quad 2r_1\mathcal{F}^i_j(\partial_1\nabla^j_i u + r_1 R^1_{i1j})]. \end{aligned}$$

We compute the third derivative. In the Fermi coordinates, the determinant of the positive matrix $(g_{ij})_{1\le i,j\le m}$ is denoted by $|g|$. Recall the equation (5.29). By definition of $\det d_{\nabla_m u}\exp_m$, the potential function u satisfies the Monge-Ampère type equation

$$\det \mathrm{Hess}^{(c)} u = \frac{\sqrt{|g|(z)}\rho_0(z)}{\sqrt{|g|(Z)}\det(D_v Z)\rho_t(z)}. \tag{5.46}$$

By taking the logarithm and differentiating the associated equation with respect to the partial variable z^k at the point $(0, r_1, 0, s_1, 0)$, we obtain

$$\begin{aligned} \mathcal{F}^i_j\partial_k\nabla^j_i u &= \frac{\partial_k\rho_0}{\rho_0} - \frac{(1-t)\partial_k\rho_0 + t\partial_p\rho_1(\partial_k Z^p + D_l Z^p\nabla^l_k u)}{\rho_t} - \\ &\quad B^p_q(\partial_k D_p Z^q + D^2_{pl}Z^q\nabla^l_k u) + \mathcal{F}^i_j\mathcal{S}^q_i\mathcal{S}^j_p\partial_k A^p_q + \\ &\quad \mathcal{F}^i_j\mathcal{S}^q_i\mathcal{S}^j_p D_l A^p_q\nabla^l_k u \\ &= \frac{\partial_k\rho_0}{\rho_0} - \frac{(1-t)\partial_k\rho_0 + t\partial_p\rho_1\partial_k Z^p}{\rho_t} - B^p_q\partial_k D_p Z^q + \\ &\quad \mathcal{F}^i_j\mathcal{S}^q_i\mathcal{S}^j_p\partial_k A^p_q + (-\frac{t\partial_p\rho_1 D_l Z^p}{\rho_t} - B^p_q D^2_{pl}Z^q + \\ &\quad \mathcal{F}^i_j\mathcal{S}^q_i\mathcal{S}^j_p D_l A^p_q)\nabla^l_k u, \end{aligned}$$

where the matrix (B^p_q) is the inverse of the matrix $(D_j Z^i)$ and (A^p_q) is the inverse of the matrix $(\mathcal{S}^i_j)$. Thus we get the following simplified expressions:

$$\begin{aligned} VI_2 &= \frac{1}{r_2+r_3}\{[\frac{\partial_k\rho_0}{\rho_0} - \frac{(1-t)\partial_k\rho_0 + t\partial_p\rho_1\partial_k X^p}{\rho_t} - B^p_q\partial_k D_p X^q + \\ &\quad \mathcal{F}^i_j\mathcal{S}^q_i\mathcal{S}^j_p\partial_k A^p_q]D_k\mathcal{J}^{\alpha_1}_{\beta_1}\tilde{\xi}^{\alpha_1}_0\tilde{\xi}^{\beta_1}_0 + (-\frac{t\partial_p\rho_1 D_l X^p}{\rho_t} - B^p_q D^2_{pl}X^q + \\ &\quad \mathcal{F}^i_j\mathcal{S}^q_i\mathcal{S}^j_p D_l A^p_q)\nabla^l_k u D_k\mathcal{J}^{\alpha_1}_{\beta_1}\tilde{\xi}^{\alpha_1}_0\tilde{\xi}^{\beta_1}_0 + r_1\mathcal{F}^i_j R^k_{i1j}D_k\mathcal{J}^{\alpha_1}_{\beta_1}\tilde{\xi}^{\alpha_1}_0\tilde{\xi}^{\beta_1}_0 - \\ &\quad 2s_1[\frac{\partial_{n_1+1}\rho_0}{\rho_0} - \frac{(1-t)\partial_{n_1+1}\rho_0 + t\partial_p\rho_1\partial_{n_1+1}X^p}{\rho_t} - B^p_q\partial_{n_1+1}D_p X^q + \\ &\quad \mathcal{F}^i_j\mathcal{S}^q_i\mathcal{S}^j_p\partial_{n_1+1}A^p_q] + 2s_1(\frac{t\partial_p\rho_1 D_l X^p}{\rho_t} + B^p_q D^2_{pl}X^q - \\ &\quad \mathcal{F}^i_j\mathcal{S}^q_i\mathcal{S}^j_p D_l A^p_q)\nabla^l_{n_1+1}u - 2s_1^2\mathcal{F}^i_j R^{n_1+1}_{i(n_1+1)j}\} + \\ &\quad \frac{1}{s_2+s_3}\{[\frac{\partial_k\rho_0}{\rho_0} - \frac{(1-t)\partial_k\rho_0 + t\partial_p\rho_1\partial_k X^p}{\rho_t} - B^p_q\partial_k D_p X^q + \\ &\quad \mathcal{F}^i_j\mathcal{S}^q_i\mathcal{S}^j_p\partial_k A^p_q]D_k\mathcal{J}^{\alpha_2}_{\beta_2}\hat{\xi}^{\alpha_2}_0\hat{\xi}^{\beta_2}_0 + (-\frac{t\partial_p\rho_1 D_l X^p}{\rho_t} - B^p_q D^2_{pl}X^q + \\ &\quad \mathcal{F}^i_j\mathcal{S}^q_i\mathcal{S}^j_p D_l A^p_q)\nabla^l_k u D_k\mathcal{J}^{\alpha_2}_{\beta_2}\hat{\xi}^{\alpha_2}_0\hat{\xi}^{\beta_2}_0 + s_1\mathcal{F}^i_j R^k_{i1j}D_k\mathcal{J}^{\alpha_2}_{\beta_2}\hat{\xi}^{\alpha_2}_0\hat{\xi}^{\beta_2}_0 - \\ &\qquad 2r_1[\frac{\partial_1\rho_0}{\rho_0} - \frac{(1-t)\partial_1\rho_0 + t\partial_p\rho_1\partial_1 X^p}{\rho_t} - B^p_q\partial_1 D_p X^q + \\ &\qquad \mathcal{F}^i_j\mathcal{S}^q_i\mathcal{S}^j_p\partial_1 A^p_q] + 2r_1(\frac{t\partial_p\rho_1 D_l X^p}{\rho_t} + B^p_q D^2_{pl}X^q - \\ &\qquad \mathcal{F}^i_j\mathcal{S}^q_i\mathcal{S}^j_p D_l A^p_q)\nabla^l_1 u - 2r_1^2\mathcal{F}^i_j R^1_{i1j}\}. \end{aligned}$$

By the critical condition (5.35), Lemma 4.3, Lemma 4.4, Proposition 5.1, Cauchy-Schwarz inequality, we deduce that there exists a positive constant such that the following estimate holds:

$$\begin{aligned} VI_2 \leq & \frac{1}{r_2+r_3}\{C\mathfrak{M}+C(\varepsilon+\delta)\mathcal{F}_1^1+[C(\varepsilon+\delta)-\pi^2]\mathcal{F}_{\alpha_1}^{\alpha_1}+ \\ & C(\varepsilon+\delta)\mathcal{F}_j^i\nabla_i^{\alpha_1}u\nabla_{\alpha_1}^j u+C(\varepsilon+\delta)\mathcal{F}_{n_1+1}^{n_1+1}+ \\ & [C(\varepsilon+\delta)-2\pi^2]\mathcal{F}_{\alpha_2}^{\alpha_2}+C(\varepsilon+\delta)\mathcal{F}_j^i\nabla_i^{\alpha_2}u\nabla_{\alpha_2}^j u\}+ \\ & \frac{1}{s_2+s_3}\{C\mathfrak{M}+C(\varepsilon+\delta)\mathcal{F}_{n_1+1}^{n_1+1}+[C(\varepsilon+\delta)-\pi^2]\mathcal{F}_{\alpha_2}^{\alpha_2}+ \\ & C(\varepsilon+\delta)\mathcal{F}_j^i\nabla_i^{\alpha_2}u\nabla_{\alpha_2}^j u+C(\varepsilon+\delta)\mathcal{F}_1^1+ \\ & [C(\varepsilon+\delta)-2\pi^2]\mathcal{F}_{\alpha_1}^{\alpha_1}+C(\varepsilon+\delta)\mathcal{F}_j^i\nabla_i^{\alpha_1}u\nabla_{\alpha_1}^j u\}. \end{aligned}$$

where $\mathfrak{M}=\max\{\max_{M_1\times M_2}|d\log\rho_0|,\max_{M_1\times M_2}|d\log\rho_1|\}$.

Plugging the upper bounds (5.40),(5.41),(5.42),(5.44),(5.45),(5.47) into the inequality (5.36),we obtain the following inequality:

$$\begin{aligned} 0 \leq & \frac{1}{r_2+r_3}\{8+C\mathfrak{M}+[4C(\varepsilon+\delta)-2]\mathcal{F}_1^1+[3C(\varepsilon+\delta)-\pi^2]\mathcal{F}_{\alpha_1}^{\alpha_1}+ \\ & [3C(\varepsilon+\delta)+\frac{2r_2}{r_1^2}-1]\mathcal{F}_j^i\nabla_i^{\alpha_1}u\nabla_{\alpha_1}^j u+[C(\varepsilon+\delta)-2]\mathcal{F}_{n_1+1}^{n_1+1}+ \\ & [C(\varepsilon+\delta)-2\pi^2]\mathcal{F}_{\alpha_2}^{\alpha_2}+[C(\varepsilon+\delta)-2]\mathcal{F}_j^i\nabla_i^{\alpha_2}u\nabla_{\alpha_2}^j u\}+ \\ & \frac{1}{s_2+s_3}\{8+C\mathfrak{M}+[4C(\varepsilon+\delta)-2]\mathcal{F}_{n_1+1}^{n_1+1}+[3C(\varepsilon+\delta)-\pi^2]\mathcal{F}_{\alpha_2}^{\alpha_2}+ \\ & [3C(\varepsilon+\delta)+\frac{2s_2}{s_1^2}-1]\mathcal{F}_j^i\nabla_i^{\alpha_2}u\nabla_{\alpha_2}^j u+[C(\varepsilon+\delta)-2]\mathcal{F}_1^1+ \\ & [C(\varepsilon+\delta)-2\pi^2]\mathcal{F}_{\alpha_1}^{\alpha_1}+[C(\varepsilon+\delta)-2]\mathcal{F}_j^i\nabla_i^{\alpha_1}u\nabla_{\alpha_1}^j u\}. \end{aligned}$$

Fixing $\varepsilon<\frac{1}{24C},\delta<\frac{1}{24C}$, we get the inequality:

$$\begin{aligned} 0 \leq & \frac{1}{r_2+r_3}[8+C\mathfrak{M}-\frac{5}{3}\mathcal{F}_1^1-\frac{1}{4}\mathcal{F}_j^i\mathcal{S}_i^{\alpha_1}\mathcal{S}_{\alpha_1}^j+ \\ & (\frac{2r_2}{r_1^2}-\frac{1}{2})\mathcal{F}_j^i\nabla_i^{\alpha_1}u\nabla_{\alpha_1}^j u-\frac{1}{4}\mathcal{F}_j^i\mathcal{S}_i^{k_2}\mathcal{S}_{k_2}^j]+ \\ & \frac{1}{s_2+s_3}[8+C\mathfrak{M}-\frac{5}{3}\mathcal{F}_1^1-\frac{1}{4}\mathcal{F}_j^i\mathcal{S}_i^{\alpha_2}\mathcal{S}_{\alpha_2}^j+ \\ & (\frac{2s_2}{s_1^2}-\frac{1}{2})\mathcal{F}_j^i\nabla_i^{\alpha_2}u\nabla_{\alpha_2}^j u-\frac{1}{4}\mathcal{F}_j^i\mathcal{S}_i^{k_1}\mathcal{S}_{k_1}^j] \\ \leq & \frac{1}{r_2+r_3}[8+C\mathfrak{M}-\frac{1}{4}\mathcal{F}_j^i\mathcal{S}_i^k\mathcal{S}_k^j+ \\ & (\frac{2r_2}{r_1^2}-\frac{1}{2})\mathcal{F}_j^i\nabla_i^{\alpha_1}u\nabla_{\alpha_1}^j u]+ \\ & \frac{1}{s_2+s_3}[8+C\mathfrak{M}-\frac{1}{4}\mathcal{F}_j^i\mathcal{S}_i^k\mathcal{S}_k^j+ \\ & (\frac{2s_2}{s_1^2}-\frac{1}{2})\mathcal{F}_j^i\nabla_i^{\alpha_2}u\nabla_{\alpha_2}^j u]. \end{aligned}$$

If δ is small enough, we can assume $\max\{r_2,s_2\}<\frac{(\pi-\delta)^2}{4}$. Hence, we have

$$0 \leq 8+C\mathfrak{M}-\frac{1}{4}\mathcal{F}_j^i\mathcal{S}_i^k\mathcal{S}_k^j.$$

By the inequality of arithmetic geometric means and the equation (5.46), we get that there exists a positive number $\tilde{C}$ such that

$$\mathcal{F}_j^i\mathcal{S}_i^k\mathcal{S}_k^j\geq\tilde{C}(\min\{1,\frac{\rho_1}{\rho_0}\}^{\frac{1}{n_1+n_2}}\frac{1}{(r_2s_2)^{\frac{1}{n_1+n_2}}}.$$

Thus $r_2 s_2$ is bounded from below by $\frac{\tilde{C}^{n_1+n_2}\min\{1,\frac{\rho_1}{\rho_0}\}}{(32+4C\mathfrak{M})^{n_1+n_2}}$. This ends the proof of the Claim 5.1.

Proof of estimate (5.2). We will prove the estimate (5.2) in three steps. Fix $(\tilde{m},\hat{m}) \in M_1 \times M_2$.

Step 1.We first treat the case:$|\widetilde{\nabla}_{\tilde{m}}u| \leq \widetilde{\eta}_1, |\hat{\nabla}_{\hat{m}}u| \leq \hat{\eta}_1$ for some constants $\tilde{\eta}_1, \hat{\eta}_1 \in (0,\pi)$. Using Lemma 1.2 and the definition of determinant, there exists a positive constant $C > 0$ such that

$$\begin{aligned} \left(\frac{\sin|\widetilde{\nabla}_{\tilde{m}}u|}{|\widetilde{\nabla}_{\tilde{m}}u|}\right)^{n_1-1} - C\varepsilon &\leq \det \tilde{J}_0, \\ \left(\frac{\sin|\widehat{\nabla}_{\hat{m}}u|}{|\widehat{\nabla}_{\hat{m}}u|}\right)^{n_2-1} - C\varepsilon &\leq \det \hat{J}_0. \end{aligned}$$

Recall that real function $\frac{\sin s}{s}$ is increasing in the interval $(0,\pi)$. By choosing ε small enough, we get the estimate.

Step 2.In this step, we will examine the case:$\min\{|\widetilde{\nabla}_{\tilde{m}}u|, |\widehat{\nabla}_{\hat{m}}u|\} > \pi - \delta$, for $\delta \in (0, \frac{\pi}{4})$ small enough. Using Claim 5.1, by choosing δ small enough, we derive that both the second eigenvalue of $\widetilde{\mathcal{J}}$ and the second eigenvalue of $\widehat{\mathcal{J}}$ have some positive lower bounds. By definition of $\widetilde{\mathcal{J}}$ and $\widehat{\mathcal{J}}$, we obtain that both $\det \tilde{J}_0$ and $\det \hat{J}_0$ have positive lower bounds.

Step 3.In the last step, we address the case $|\widetilde{\nabla}_{\tilde{m}}u| \geq \pi - \eta_0, |\widehat{\nabla}_{\hat{m}}u| \leq \hat{\eta}_2$ for some constants $\eta_0 \in (0,\frac{\pi}{4}), \hat{\eta}_2 \in (0,\pi)$ or $|\widetilde{\nabla}_{\tilde{m}}u| \leq \tilde{\eta}_2, |\widehat{\nabla}_{\hat{m}}u| \geq \pi - \eta_0$ for some constants $\eta_0 \in (0,\frac{\pi}{4}), \tilde{\eta}_2 \in (0,\pi)$. Without loss of generality, we assume that $|\widetilde{\nabla}_{\tilde{m}}u| \geq \pi - \eta_0, |\widehat{\nabla}_{\hat{m}}u| \leq \hat{\eta}_2$ for some constants $\eta_0 \in (0,\frac{\pi}{4}), \hat{\eta}_2 \in (0,\pi)$. The constant η_0 will be determined later.

As same as step 1, for $|\widehat{\nabla}_{\hat{m}}u| \leq \hat{\eta}_2$, the determinant $\det \hat{J}_0(\hat{m}, \hat{\nabla}_{\hat{m}}u, 1)$ has a positive uniform lower bound. To complete the proof, it suffices to show that the determinant $\det \tilde{J}_0(\tilde{m}, \tilde{\nabla}_{\tilde{m}}u, 1)$ also has a positive lower bound.

Note that one can prove that the minimization problem $\min\{\langle \widetilde{\mathcal{J}}\xi,\xi\rangle : (\tilde{m},\hat{m}) \in M_1 \times M_2, \xi = \tilde{\xi} + \hat{\xi}, \tilde{\xi} \in T_{\tilde{m}}M_1, \hat{\xi} \in T_{\hat{m}}M_2, |\tilde{\xi}|_{\tilde{m}} = 1 = |\hat{\xi}|, \tilde{g}(\tilde{\xi}, \tilde{\nabla}_{\tilde{m}}u) = 0, |\tilde{\nabla}_{\tilde{m}}u| \geq \frac{3\pi}{4}\}$ also has a positive lower bound. Thus as same as step 2, $\det \hat{J}_0(\hat{m}, \hat{\nabla}_{\hat{m}}u, 1)$ has a positive uniform lower bound. This completes the proof of the estimate (5.2).

5.3.3 Proof of Theorem 5.2

This subsection is devoted to the proof of Theorem 5.2. It is known that the sectional curvatures of $M_1 \times M_2$ are non-negative and may vanish, besides its c-curvature is non-negative and may vanish on some directions. The vanishing of c-curvature are the main obstacles of the smoothness. We will derive the uniformly C^2 estimate by estimate (5.2). Then the method of continuity implies the smoothness of the optimal transport map.

To begin with, we give a basic lemma.

Lemma 5.5. *Let (M,g) be a closed Riemannian manifold of dimension $n \geq 2$. Suppose that (M,g) satisfies (1.3). Then there exists a positive constant Λ_0 depending only on n such that, for $0 < \varepsilon < \Lambda_0$,if*

$$\|Riem - \frac{1}{2} g \owedge g\|_{C^2(M,g)} < \varepsilon, \tag{5.47}$$

then for every $m \in M, \nu \in I(m)$ and every positive definite linear operator $\mathcal{F}(m,\nu) : T_mM \to T_mM$, the following estimate holds

$$tr\mathcal{S}\mathcal{F}\mathcal{S} + tr\mathcal{F}\mathcal{R} \geq \frac{1}{2} tr\mathcal{F},$$

where $\mathcal{R}(\cdot) = R(\cdot,\nu)\nu$.

Proof. It is clear that left side is equal to $tr\mathcal{F}$ when $\nu = 0$. So we assume that $\nu \neq 0$. Taking the Fermi coordinate system along the geodesic $\exp_m(s\frac{\nu}{|\nu|})$.
If $|\nu| \geq \frac{3\pi}{4}$, set $\bar{\mathcal{R}}(\cdot) = \bar{R}(\cdot,\nu)\nu$. In view of the curvature assumption (5.47) and the positive definiteness of the linear operator $\mathcal{F}$, we have

$$\begin{aligned} tr\mathcal{S}\mathcal{F}\mathcal{S} + tr\mathcal{F}\mathcal{R} &\geq \mathcal{F}_1^1 + tr\mathcal{F}\bar{\mathcal{R}} - \varepsilon tr\mathcal{F} \\ &= \mathcal{F}_1^1 + |\nu|^2\sum_{i=2}^n \mathcal{F}_i^i - \varepsilon tr\mathcal{F} \\ &\geq (1-\varepsilon)tr\mathcal{F} \\ &\geq \frac{1}{2}tr\mathcal{F}, \end{aligned}$$

where the last inequality yields provided $\varepsilon < \frac{1}{2}$.
If $|\nu| \leq \frac{3\pi}{4}$, the condition of Theorem 3.1 is satisfied provided $\varepsilon < \frac{1}{3\pi\sqrt{2(n-1)}}$. From (3.10), we know that

$$\begin{aligned} |J_0^{-1} - \bar{J}_0^{-1}| &\leq 4\sqrt{n-1}(\frac{|\nu|}{\sin|\nu|})^2\varepsilon \\ &\leq \frac{9}{2}\pi^2\sqrt{n-1}\varepsilon, \end{aligned} \tag{5.48}$$

where the last inequality follows from the fact that the function $\frac{t}{\sin t}$ is increasing in $(0,\pi)$. Observe that

$$\mathcal{S} - \bar{\mathcal{S}} = (J_0^{-1} - \bar{J}_0^{-1})J_1 + \bar{J}_0^{-1}(J_1 - \bar{J}_1).$$

By (5.48), Lemma 1.1 and Lemma 1.2, we derive that there exists a positive constant C depending only on n such that

$$|\mathcal{S} - \bar{\mathcal{S}}| \leq C\varepsilon.$$

It is readily to see that $\mathcal{S}$ is bounded. Thus

$$\begin{aligned} tr\mathcal{S}\mathcal{F}\mathcal{S} + tr\mathcal{F}\mathcal{R} &\geq tr\bar{\mathcal{S}}\mathcal{F}\bar{\mathcal{S}} + tr\mathcal{F}\bar{\mathcal{R}} - C\varepsilon tr\mathcal{F} \\ &= \mathcal{F}_1^1 + (\frac{|\nu|\cos|\nu|}{\sin|\nu|})^2\sum_{i=2}^n \mathcal{F}_i^i + |\nu|^2\sum_{i=2}^n \mathcal{F}_i^i - C\varepsilon tr\mathcal{F} \\ &= \mathcal{F}_1^1 + \frac{|\nu|^2}{\sin^2|\nu|}\sum_{i=2}^n \mathcal{F}_i^i - C\varepsilon tr\mathcal{F} \\ &\geq (1-C\varepsilon)tr\mathcal{F}, \end{aligned}$$

The desired inequality follows if we choose $\varepsilon < \min\{\frac{1}{2C}, \frac{1}{3\pi\sqrt{2(n-1)}}\}$. □

We now prove Theorem 5.2 by the continuity method. Assume that the condition of Theorem 5.2 is satisfied.

Let $\mathcal{I}$ be the set of the parameter $t \in [0,1]$ for which there exists a $C^{k+2,\alpha}$ solution u_t of the Monge-Ampère type equation (5.29) with ρ_1 replaced by $\rho_t = (1-t)\rho_0 + t\rho_1$. To ensure the uniqueness, we assume that $\int_{M_1\times M_2} u_t dvol = 0$.

It is clear that $0 \in \mathcal{I}$, so the set $\mathcal{I}$ is nonempty. The openness is derived by an implicit function theorem [53]. The connectedness of the set $[0,1]$ will prove the equation (5.29) admits a $C^{k,\alpha}$ solution if $\mathcal{I}$ is closed.

In subsection 5.3.2, we have proved the estimate (5.2). By Lemma 5.1, in order to prove Theorem 5.2, it is sufficient to prove that there exists a positive constant C such that

$$\forall t \in \mathcal{I}, \max_{m\in M_1\times M_2} |\nabla_m^2 u_t + \mathcal{S}(m, \nabla_m u_t, 1)| \leq C. \tag{5.49}$$

From Corollary 3.1, we know that the MTW tensor on $M_1 \times M_2$ satisfies the A3W condition but not A3S condition. Moreover, it is non-negative. Thus the condition of Theorem 6.1 in Delanoë [27] is not satisfied. We have to prove the second derivative estimate.

Proof of Theorem 5.2. Suppose that the condition of Theorem 5.2 is established. As mentioned before, it is sufficient to prove (5.49). Let

$$\mathcal{H} = \nabla^2 u + \nabla^2 \frac{d_G^2}{2},$$

where G is the optimal transport map and the derivative of $\frac{d_p^2}{2}(m) = \frac{d^2}{2}(m,p)$ is with respect to m and then to take value at $p = G(m)$.

Consider the maximization problem

$$\max\{\langle \mathcal{H}\xi, \xi\rangle e^{\frac{\beta}{2}|\nabla u|_m^2} : m \in M_1 \times M_2, \xi \in T_m(M_1 \times M_2), |\xi|_m = 1\},$$

where β is a positive constant to be determined later.

Assume that the maximum is achieved at the point (p_0, ξ_0). Fixing $m \in M_1 \times M_2, \xi \in T_m(M_1 \times M_2)\backslash\{0\}$, we consider the test function:

$$h\,(m,\xi) = \frac{\langle \mathcal{H}\xi, \xi\rangle}{|\xi|^2} e^{\frac{\beta}{2}|\nabla u|_m^2}.$$

It is clear that h attains the maximum at the point (m_0, ξ_0).

Take the coordinate system $z = (x, y)$, where x is normal coordinate system in M_1 around the point $\tilde{p}_0$ and y is normal coordinate system in M_2 around the point $\hat{p}_0$. The associated coordinate system in tangent bundle $T(M_1 \times M_2)$ is denoted by (z, v), where $v = v^i \frac{\partial}{\partial z^i}$.

Components of tensors will be denoted by:

$$\begin{aligned} &\nabla_p u = \nabla^i u(p) \frac{\partial}{\partial z^i}, \nabla_p^2 u = \nabla_j^i u(p) dz^j \otimes \frac{\partial}{\partial z^i}, \\ &\mathcal{S} = \mathcal{S}_j^i(p,\nu,1) dz^j \otimes \frac{\partial}{\partial z^i}, \mathcal{H} = \mathcal{H}_j^i(p) dz^j \otimes \frac{\partial}{\partial z^i}, \\ &\mathcal{F} = \mathcal{F}_j^i(p) dz^j \otimes \frac{\partial}{\partial z^i}, \end{aligned}$$

where $\mathcal{H}_k^i \mathcal{F}_j^k = \delta_j^i$.

Suppose that the tangent vector $\xi_0 = \frac{\partial}{\partial z^1}\Big|_{p_0}$. We may also assume that $(\mathcal{H}_j^i)$ is diagonal at the point p_0.

Let m be in the domain of the coordinate system z. The associated coordinate is given by $z = (z^1, \cdots, z^{n_1+n_2})$. Fix $\xi \in T_m(M_1 \times M_2)$. The associated coordinate of ξ is denoted by $\xi = \xi^i \frac{\partial}{\partial z^i}$. Then

$$\langle \mathcal{H}\xi, \xi\rangle = \mathcal{H}_b^a g_{ap} \xi^b \xi^p, |\xi|^2 = g_{ab}\xi^a \xi^b. \tag{5.50}$$

In the following all terms are evaluated at the point (p_0, ξ_0). It will be implicitly understood throughout the calculations.

It is clear that the function $\log h$ also attains its maximum at the point (p_0, ξ_0).

The first derivative condition

From (5.50), by differentiating the function $\log h$ with respect to z^i, the first derivative condition could be read as:

$$\frac{\nabla_i \mathcal{H}_1^1}{\mathcal{H}_1^1} + \beta \nabla_i^k u \nabla_k u = 0. \tag{5.51}$$

The second derivative condition

Differentiating twice the test function $\log h$ with respect to z^i, z^j, the second derivative condition can be written as follows:

$$0 \geq I_3 + II_3 + III_3, \tag{5.52}$$

where

$$
\begin{aligned}
I_3 &= \frac{1}{\mathcal{H}_1^1}\mathcal{F}_j^i\nabla_i^j\nabla_1^1 u - \frac{1}{(\mathcal{H}_1^1)^2}\mathcal{F}_j^i\nabla_i\mathcal{H}_1^1\nabla^j\mathcal{H}_1^1,\\
II_3 &= \beta\mathcal{F}_j^i\nabla^j\nabla_i^k u\nabla_k u + \frac{1}{\mathcal{H}_1^1}\mathcal{F}_j^i\nabla_i^j\mathcal{S}_1^1,\\
III_3 &= \beta\mathcal{F}_j^i\nabla_i^k u\nabla_k^j u + \frac{1}{\mathcal{H}_1^1}\mathcal{F}_j^i\partial_{ij}^2 g_{k1}\mathcal{H}_1^k - \mathcal{F}_j^i\partial_{ij}^2 g_{11}.
\end{aligned}
$$

We denote by I_3, II_3 and III_3 these terms.

Term I_3 The term I_3 involves the fourth derivative of the potential function u. Making use of the covariant derivative commutative formula

$$
\begin{aligned}
\nabla_{klij}u &= \nabla_{ijkl}u + (\nabla_j R_{kli}^s + \nabla_l R_{ikj}^s)\nabla_s u +\\
&\quad R_{sklj}\nabla_i^s u + R_{skli}\nabla_j^s u +\\
&\quad R_{silj}\nabla_k^s u + R_{sikj}\nabla_l^s u,
\end{aligned}
$$

we get:

$$
\begin{aligned}
\mathcal{F}_j^i\nabla_i^j\nabla_1^1 u &= \mathcal{F}_j^i\nabla_1^1\nabla_i^j u + \mathcal{F}_j^i(\nabla_j R_{11i}^s + \nabla_1 R_{i1j}^s)\nabla_s u +\\
&\quad 2\mathcal{F}_j^i(R_{s11j}\nabla_i^s u + R_{si1j}\nabla_1^s u).
\end{aligned}
$$

Thus

$$
\begin{aligned}
I_3 &= \frac{1}{\mathcal{H}_1^1}(\mathcal{F}_j^i\nabla_1^1\nabla_i^j u - \frac{1}{\mathcal{H}_1^1}\mathcal{F}_j^i\nabla_i\mathcal{H}_1^1\nabla^j\mathcal{H}_1^1) +\\
&\quad \frac{1}{\mathcal{H}_1^1}[\mathcal{F}_j^i(\nabla_1 R_{i1j}^s + \nabla_j R_{11i}^s)\nabla_s u +\\
&\quad 2\mathcal{F}_j^i(R_{si1j}\mathcal{H}_1^s + R_{s11j}\mathcal{H}_i^s) -\\
&\quad 2\mathcal{F}_j^i(R_{si1j}\mathcal{S}_1^s + R_{s11j}\mathcal{S}_i^s)].
\end{aligned}
$$

Differentiating the equation (5.29) twice, we derive

$$
\begin{aligned}
\mathcal{F}_j^i\nabla_1^1\nabla_i^j u &= \mathcal{F}_s^i\nabla_1\mathcal{H}_q^s\mathcal{F}_j^q\nabla_1\mathcal{H}_i^j + \nabla_1^1\phi - \mathcal{F}_j^i\nabla_1^1\mathcal{S}_j^i\\
&= \mathcal{F}_i^i\mathcal{F}_j^j(\nabla_1\mathcal{H}_i^j)^2 + \nabla_1^1\phi - \mathcal{F}_j^i\nabla_1^1\mathcal{S}_j^i.
\end{aligned}
$$

The first term in the above expression is non-negative. Moreover, it can control the negative term in I_3. Indeed,

$$
\begin{aligned}
&\mathcal{F}_i^i\mathcal{F}_j^j(\nabla_1\mathcal{H}_i^j)^2 - \frac{1}{\mathcal{H}_1^1}\mathcal{F}_i^i(\nabla_i\mathcal{H}_1^1)^2 - \frac{1}{2\mathcal{H}_1^1}\sum_{i\geq 2}\mathcal{F}_i^i(\nabla_i\mathcal{H}_1^1)^2\\
=\ &\mathcal{F}_i^i\mathcal{F}_j^j(\nabla_1\mathcal{H}_i^j)^2 - \frac{1}{(\mathcal{H}_1^1)^2}(\nabla_1\mathcal{H}_1^1)^2 - \frac{3}{2\mathcal{H}_1^1}\sum_{i\geq 2}\mathcal{F}_i^i(\nabla_i\mathcal{H}_1^1)^2\\
\geq\ &\frac{1}{2\mathcal{H}_1^1}\sum_{i\geq 2}\mathcal{F}_i^i(\nabla_i\mathcal{H}_1^1)^2 + \frac{2}{\mathcal{H}_1^1}\sum_{i\geq 2}\mathcal{F}_i^i[(\nabla_1\mathcal{H}_i^1)^2 - (\nabla_i\mathcal{H}_1^1)^2]\\
\geq\ &-\frac{6}{\mathcal{H}_1^1}\sum_{i\geq 2}\mathcal{F}_i^i(\nabla_1\mathcal{H}_i^1 - \nabla_i\mathcal{H}_1^1)^2.
\end{aligned}
$$

where the last inequality follows from

$$\begin{aligned}
& \frac{2}{\mathcal{H}_1^1}\sum_{i\geq 2}\mathcal{F}_i^i[(\nabla_1\mathcal{H}_i^1)^2-(\nabla_i\mathcal{H}_1^1)^2] \\
= & \frac{2}{\mathcal{H}_1^1}\sum_{i\geq 2}\mathcal{F}_i^i[\nabla_1\mathcal{H}_i^1-\nabla_i\mathcal{H}_1^1][\nabla_1\mathcal{H}_i^1+\nabla_i\mathcal{H}_1^1] \\
= & \frac{2}{\mathcal{H}_1^1}\sum_{i\geq 2}\mathcal{F}_i^i[\nabla_1\mathcal{H}_i^1-\nabla_i\mathcal{H}_1^1][\nabla_1\mathcal{H}_i^1-\nabla_i\mathcal{H}_1^1+2\nabla_i\mathcal{H}_1^1] \\
\geq & -\frac{1}{2\mathcal{H}_1^1}\sum_{i\geq 2}\mathcal{F}_i^i(\nabla_i\mathcal{H}_1^1)^2-\frac{6}{\mathcal{H}_1^1}\sum_{i\geq 2}\mathcal{F}_i^i(\nabla_1\mathcal{H}_i^1-\nabla_i\mathcal{H}_1^1)^2.
\end{aligned}$$

As a consequence, we have

$$\begin{aligned}
I_3 \geq & -\frac{6}{(\mathcal{H}_1^1)^2}\sum_{i\geq 2}\mathcal{F}_i^i(\nabla_1\mathcal{H}_i^1-\nabla_i\mathcal{H}_1^1)^2+\frac{1}{\mathcal{H}_1^1}(\nabla_1^1\phi-\mathcal{F}_j^i\nabla_1^1\mathcal{S}_j^i)+ \\
& \frac{1}{\mathcal{H}_1^1}[\mathcal{F}_j^i(\nabla_1 R_{i1j}^s+\nabla_j R_{11i}^s)\nabla_s u+2\mathcal{F}_j^i(R_{si1j}\mathcal{H}_1^s+R_{s11j}\mathcal{H}_i^s)- \\
& 2\mathcal{F}_j^i(R_{si1j}\mathcal{S}_1^s+R_{s11j}\mathcal{S}_i^s)] \\
= & I_{31}+I_{32}+I_{33}.
\end{aligned} \tag{5.53}$$

We first deal with I_{31}. After commuting the third derivatives, we see that

$$\begin{aligned}
\nabla_1\mathcal{H}_i^1-\nabla_i\mathcal{H}_1^1 = & R_{1i1}^k\nabla_k u+\partial_1\mathcal{S}_i^1-\partial_i\mathcal{S}_1^1+ \\
& D_k\mathcal{S}_i^1\nabla_1^k u-D_k\mathcal{S}_1^1\nabla_i^k u \\
= & R_{1i1}^k\nabla_k u+\partial_1\mathcal{S}_i^1-\partial_i\mathcal{S}_1^1- \\
& D_k\mathcal{S}_i^1\mathcal{S}_1^k+D_k\mathcal{S}_1^1\mathcal{S}_i^k+ \\
& D_k\mathcal{S}_i^1\mathcal{H}_1^k-D_k\mathcal{S}_1^1\mathcal{H}_i^k \\
= & R_{1i1}^k\nabla_k u+\partial_1\mathcal{S}_i^1-\partial_i\mathcal{S}_1^1- \\
& D_k\mathcal{S}_i^1\mathcal{S}_1^k+D_k\mathcal{S}_1^1\mathcal{S}_i^k+ \\
& D_1\mathcal{S}_i^1\mathcal{H}_1^1-D_i\mathcal{S}_1^1\mathcal{H}_i^i,
\end{aligned}$$

where the last equality follows from the fact that the matrix $(\mathcal{H}_j^i)$ is diagonal at p_0.

Note that $\mathcal{H}_1^1$ is the maximal eigenvalue of $(\mathcal{H}_j^i)$. Assume that $\mathcal{H}_1^1\geq 1$. From Lemma 5.2, we have

$$I_{31} \geq -C-Ctr\mathcal{F}. \tag{5.54}$$

We now treat I_{32}. By a lengthy computation, we get

$$\begin{aligned}
\nabla_1^1\phi = & \partial_{11}^2\phi+2\partial_1 D_k\phi\nabla_1^k u+D_{pq}^2\phi\nabla_1^p u\nabla_1^q u+ \\
& D_k\phi(\partial_1\nabla_1^k u-\partial_1\Gamma_{1s}^k\nabla^s u).
\end{aligned}$$

After commuting the third derivative, we obtain

$$\begin{aligned}
\nabla_1^1\phi = & \partial_{11}^2\phi+2\partial_1 D_k\phi\nabla_1^k u+D_{pq}^2\phi\nabla_1^p u\nabla_1^q u+ \\
& D_k\phi(\partial_k\nabla_1^1 u+R_{1k1}^s\nabla_s u-\partial_1\Gamma_{1s}^k\nabla^s u) \\
= & \partial_{11}^2\phi+2\partial_1 D_k\phi\nabla_1^k u+D_{pq}^2\phi\nabla_1^p u\nabla_1^q u+ \\
& D_k\phi(\nabla^k\mathcal{H}_1^1+R_{1k1}^s\nabla^s u-\partial_k\mathcal{S}_1^1- \\
& D_p\mathcal{S}_1^1\nabla_k^p u-\partial_1\Gamma_{1s}^l\nabla^s u).
\end{aligned}$$

Making use of the critical condition (5.51) again, it follows that

$$\begin{aligned}
\nabla_1^1\phi &= \partial_{11}^2\phi + 2\partial_1 D_k\phi\nabla_1^k u + D_{pq}^2\phi\nabla_1^p u\nabla_1^q u - \\
&\quad \beta\mathcal{H}_1^1 D_k\phi\nabla_l^k u\nabla^l u + D_k\phi(R_{1k1}^s\nabla^s u - \\
&\quad \partial_k\mathcal{S}_1^1 - D_p\mathcal{S}_1^1\nabla_k^p u - \partial_1\Gamma_{1s}^l\nabla^s u) \\
&= \partial_{11}^2\phi + 2\partial_1 D_k\phi\mathcal{H}_1^k - 2\partial_1 D_k\phi\mathcal{S}_1^k + \\
&\quad D_{pq}^2\phi(\mathcal{H}_1^p - \mathcal{S}_1^p)(\mathcal{H}_1^q - \mathcal{S}_1^q) - \\
&\quad \beta\mathcal{H}_1^1 D_k\phi\nabla_l^k u\nabla^l u + D_k\phi(R_{1k1}^s\nabla^s u - \\
&\quad \partial_k\mathcal{S}_1^1 - D_p\mathcal{S}_1^1\mathcal{H}_k^p + D_p\mathcal{S}_1^1\mathcal{S}_k^p - \partial_1\Gamma_{1s}^l\nabla^s u).
\end{aligned}$$

In view of Lemma 5.2, we derive

$$\frac{1}{\mathcal{H}_1^1}\nabla_1^1\phi \geq -C\|\phi\| - C\|\phi\|\mathcal{H}_1^1 - \beta D_k\phi\nabla_l^k u\nabla^l u. \tag{5.55}$$

Similarly,

$$\begin{aligned}
-\mathcal{F}_j^i\nabla_1^1\mathcal{S}_i^j &= -\mathcal{F}_j^i\partial_{11}^2\mathcal{S}_i^j - 2\mathcal{F}_j^i\partial_1 D_k\mathcal{S}_i^j\nabla_1^k u - \mathcal{F}_j^i D_{pq}^2\mathcal{S}_i^j\nabla_1^p u\nabla_1^q u - \\
&\quad \mathcal{F}_j^i D_k\mathcal{S}_i^j(\nabla^k\mathcal{H}_1^1 + R_{1k1}^s\nabla_s u - \partial_k\mathcal{S}_1^1 - \\
&\quad D_p\mathcal{S}_1^1\nabla_k^p u - \partial_1\Gamma_{1s}^k\nabla^s u) \\
&= -\mathcal{F}_j^i\partial_{11}^2\mathcal{S}_i^j - 2\mathcal{F}_j^i\partial_1 D_k\mathcal{S}_i^j\nabla_1^k u - \mathcal{F}_j^i D_{pq}^2\mathcal{S}_i^j\nabla_1^p u\nabla_1^q u + \\
&\quad \beta\mathcal{H}_1^1\mathcal{F}_j^i D_k\mathcal{S}_i^j\nabla_l^k u\nabla^l u + \mathcal{F}_j^i D_k\mathcal{S}_i^j(-R_{1k1}^s\nabla_s u + \\
&\quad \partial_k\mathcal{S}_1^1 + D_p\mathcal{S}_1^1\nabla_k^p u - \partial_1\Gamma_{1s}^k\nabla^s u).
\end{aligned}$$

Note that the *MTW tensor* on $M_1 \times M_2$ is non-negative. Thus

$$-\frac{1}{\mathcal{H}_1^1}\mathcal{F}_j^i\nabla_1^1\mathcal{S}_i^j \geq -Ctr\mathcal{F} + \beta\mathcal{F}_j^i D_k\mathcal{S}_i^j\nabla_l^k u\nabla^l u. \tag{5.56}$$

By virtue of (5.55) and (5.56), it follows that

$$\begin{aligned}
I_{32} &\geq -C\|\phi\| - Ctr\mathcal{F} - C\|\phi\|\mathcal{H}_1^1 - \\
&\quad \beta D_k\phi\nabla_l^k u\nabla^l u + \beta\mathcal{F}_j^i D_k\mathcal{S}_i^j\nabla_l^k u\nabla^l u.
\end{aligned} \tag{5.57}$$

For I_{33}, using Lemma 5.2, we get

$$\begin{aligned}
I_{33} &= \frac{1}{\mathcal{H}_1^1}[\mathcal{F}_i^i(\nabla_1 R_{i1i}^s + \nabla_j R_{11i}^s)\nabla_s u + 2\mathcal{F}_i^i(R_{1i1i}\mathcal{H}_1^1 + R_{i11i}\mathcal{H}_i^i) - \\
&\quad 2\mathcal{F}_i^i(R_{si1i}\mathcal{S}_1^s + R_{s11i}\mathcal{S}_i^s)] \\
&\geq -Ctr\mathcal{F}.
\end{aligned} \tag{5.58}$$

Substituting (5.54)(5.57)(5.58) into (5.53), we see that

$$\begin{aligned}
I_3 &\geq -C - C\|\phi\| - Ctr\mathcal{F} - C\|\phi\|\mathcal{H}_1^1 - \\
&\quad \beta D_k\phi\nabla_l^k u\nabla^l u + \beta\mathcal{F}_j^i D_k\mathcal{S}_i^j\nabla_p^k u\nabla^p u.
\end{aligned} \tag{5.59}$$

The term II_3 We are in position to deal with the term II_3. It is readily to see that

$$\begin{aligned}
II_3 &= \beta\mathcal{F}_j^i\nabla^j\nabla_i^k u\nabla_k u + \frac{1}{\mathcal{H}_1^1}\mathcal{F}_j^i(\partial_{ij}^2\mathcal{S}_1^1 + 2\partial_i D_k\mathcal{S}_1^1\nabla_j^k u + \\
&\quad D_{kl}^2\mathcal{S}_1^1\nabla_i^k u\nabla_j^l u + D_k\mathcal{S}_1^1\partial_j\nabla_i^k u - D_k\mathcal{S}_1^1\partial_j\Gamma_{iq}^k\nabla^q u).
\end{aligned}$$

After commuting the derivatives, we find

$$\begin{aligned} II_3 &= \beta\mathcal{F}^i_j\nabla_k\nabla^j_i u\nabla^k u + \beta\mathcal{F}^i_j R^l_{ikj}\nabla_l u\nabla^k u + \\ &\quad \frac{1}{\mathcal{H}^1_1}\mathcal{F}^i_j(\partial^2_{ij}\mathcal{S}^1_1 + 2\partial_i D_k\mathcal{S}^1_1\nabla^k_j u + \\ &\quad D^2_{kl}\mathcal{S}^1_1\nabla^k_i u\nabla^l_j u + D_k\mathcal{S}^1_1\nabla_k\nabla^j_i u + \\ &\quad D_k\mathcal{S}^1_1 R^s_{ikj}\nabla_s u - D_k\mathcal{S}^1_1\partial_j\Gamma^k_{iq}\nabla^q u) \\ &= \beta\mathcal{F}^i_j\nabla_k\nabla^j_i u\nabla^k u + \beta\mathcal{F}^i_j R^l_{ikj}\nabla_l u\nabla^k u + \\ &\quad \frac{1}{\mathcal{H}^1_1}\mathcal{F}^i_j\nabla_k\nabla^j_i u D_k\mathcal{S}^1_1 + \frac{1}{\mathcal{H}^1_1}\mathcal{F}^i_j(\partial^2_{ij}\mathcal{S}^1_1 + \\ &\quad 2\partial_i D_k\mathcal{S}^1_1\nabla^k_j u + D^2_{kl}\mathcal{S}^1_1\nabla^k_i u\nabla^l_k u + \\ &\quad D_k\mathcal{S}^1_1 R^s_{ikj}\nabla_s u - D_k\mathcal{S}^1_1\partial_j\Gamma^l_{iq}\nabla^q u). \end{aligned}$$

Differentiating the equation (5.29) with respect to z^k, we deduce

$$\mathcal{F}^i_j\nabla_k\nabla^j_i u = \partial_k\phi + D_l\phi\nabla^l_k u - \mathcal{F}^i_j\partial_k\mathcal{S}^i_j - \mathcal{F}^i_j D_l\mathcal{S}^i_j\nabla^l_k u.$$

Thus

$$\begin{aligned} II_3 &= \beta(\partial_k\phi + D_l\phi\nabla^l_k u - \mathcal{F}^i_j\partial_k\mathcal{S}^i_j - \mathcal{F}^i_j D_l\mathcal{S}^i_j\nabla^l_k u)\nabla^k u + \\ &\quad \beta\mathcal{F}^i_j R^l_{ikj}\nabla_l u\nabla^k u + \frac{1}{\mathcal{H}^1_1}D_k\mathcal{S}^1_1(\partial_k\phi + D_q\phi\nabla^q_k u - \\ &\quad \mathcal{F}^i_j\partial_k\mathcal{S}^j_i - \mathcal{F}^i_j D_q\mathcal{S}^j_i\nabla^q_k u) + \frac{1}{\mathcal{H}^1_1}\mathcal{F}^i_j(\partial^2_{ij}\mathcal{S}^1_1 + \\ &\quad 2\partial_i D_k\mathcal{S}^1_1\nabla^k_j u + D^2_{kl}\mathcal{S}^1_1\nabla^k_i u\nabla^j_l u + \\ &\quad D_k\mathcal{S}^1_1 R^s_{ikj}\nabla_s u - D_k\mathcal{S}^1_1\partial_j\Gamma^l_{iq}\nabla^q u). \end{aligned}$$

Let us observe that

$$\mathcal{F}^i_j\nabla^k_i u = \delta^k_j - \mathcal{F}^i_j\mathcal{S}^k_i, \tag{5.60}$$

$$\mathcal{F}^i_j\nabla^k_i u\nabla^j_l u = \mathcal{H}^k_l - 2\mathcal{S}^k_l + \mathcal{F}^i_j\mathcal{S}^k_i\mathcal{S}^j_l. \tag{5.61}$$

Together with Lemma 5.2, we infer

$$\begin{aligned} II_3 &\geq -C - C(1+\beta)\|\phi\|_{C^1} - Ctr\mathcal{F} + \\ &\quad \beta D_l\phi\nabla^l_k u\nabla^k u - \beta\mathcal{F}^i_j\partial_k\mathcal{S}^i_j\nabla^k u - \\ &\quad \beta\mathcal{F}^i_j D_l\mathcal{S}^j_i\nabla^l_k u\nabla^k u + \beta\mathcal{F}^i_j R^l_{ikj}\nabla_l u\nabla^k u. \end{aligned} \tag{5.62}$$

The term III_3 In view of (5.61), we get

$$\begin{aligned} III_3 &= \beta tr\mathcal{H} - 2\beta tr\mathcal{S} + \beta tr\mathcal{S}\mathcal{F}\mathcal{S} + \frac{1}{\mathcal{H}^1_1}\mathcal{F}^i_j\partial^2_{ij}g_{k1}\mathcal{H}^k_1 - \mathcal{F}^i_j\partial^2_{ij}g_{11} \\ &\geq -C\beta - Ctr\mathcal{F} + \beta\mathcal{H}^1_1 + \beta tr\mathcal{S}\mathcal{F}\mathcal{S}. \end{aligned} \tag{5.63}$$

Substituting (5.59)(5.62)(5.63) into (5.52), we see that

$$\begin{aligned} 0 &\geq -C(1+\beta) - C(2+\beta)\|\phi\| - 3Ctr\mathcal{F} + \\ &\quad (\beta - C\|\phi\|)\mathcal{H}^1_1 - \beta\mathcal{F}^i_j\partial_k\mathcal{S}^i_j\nabla^k u + \\ &\quad \beta(tr\mathcal{S}\mathcal{F}\mathcal{S} + tr\mathcal{F}\mathcal{R}). \end{aligned}$$

where $\mathcal{R}(\cdot) = R(\cdot,\nabla u)\nabla u$.

Note that Theorem 4.1 and (3.10) imply that there exists a positive constant Λ_1 depending only on n_1, n_2 such that

$$|\mathcal{F}\partial_x\mathcal{S}| \leq \Lambda_1\varepsilon\max\{tr\mathcal{F}, tr\mathcal{S}\mathcal{F}\mathcal{S}\}.$$

Choosing $0 < \varepsilon < \Lambda_0$, due to Lemma 5.5, we derive

$$\begin{aligned} 0 \quad \geq \quad & -C(1+\beta) - C(2+\beta)\|\phi\| + (\beta - C\|\phi\|)\mathcal{H}_1^1 + \\ & [(\frac{1}{4} - \pi\Lambda_1\varepsilon)\beta - 3C]tr\mathcal{F} + (\frac{1}{2} - \pi\Lambda_1\varepsilon)tr\mathcal{SFS}. \end{aligned}$$

Taking $\varepsilon < \frac{1}{8\pi\Lambda_1}$ and $\beta \geq \max\{2C\|\phi\|, 24C\}$, we get

$$\|\phi\|\mathcal{H}_1^1 \quad \leq \quad (1+\beta) + (2+\beta)\|\phi\|. \tag{5.64}$$

Thus the $\mathcal{H}_1^1$ is bounded above at the point p_0. By the positivity of $\mathcal{H}$, thus $|\mathcal{H}|$ is bounded from above. This finishes the proof of Theorem 5.2.

5.4 The smoothness on C^4 perturbation of product Riemannian manifold

In this section, we prove Theorem 5.3. Let $(M_1, \tilde{g})$ and $(M_2, \hat{g})$ be two closed Riemannian manifolds of dimension $n_1 \geq 2$ and $n_2 \geq 2$ respectively. Let $\rho_0 dvol$ and $\rho_1 dvol$ be two smooth positive Borel probability measures on $M_1 \times M_2$. Set $(M_1 \times M_2, g^\times)$ be Riemannian product of $(M_1, \tilde{g})$ and $(M_2, \hat{g})$ and Riem the corresponding (4,0)-th Riemann curvature tensor.

Note that Theorem 5.3 is trivial if the optimal transport map on $(M_1 \times M_2, g^\times)$ is not smooth. Indeed, we just take $g = g^\times$. Without generality, assume that the optimal transport map on $(M_1 \times M_2, g^\times)$ is smooth.

Let h be a non-trivial C^4 smooth function on M_1. Consider the conformal metric $(M_1 \times M_2, g = e^{-2u} g^\times)$ with $u = -\frac{1}{2}\log(1 + \varepsilon h^2)$. Note that u can be viewed as a function on $M_1 \times M_2$.

It is clear that g is C^4 perturbation of $g^\times$ for ε sufficiently small, i.e.

$$\|g - g^\times\|_{C^4} < \varepsilon, \varepsilon \text{ sufficiently small.}$$

Let $\tilde{m}_0 \in M_1$ be a point such that $\widetilde{\Delta} u(\tilde{m}_0) < 0$. The existence of $\tilde{m}_0$ follows from the method of integration by parts. Indeed, if $\widetilde{\Delta} u$ is non-negative on M_1. From the integration by parts, we know that

$$\int_{M_1} (u - \min_{M_1} u)\widetilde{\Delta} u = -\int_{M_1} |\widetilde{\nabla} u|^2.$$

Thus h is trivial, this gives the contradiction.

Fix a point $\hat{m}_0 \in M_2$. Let $z = (x, y)$ be a local coordinate system where x is the geodesic normal coordinate system in M_1 centered at $\tilde{m}_0$ and y is the geodesic normal coordinate system in M_2 centered at $\hat{m}_0$.

It is known that the Riemann curvature tensor Riem^u of the conformal metric g is given by

$$\text{Riem}^u = e^{-2u}[\text{Riem} + (\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g^\times) \owedge g^\times].$$

Fix $i \in \{1, \cdots, n_1\}$. Thus

$$\text{Riem}^u_{i(n_1+1)i(n_1+1)} = e^{-2u}[\widetilde{\nabla}^i_i u + (\widetilde{\nabla}_i u)^2 - |\widetilde{\nabla} u|^2].$$

As a direct consequence,

$$\sum_{i=1}^{n_1} \text{Riem}^u_{i(n_1+1)i(n_1+1)}(\tilde{m}_0, \hat{m}_0) = e^{-2u}[\widetilde{\Delta} u - (n_1 - 1)|\widetilde{\nabla} u|^2](\tilde{m}_0) < 0. \tag{5.65}$$

It is known that the $A3W$ condition is a necessary condition of the continuity of the optimal transport map. Note that the $A3W$ condition implies that the sectional curvature on plane which is spanned by the mutually perpendicular vector is non-negative. Using (5.65), we know that there exists some optimal transport map on $(M_1 \times M_2, e^{-2u} g^\times)$ with the regular positive probability measures is not continuous.

In conclusion, we derive Theorem 5.3.

Chapter 6

Perspectives

In this thesis, we mainly prove the smoothness of the optimal transport map on two classes of compact Riemannian manifold which are nearly spherical manifolds and Riemannian products of nearly spherical manifolds. It is interesting to find other manifolds such that the corresponding optimal transport map is smooth.

We list some open questions about the optimal transportation in references.

(1) Does the A3W condition imply that the injectivity domain is convex?

Loeper-Villani [75] showed that the A3S condition implied that injectivity domain is uniformly convex in case of non-focal Riemannian manifold. Figalli-Gallouët-Rifford [38] showed that the A3W condition deduced that injectivity domain is convex in case of non-focal Riemannian manifold.

(2) Is the continuity of the optimal transport map equivalent to the A3W condition and the convexity of the injectivity domain?

Figalli-Rifford-Villani [46] proved that the A3W condition and the convexity of the injectivity domain is necessary for the continuity of the optimal transport map and also sufficient in dimension 2.

(3) Does one has the control on the Hausdorff dimension of the singular set?

Figalli [37] proved that the singular set is a 1-dimensional manifold of class C^1 out of a countable set in the plane when the target is not convex. But the result is not known in high dimension.

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